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Cable Braid Electromagnetic Penetration Model

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Abstract

The model for penetration of a wire braid is rigorously formulated. Integral formulas are developed from energy principles and reciprocity for both self and transfer immittances in terms of potentials for the fields. The detailed boundary value problem for the wire braid is also set up in a very efficient manner; the braid wires act as sources for the potentials in the form of a sequence of line multipoles with unknown coefficients that are determined by means of conditions arising from the wire surface boundary conditions. Approximations are then introduced to relate the local properties of the braid wires to a simplified infinite periodic planar geometry. This is used in the preceding integral formulas to be able to treat nonuniform coaxial geometries including eccentric interior coaxial arrangements and an exterior ground plane.

Acknowledgments

The authors would like to thank R. E. Jorgenson for many discussions regarding cable penetration models. The authors would like to thank Rebecca S. Coats for the setup of the braid geometry. The authors would also like to thank H. G. Hudson for many discussions regarding cable penetration experiments.

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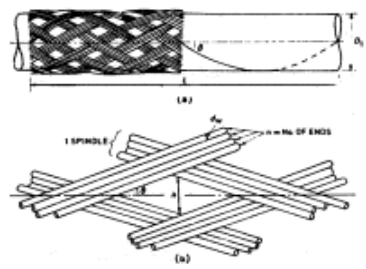

Figure 1. Braid structure on cylinder (figure is from [1]).

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1 SUMMARY

This report discusses the formulation of the cable braid penetration model. Figure 1 shows a typical cylindrical cable braid. We begin with the electric problem by defining the self and transfer admittances of the braid. Next the perfectly conducting self and transfer impedances of the braid are defined for the magnetic problem. Finally, the finitely conducting transfer impedances are setup. As an asymptotic simplification we examine the planar approximation to the cylindrical braid and extract the electric potential and magnetic flux penetration properties which can then be applied to generalized coaxial problems through integral formulas (many of which are derived from reciprocity). Examples of both uniform cylindrical and asymmetric coaxial geometries (exterior ground planes and eccentric interiors) are examined.

1.1 Transfer Admittance

The braid transfer admittance (or capacitance) arises from the electric field penetration of the braid as shown in Figure 2. This penetration can play a role, particularly with low optical coverage cable braids, an example of which is shown in Figure 3. The electric scalar potential is used to formulate the self and transfer admittances of the braid.

The planar braid approximation involves two key quantities. First, is the potential at a large distance from the braid on the shadow side when the opposite side is driven by a normal uniform electric field; this potential enters the transfer admittance problem. Second, is the spatial shift in potential reference on the illuminated side of the braid relative to the braid center; this potential shift enters the self admittance problem.

A line charge multipole basis is introduced to facilitate the detailed electric solution of the periodic

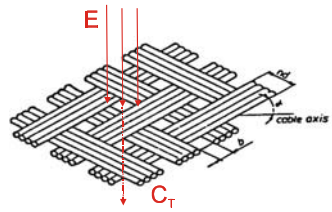

Figure 2. Electric field penetration of a planar approximation to the braid, which is connected to the transfer admittance or transfer capacitance C_T (original braid figure from [7]).

planar braid problem. Different dielectric materials adjacent to the braid are also treated by means of images.

1.2 Transfer Impedance

The braid transfer impedance (or inductance for perfectly conducting braid wires) arises from the magnetic field penetration of the braid as shown in Figures 4 (the braid holes can be seen in Figure 5) and 6. The magnetic penetration of the braid holes shown in Figure 4 can be significant for medium levels of optical coverage, an example of which is shown in Figure 5. The intra-braid coupling known as porpoising shown in Figure 6 is important for both medium levels of optical coverage (where it opposes the magnetic hole penetration) and for high values of optical coverage, an example of which is shown in Figure 7. Both magnetic vector and scalar potentials are examined for the formulation of the perfectly conducting magnetic braid problem.

The planar braid approximation again involves two key quantities. First, is the magnetic flux penetrating the braid on the shadow side when the opposite side is driven by a tangential uniform magnetic field; the vector potential representing this flux enters the transfer impedance problem. Second, is the magnetic potential shift in spatial reference on the illuminated side of the braid relative to the braid center; this potential shift enters the self impedance problem.

A line multipole basis is introduced to facilitate the detailed magnetic solution of the periodic planar magnetic braid problem with the net currents represented by the vector potential part.

1.2.1 Finite Conductivity Braid

The finite conductivity of the braid wires requires us to treat the internal problem of the braid wires. We use a complex power approach in addition to a reciprocity approach based on the preceding perfectly conducting treatment to formulate the contribution from the inside of the wires to the impedance per unit length of the transmission line and the transfer impedance per unit length, which include losses within the wires and perturbations of the magnetic field both inside and outside the wires. We represent the internal problem by means of the axial components of the two Hertz potentials. The internal normal component of the electric current density is taken to approximately vanish at the braid wire surface. We also match the internal magnetic field to the magnetic potentials of the outer problem at the braid wire surfaces.

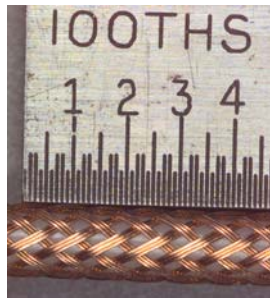

Figure 3. Relatively low (≈60%) optical coverage braid illustrating the inherent braid apertures (figure for Remee 1400 cable from [1]).

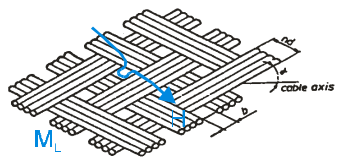

Figure 4. Magnetic penetration of braid apertures illustrated with approximate planar braid model (original braid figure from [7]). This is connected with the transfer impedance. The inductance M_L is part of the transfer inductance L_T.

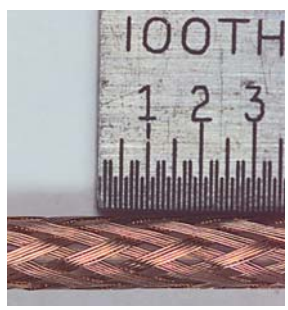

Figure 5. Medium (≈78%) optical coverage braid (figure for Belden 9201 cable from [1]).

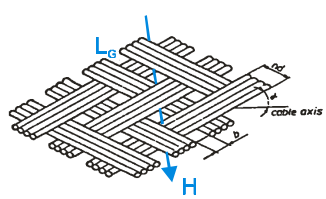

Figure 6. Intrabraid magnetic porpoising penetration (original braid figure from [7]). This is also connected with the transfer impedance of the braid. The inductance L_G is part of the transfer inductance L_T.

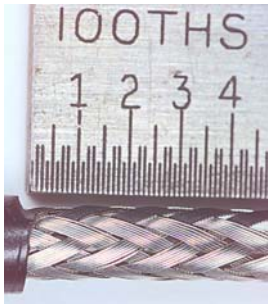

Figure 7. High (≈95%) optical coverage braid where the intrabraid porpoising contribution is the dominant transfer impedance contribution (figure for Belden 8240 cable from [1]).

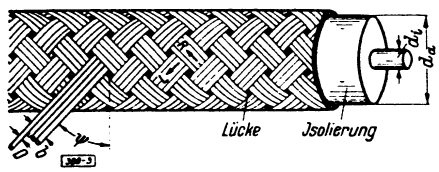

Figure 8. Illustration of a coaxial cable with braided shield (figure from [4]).

2 INTRODUCTION

In this paragraph we give a very brief list of some of the references on cable braid penetration. Electromagnetic penetration of shielded cables is an important and interesting subject with a long history. Early work on solid shields can be found in Schelkunoff's paper [2]. Eddy current penetration of cable shields is discussed in Kaden's book [3] along with some models for apertures in thick screens. Measurements of braid coupling are given in [4]. A classic text on cable braids is Vance's book [5]. Lee's book [6] also has valuable information on cables and shielding. The porpoising contribution to the transfer inductance of a cable braid was introduced by Tyni [7]. Various improvements in the geometrical description were made by Sali [8] and a discussion of the low frequency diffusion is given by Guofu [9]. Kley [10] improved and assembled all these contributions into a complete semi-empirical model where some parameters were based on measurements of typical commercial cables. The book by Tesche [11] also has a nice summary of these models. These models are quite useful and identify the fundamental penetration mechanisms. Nevertheless, a first principles model directly based on the braid geometry would be desirable, particularly when a selected cable deviates at all from typical geometries employed in commercial cables.

This report rigorously formulates the cable braid penetration problem. We start with a coaxial topology and use the electric energy to define the self capacitance and the transfer capacitance. We then discuss simplified scalar reciprocity approach following a similar procedure to that of Latham [12]; this method has advantages in that breaks in symmetry of the coaxial region or in the exterior drives (as noted at the end of this paragraph) can be straightforwardly included; approximations to the energy arguments are also used to arrive at the same results. The integral quantities for the transfer immittances that describe the coupling are identified. These depend on the external (outside the metallic conductors) scalar potentials representing the fields. A line multipole representation for the wire charges and currents is used to simplify the description; in the past we have found line multipoles to be particularly efficient in modeling wire currents, such as arrays [13]. Images of these charges are used to treat adjacent dielectric material surfaces. The coaxial problem is next approximately cast into a planar geometry with periodic symmetries to simplify the analysis of the penetrations. The electric coupling is treated first. Magnetic coupling is then treated; both hole and porpoising contributions are included in a self-consistent way. Magnetic energy is used to define the self and transfer inductances. Next, the magnetic diffusion into the conductor is introduced in the magnetic problem. The formulation begins with a symmetric coaxial geometry, but nonsymmetric geometries, including an exterior ground plane and an eccentric interior, are also examined. Figure 8 shows an example of a braided shield. Figure 9 shows a more typical braid rolled out on a flat surface. Figure 10 shows a simplified cylindrical topology of the center conductor - braided shield - chassis system. Figure 11 shows the transfer immittance sources associated with the braid penetration.

3 ELECTROSTATIC COUPLING FORMULATION

The goal of this section is to formulate the transfer admittance per unit length, or transfer capacitance per unit length C_T, of the braid penetration, where

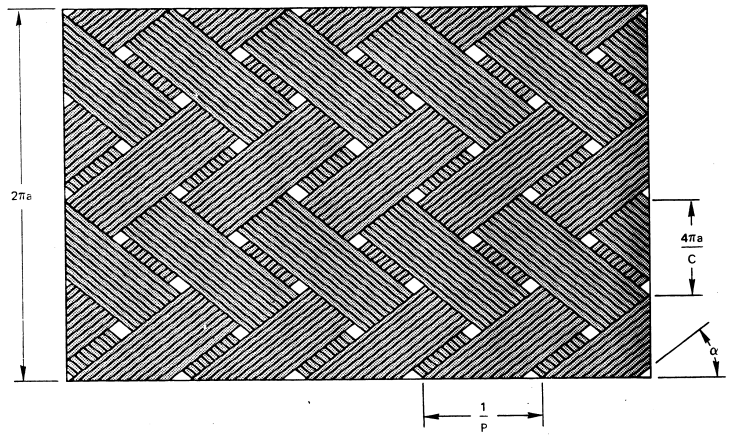

Figure 9. Expanded view of a braided shield (figure from [5])

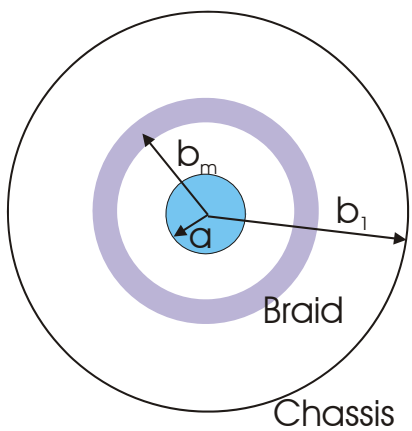

Figure 10. Cylindrical topology of center conductor - braided shield - chassis system.

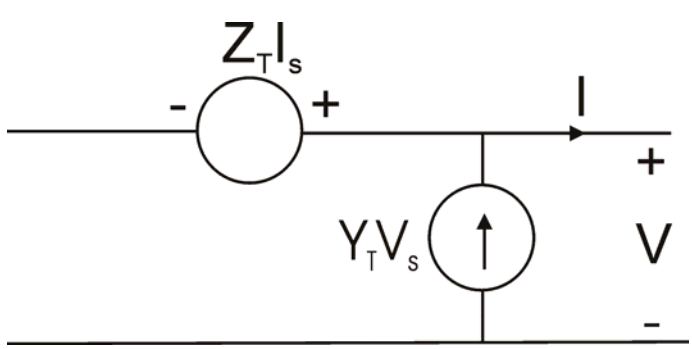

Figure 11. Transfer admittance and transfer impedance transmission line source model for braided cable. The upper conductor in the depicted model represents, for example, the center conductor of the inner coaxial transmission line, and the direction of propagation is taken to the right, in positive z.

$$Y_T = -i\omega C_T \tag{1}$$

and time dependence $e^{-i\omega t}$ is suppressed, as well as the inner coaxial admittance per unit length Y_1, or capacitance per unit length C_1, where

$$Y_1 = -i\omega C_1 \tag{2}$$

Even though the braid has a large but finite conductivity, the asymptotic form of these local quantities for large conductivities can be arrived at by treating the braid wires as perfect electric conductors.

We will assume that there is enough incidental contact for the braid wires to locally be at an equal potential, which we usually take to vanish.

3.1 Capacitance Per Unit Length

First we use the electrostatic energy to define the self capacitance. The electric displacement is

$$\underline{D} = \varepsilon \underline{E} \tag{3}$$

where ε is the electric permittivity (for example, it could be assumed to be piecewise constant throughout the volume with values ε_j in subregions $j = 1, ..., J$). The capacitance in a system is determined by equating the electrostatic energies

$$\frac{1}{2}CV^2 = \frac{1}{2}\int_V \underline{D} \cdot \underline{E} dV = \frac{1}{2}\int_V DEdV = \frac{1}{2}\int_V \varepsilon E^2 dV \tag{4}$$

Using the potential ϕ where the electric field is

$$\underline{E} = -\nabla\phi \tag{5}$$

we find

$$\frac{1}{2}CV^2 = \frac{1}{2}\int_V \varepsilon \nabla\phi \cdot \nabla\phi dV \tag{6}$$

Noting that

$$\nabla \cdot (\phi\varepsilon\nabla\phi) = \varepsilon\nabla\phi \cdot \nabla\phi + \phi\nabla \cdot (\varepsilon\nabla\phi) \tag{7}$$

and by virtue of Gauss's law

$$-\nabla \cdot (\varepsilon\nabla\phi) = \nabla \cdot (\varepsilon\underline{E}) = \nabla \cdot \underline{D} = \rho_v \tag{8}$$

where ρ_v is the volumetric electric charge density, gives

$$CV^2 = -\oint_S \phi\underline{D} \cdot \underline{n} dS + \int_V \phi\rho_v dV \tag{9}$$

where the unit outward normal to the volume is $\underline{n}$. If we stay outside the conductors the final term vanishes

$$CV^2 = -\oint_S \phi D_n dS \tag{10}$$

Suppose we break up the surface into the several parts, the center conductor (S_c is a cylindrical surface just outside the center conductor), the braid wires (S_w consists of meandering cylindrical surfaces just outside each of the braid wires), and the return (chassis) ($S_{chassis}$ is a surface just inside the outer chassis return if the chassis is a simple cylinder, otherwise it could be any conforming surface, such as a plane above a planar chassis return)

$$CV^2 = -\int_{S_c} \phi D_n dS - \int_{S_w} \phi D_n dS - \int_{S_{chassis}} \phi D_n dS \tag{11}$$

where we have dropped the end surfaces because we apply this over a short axial period ℓ and we assume the field and potential are periodic (with the normal direction $\underline{n}$ reversing on the two end surfaces). The potential is a constant on the various conductors. Hence the integral of the normal displacement produces the negative of the electric charge on the particular conductor (negative because the unit normal is pointing into the conductor)

$$CV^2 = V_c Q_c + V_w Q_w + V_{chassis} Q_{chassis} \tag{12}$$

Taking the braid wires to be at zero potential $V_w = 0$ gives

$$CV^2 = V_c Q_c + V_{chassis} Q_{chassis} \tag{13}$$

Now for the interior coaxial mode we take $V_{chassis} = 0$ and set $V_c = V_1 = V$ with $Q_c = q_1 \ell$, and ℓ is an axial braid period. Then with $C = C_1$

$$C_1 V_1 = q_1 \tag{14}$$

we obtain the usual definition of the self capacitance per unit length C_1. It is frequently convenient to take the chassis sufficiently far away to ignore the charge $Q_{chassis} \to 0$ (although this could be included to obtain coupling to the outer line from excitation of the inner line).

3.1.1 Transfer Capacitance Per Unit Length

The capacitance matrix for a two conductor system (with a reference, the braid) can be written as

$$Q_1 = C_{11} V_1 + C_{12} V_2 \tag{15}$$

$$Q_2 = C_{21} V_1 + C_{22} V_2 \tag{16}$$

We intend to take conductor 1 as the center conductor and conductor 2 as the chassis. In a reciprocal media (which we are assuming throughout), the cross terms are equal $C_{12} = C_{21} = C_m$, where C_m is called the mutual capacitance. The continuity equation is

$$\nabla \cdot \underline{J} = -\frac{\partial}{\partial t} \rho_v \tag{17}$$

or in integral form

$$\oint_S \underline{J} \cdot \underline{n} dS = -\frac{\partial}{\partial t} \int_V \rho_v dV \tag{18}$$

This yields

$$\frac{\partial I}{\partial z} = -\frac{\partial q}{\partial t} \tag{19}$$

where I is the net current on a conductor and q is the charge per unit length on a conductor. Hence, from the continuity equation we can determine the current changes over a short periodic section along the line as

$$I_1(z+\ell) - I_1(z) = -\frac{dQ_1}{dt} = -\frac{d}{dt}(\ell q_1) = -C_{11}\frac{dV_1}{dt} - C_{12}\frac{dV_2}{dt} \tag{20}$$

$$I_2(z+\ell) - I_2(z) = -\frac{dQ_2}{dt} = -\frac{d}{dt}(\ell q_2) = -C_{21}\frac{dV_1}{dt} - C_{22}\frac{dV_2}{dt} \tag{21}$$

The power removed from a periodic section of line is minus the derivative of the stored electric energy

$$V_1[I_1(z+\ell) - I_1(z)] + V_2[I_2(z+\ell) - I_2(z)] = -\frac{d}{dt}\left(\frac{1}{2}C_{11}V_1^2 + C_m V_1 V_2 + \frac{1}{2}C_{22}V_2^2\right) \tag{22}$$

If we take two sources, denoting the resulting field due to these by subscripts 1 and 2, then by superposition we can write the total field as

$$\underline{E} = \underline{E}_1 + \underline{E}_2 \tag{23}$$

Equating electric energies in a region gives

$$\frac{1}{2}C_{11}V_1^2 + C_m V_1 V_2 + \frac{1}{2}C_{22}V_2^2 = \frac{1}{2}\int_V \underline{D} \cdot \underline{E} dV = \frac{1}{2}\int_V DE dV$$

$$= \frac{1}{2}\int_V D_1 E_1 dV + \frac{1}{2}\int_V \underline{D}_1 \cdot \underline{E}_2 dV + \frac{1}{2}\int_V \underline{E}_1 \cdot \underline{D}_2 dV + \frac{1}{2}\int_V D_2 E_2 dV$$

$$= \frac{1}{2}\int_V \varepsilon E_1^2 dV + \int_V \varepsilon \underline{E}_1 \cdot \underline{E}_2 dV + \frac{1}{2}\int_V \varepsilon E_2^2 dV \tag{24}$$

Here we can identify the self capacitance terms $C_{11} = \ell C_1$ and $C_{22} = \ell C_{sh}$, as well as the mutual capacitance term C_m, with the field integrals. Taking $C_m = -\ell C_T$ and $Q_2 = -Q_{sh}$ and $V_2 = -V_{sh}$

$$Q_1 = C_{11}V_1 - C_m V_{sh} = \ell C_1 V_1 + \ell C_T V_{sh} \tag{25}$$

$$Q_{sh} = -C_m V_1 + C_{22}V_{sh} = \ell C_T V_1 + \ell C_{sh} V_{sh} \tag{26}$$

Now setting the center conductor voltage to zero (the same as the braid) $V_1 = V_c = 0$, $V_2 = -V_{sh}$, and taking $Q_2 \approx -\ell q_{sh}$ with $Q_1 = \ell q_1 = \ell q_T$ gives

$$q_1 = C_T V_{sh} = q_T \tag{27}$$

$$q_{sh} = C_{sh} V_{sh} \tag{28}$$

where C_{sh} is the outer line capacitance per unit length and q_{sh} is the outer line charge per unit length on the braid (again these are defined with a potential difference V_{sh} on the outer line, potential $-V_{sh} = V_{chassis}$ on the chassis, with the center conductor at zero potential, the same as the braid wires). Then we find the current source as

$$I_1(z+\ell) - I_1(z) = i_s \ell = -\frac{dQ_1}{dt} = -\frac{d}{dt}(\ell q_1) = -\frac{d}{dt}(\ell q_T) \tag{29}$$

or

$$i_s = -\frac{dq_T}{dt} = i\omega q_T = i\omega C_T V_{sh} = -Y_T V_{sh} \tag{30}$$

If we place a negative voltage $V_{chassis} = -V_{sh}$ on the ground return with the braid at zero potential, we expect the center conductor charge when held at zero potential (the same as the braid) to be positive. This means that C_m is negative, and with our preceding definition, transfer capacitance is a positive quantity.

The transfer capacitance ℓC_T is

$$\ell C_T = -\frac{1}{V_1 V_2}\int_V \underline{D}_1 \cdot \underline{E}_2 dV = -\frac{1}{V_1 V_2}\int_V \underline{D}_2 \cdot \underline{E}_1 dV = -\frac{1}{V_1 V_2}\int_V \varepsilon \underline{E}_1 \cdot \underline{E}_2 dV \tag{31}$$

Taking the fields to be given by the gradient of a scalar potential and using Gauss's law we can write

$$\underline{E}_1 = -\nabla\phi_1 \tag{32}$$

$$-\nabla\cdot(\varepsilon\nabla\phi_1) = \nabla\cdot(\varepsilon\underline{E}_1) = \nabla\cdot\underline{D}_1 = \rho_{v1} \tag{33}$$

$$\underline{E}_2 = -\nabla\phi_2 \tag{34}$$

$$-\nabla\cdot(\varepsilon\nabla\phi_2) = \nabla\cdot(\varepsilon\underline{E}_2) = \nabla\cdot\underline{D}_2 = \rho_{v2} \tag{35}$$

Noting that

$$\underline{D}_1 \cdot \underline{E}_2 = -\underline{D}_1 \cdot \nabla\phi_2 = -\nabla\cdot(\phi_2\underline{D}_1) + \phi_2\nabla\cdot\underline{D}_1 = -\nabla\cdot(\phi_2\underline{D}_1) + \phi_2\rho_{v1} \tag{36}$$

$$\underline{E}_1 \cdot \underline{D}_2 = \underline{D}_2 \cdot \nabla\phi_1 = -\nabla\cdot(\phi_1\underline{D}_2) + \phi_1\nabla\cdot\underline{D}_2 = -\nabla\cdot(\phi_1\underline{D}_2) + \phi_1\rho_{v2} \tag{37}$$

and using the divergence theorem gives

$$\begin{aligned} C_T &= \frac{1}{\ell V_1 V_2}\oint_S \phi_2 \underline{D}_1 \cdot \underline{n} dS - \frac{1}{\ell V_1 V_2}\int_V \phi_2 \rho_{v1} dV \\ &= \frac{1}{\ell V_1 V_2}\oint_S \phi_1 \underline{D}_2 \cdot \underline{n} dS - \frac{1}{\ell V_1 V_2}\int_V \phi_1 \rho_{v2} dV \end{aligned} \tag{38}$$

Applying this outside the metallic surfaces the volume integral vanishes

$$C_T = \frac{1}{\ell V_1 V_2} \oint_S \phi_2 \underline{D}_1 \cdot \underline{n} dS = \frac{1}{\ell V_1 V_2} \oint_S \phi_1 \underline{D}_2 \cdot \underline{n} dS \tag{39}$$

Using the second equality and setting $V_2 = -V_{sh}$ and $\underline{D}_2 = \underline{D}_{sh}$ (we do not reverse the sign of the potential $\phi_2 = \phi_{sh}$ in the shield circuit even though we have reversed the sign of the chassis voltage and charge per unit length relative to the 2 problem; note that we construct the potential ϕ_{sh} with $\phi_{sh} = 0$ on the braid wires and on the center conductor) and noting that $V_1 = V_c$ with $\phi_1 = 0$ on the braid wires and on the chassis, we have

$$C_T = -\int_{S_c} \frac{1}{V_{sh}} \underline{D}_{sh} \cdot \underline{n} dS/\ell = q_T/V_{sh} = C_{sh} q_T/q_{sh} \tag{40}$$

where the surface integral on the braid wires and on the chassis vanishes because $\phi_1 = 0$ on these surfaces. The preceding integral produces the charge $Q_1 = Q_T = \ell q_T$ on the center conductor over a length ℓ, given a voltage drive $-V_{sh}$ on the outer line. The current source per unit length i_s is the right hand side of the interior transmission line equation

$$\frac{dI_1}{dz} + Y_1 V_1 = i_s \tag{41}$$

Note that for an open circuited inner line we can set the center conductor charge to vanish from the original charge equation

$$Q_1 = 0 = C_{11} V_1 + C_m V_2 = \ell C_1 V_c + \ell C_T V_{sh} \tag{42}$$

which gives the required center conductor potential to create an open circuit condition

$$V_c = -\phi_c = -C_T V_{sh}/C_1 = -q_T/C_1 \tag{43}$$

In terms of this potential we can write the transfer capacitance as

$$C_T = q_T/V_{sh} = \phi_c C_1 C_{sh}/q_{sh} \tag{44}$$

3.2 Integral Forms For Perturbation Of Planar Braid Transfer And Self Capacitances Per Unit Length

We now derive integral expressions using a simplified scalar reciprocity for the transfer capacitance per unit length and for the braid correction to the inner coaxial capacitance per unit length. These are useful in that they allow us to apply the properties of a planar approximation to the braid, not only to uniform coaxial geometries, but also to nonuniform geometries such as an interior eccentric coax and an exterior ground plane.

Following the derivation in [12] we consider an auxiliary electrostatic potential of the coaxial structure ϕ_0 without shield penetrations, satisfying

$$\nabla \cdot (\varepsilon \nabla \phi_0) = 0 \tag{45}$$

with linear charge density q_0 on the center conductor and $-q_0$ on the inside surface of the auxiliary shield. The meaning of "without shield penetrations" requires some further elaboration in this case where the

braid is a set of interweaving wires. The shield in the auxiliary problem is taken as a perfectly conducting shell. The geometry of the shell should conform to the braid geometry in the following sense: it should (at least initially) be inside the original braid geometry and stood off by a small fixed distance (large compared to the order of the braid spacing but small compared the global cross section geometry), but should follow the original braid contour.

We take the electric field to be the negative gradient of the potential

$$\underline{E}_0 = -\nabla\phi_0 \tag{46}$$

and the electric displacement to be

$$\underline{D}_0 = \varepsilon\underline{E}_0 \tag{47}$$

where in the source free region

$$\nabla\cdot\underline{D}_0 = 0 \tag{48}$$

and hence this potential satisfies

$$\nabla\cdot(\varepsilon\nabla\phi_0) = 0 \tag{49}$$

The boundary conditions are set to

$$\phi_0 = 0 \text{ on auxiliary shield}$$

$$= V_0 = q_0/C_0 \text{ on center conductor} \tag{50}$$

where C_0 is the capacitance per unit length of the auxiliary coax. For a cylindrical coax with inner and outer radii, a and b, respectively, and for a constant dielectric, we have the capacitance per unit length

$$C(b) = \frac{2\pi\varepsilon}{\ln(b/a)} \tag{51}$$

The auxiliary shield has capacitance per unit length (b_0 is the radius of the solid auxiliary shield)

$$C_0 = C(b_0) = \frac{2\pi\varepsilon}{\ln(b_0/a)} \tag{52}$$

The second problem has shield penetrations. The electrostatic potential in this problem outside the braid wires also satisfies

$$\nabla\cdot(\varepsilon\nabla\phi) = 0 \tag{53}$$

and the boundary conditions

$$\phi = 0 \text{ on shield metallic surfaces (braid wires)} \tag{54}$$

and

$$\phi = V = q_1/C_1 \text{ on center conductor} \tag{55}$$

Here C_1 is the capacitance per unit length of the actual coaxial system and q_1 is the charge per unit length created by the potential difference V between center conductor and braid (not including any induced charge from exterior fields penetrating the braid). We set the potentials in the two problems to be the same value

$$V = V_0 \tag{56}$$

To have $q_1 \to q_0$ would also require $C_1 = C_0$; this equality does not hold when C_1 is taken to be the actual braided coaxial capacitance per unit length since $C_1 \neq C_0$. In this case the charge per unit length q_1 and capacitance per unit length C_1 will include not only a term for the correction of the braid structure, but also a term to correct for the arbitrary choice of the auxiliary shield position (in the cylindrical case radius b_0), in addition to the auxiliary capacitance C_0.

The difference electrostatic potential between this actual problem and the auxiliary closed shield problem

$$\phi_d = \phi - \phi_0 \tag{57}$$

also satisfies

$$\nabla \cdot (\varepsilon \nabla \phi_d) = 0 \tag{58}$$

but the boundary conditions in this case can be taken as

$$\phi_d = 0 \text{ on center conductor} \tag{59}$$

$$\phi_d = \phi \text{ on auxiliary shield} \tag{60}$$

This difference potential does not satisfy Laplace's equation at the surface of the wire conductors of the actual shield because of the electric charge density on these surfaces but does satisfy it throughout the free volume. Hence, we can write

$$\phi_0 \nabla \cdot (\varepsilon \nabla \phi_d) - \phi_d \nabla \cdot (\varepsilon \nabla \phi_0) = \nabla \cdot (\phi_0 \varepsilon \nabla \phi_d - \phi_d \varepsilon \nabla \phi_0) = 0$$

Integration over the volume V_0 of the auxiliary problem (not to be confused with the auxiliary voltage) yields

$$\oint_{S_0} \left(\phi_0 \varepsilon \frac{\partial \phi_d}{\partial n} - \phi_d \varepsilon \frac{\partial \phi_0}{\partial n} \right) dS = 0 \tag{61}$$

where the closed surface S_0 bounds the auxiliary volume V_0 and consists of the center conductor surface S_c, the outer auxiliary surface S_0 (the open cylindrical-type surface) and the closing periodic surfaces at the ends of the segment in z. The first term vanishes on the outer auxiliary shield surface S_0 since $\phi_0 = 0$ there. The second term vanishes on the center conductor surface S_c since $\phi_d = 0$. We assume that the two end disc surfaces of S are placed one braid period ℓ apart, so that these two integrals cancel with the reversal of sign of $\underline{n}$. Thus we can write

$$\int_{S_c} \phi_0 \varepsilon \frac{\partial \phi_d}{\partial n} dS = V_0 \int_{S_c} \varepsilon \frac{\partial \phi_d}{\partial n} dS = \int_{S_0} \phi_d \varepsilon \frac{\partial \phi_0}{\partial n} dS \tag{62}$$

The difference charge on the center conductor over the short length ℓ is

$$Q_d = \ell q_d = \int_{S_c} \varepsilon \frac{\partial \phi_d}{\partial n} dS = \ell \left(q - q_0\right) = Q - Q_0 \tag{63}$$

where $\underline{n}$ points into the center conductor, Q is the actual total charge, Q_0 is the total charge in the auxiliary problem, and q_d is the difference charge per unit length. Thus we can write the difference charge as

$$Q_d = (1/V_0) \int_{S_0} \phi_d \varepsilon \frac{\partial \phi_0}{\partial n} dS = \frac{C_0}{q_0} \int_{S_0} \phi_d \varepsilon \frac{\partial \phi_0}{\partial n} dS \tag{64}$$

The quantity of interest is the distributed current source resulting from this difference charge. The current source per unit length of the inner transmission line resulting from braid coupling from the current continuity equation is i_d (note that $\phi_d = \phi$ on S_0 since $\phi_0 = 0$)

$$I_d = \ell i_d = -\frac{dQ_d}{dt} = i\omega Q_d = i\omega C_0 \int_{S_0} \phi \frac{1}{q_0} \varepsilon \frac{\partial \phi_0}{\partial n} dS \tag{65}$$

For a uniform cylindrical coaxial structure by Gauss's law (where $\underline{n}$ points outward on the surface S_0)

$$\frac{1}{q_0} \varepsilon \frac{\partial \phi_0}{\partial n} = -\frac{1}{2\pi b_0} = -\frac{1}{P_0} \tag{66}$$

with perimeter

$$P_0 = 2\pi b_0 \tag{67}$$

Thus

$$i_d = -\frac{i\omega C_0}{\ell P_0} \int_{S_0} \phi dS \tag{68}$$

Now this term is taken to be

$$i_d = i_s - \Delta Y V_0 \tag{69}$$

where the source term is proportional to the exterior mode charge or voltage and the second admittance term corrects the capacitance per unit length of the actual cable from the auxiliary value C_0 (the minus sign results from the fact that the correction is a passive element). By superposition we can think of the total potential ϕ as consisting of two parts

$$\phi = \phi_s + \phi_1 \tag{70}$$

where ϕ_s is caused by the exterior potential difference V_{sh} and ϕ_1 is caused by the interior potential difference V_0, and the voltage V_0 is the voltage of the center conductor. We can write

$$\Delta Y = -i\omega \Delta C \tag{71}$$

and the capacitance per unit length of the cable as

$$C_1 = C_0 + \Delta C \tag{72}$$

$$Y_1 = -i\omega C_1 \tag{73}$$

with the charge per unit length due to the interior voltage

$$q_1 = C_1 V_0 \tag{74}$$

The exterior mode has voltage $-V_{sh}$ (here we take the voltage V_2 to have positive reference on the chassis so that the shield can remain at zero potential). Thus we can write the current source as

$$i_s = -Y_T V_{sh} = i\omega C_T V_{sh} = i\omega q_s \tag{75}$$

where C_T is the transfer capacitance per unit length of the braid and q_s is the source charge per unit length. The transmission line equation for the interior current can then be written as

$$\frac{dI}{dz} = -Y_1 V_0 + i_s \tag{76}$$

The capacitance per unit length correction is found from that part of ϕ resulting from taking the voltage $V_{sh} = 0$ and then

$$\Delta C = -\frac{C_0}{\ell P_0} \int_{S_0} (\phi_1/V_0)\, dS \tag{77}$$

The transfer capacitance per unit length is that part of ϕ found by setting $V_0 = 0$ and then

$$C_T = -\frac{C_0}{\ell P_0} \int_{S_0} (\phi_s/V_{sh})\, dS \tag{78}$$

The more general formulas, which do not assume a circular cross section for the coaxial cable, are

$$C_T = -\frac{C_0}{\ell} \int_{S_0} (\phi_s/V_{sh}) \left(-\frac{1}{q_0} \varepsilon \frac{\partial \phi_0}{\partial n} \right) dS \tag{79}$$

$$\Delta C = -\frac{C_0}{\ell} \int_{S_0} (\phi_1/V_0) \left(-\frac{1}{q_0} \varepsilon \frac{\partial \phi_0}{\partial n} \right) dS \tag{80}$$

In the next subsections we discuss the self and transfer capacitances per unit length for a simple circular coaxial geometry (but we will incorporate the simplification of a local planar approximation to the braid, which can be done when the local braid structure is small compared to the global cross section geometry) using (77) and (78), and then return to the more general nonuniform coaxial geometry with the examples of an exterior ground plane and an interior eccentric coaxial structure using (79) and (80).

3.2.1 Self Capacitance Correction

The preceding expressions are convenient to use when the potentials are known in the coaxial geometry of interest. But to make use of the potential solutions from an approximate planar rendition of the braid geometry, where the excitation will take the form of a uniform field E_0 rather than a voltage, we have to choose the radial position in the coaxial geometry to define the field strength at the braid in order to normalize the field drive in the planar geometry. To introduce the field strength into this expression for a uniform cylindrical problem we set

$$C_1 V_0 = P\varepsilon E_\rho = P\varepsilon E_0 \tag{81}$$

where we use the capacitance per unit length of the actual braid transmission line along with the mean perimeter

$$P = 2\pi b \tag{82}$$

to determine the extrapolation of the field to the braid center line, since this extrapolated field should have the best correspondence to the uniform drive field in the planar problem. Note that in a later section we will set up the local planar problem with dielectric materials and a uniform driving field; this drive field will be the electric displacement $D_0 = \varepsilon E_0$. The capacitance per unit length of the braid interior transmission line for a uniform dielectric material is approximately that of the capacitance to the mean radius b of the braid center line

$$C_1 \approx C = \frac{2\pi\varepsilon}{\ln(b/a)} \tag{83}$$

or more accurately, as we will see below in (94), $C_1 = 2\pi\varepsilon/\ln\left[\left(b + \phi_b/E_0\right)/a\right]$.

First Derivation The correction to the self capacitance per unit length is (here we will approximate the actual capacitance per unit length C_1 by the center line value C in the first order term ΔC)

$$\Delta C = -\frac{C_0 C_1}{P\varepsilon}\frac{1}{\ell P_0}\int_{S_0}\left(\phi_1/E_0\right)dS \approx -\frac{C_0 C}{P\varepsilon}\frac{1}{\ell P_0}\int_{S_0}\left(\phi_1/E_0\right)dS \tag{84}$$

The potential at a large interior distance from the braided shield wall compared to the local braid wire mesh structure (but near the shield compared to the distance $b-a$) for the $V_{sh} = 0$ part becomes

$$\phi_1 \sim E_0 x + \phi_b \tag{85}$$

and then

$$\Delta C \sim -\frac{CC_0}{P\varepsilon}x - \frac{CC_0}{P\varepsilon}\left(\phi_b/E_0\right) \tag{86}$$

Note that x changes positively when moving into the inner coaxial region from the braid; hence the effective braid surface corresponds to a shift in the solid shield geometry to $x = -\phi_b/E_0$, where the $x = 0$ position is at the braid center. The planar problem with a negative x directed field E_0 oriented normal to the shield wall of braid wires at zero potential (with center line at $x = 0$) having the potential $\phi_1 \sim E_0 x + \phi_b$, $x > 0$ implies an effective position of an equivalent ground plane at $x = -\phi_b/E_0$; hence the constant ϕ_b/E_0 is only dependent on the braid geometry and is positive for very thin wires and negative for very fat wires. Let us also write the capacitance C_0 in terms of the center line capacitance C using (83)

$$C_0 \sim C + \frac{\partial C}{\partial b}\left(b_0 - b\right) = C\left[1 - \frac{C}{P\varepsilon}\left(b_0 - b\right)\right] \tag{87}$$

Combining these gives

$$C_0 + \Delta C \sim C\left[1 - \frac{C}{P\varepsilon}\left(b_0 - b\right) - \frac{C_0}{P\varepsilon}\left(b - b_0\right) - \frac{C_0}{P\varepsilon}\left(\phi_b/E_0\right)\right]$$

$$C_1 = C_0 + \Delta C \sim C\left[1 - \frac{C - C_0}{P\varepsilon}\left(b_0 - b\right) - \frac{C_0}{P\varepsilon}\left(\phi_b/E_0\right)\right]$$

$$\approx \frac{2\pi\varepsilon}{\ln(b/a)}\left[1-\frac{1}{2\pi b\varepsilon}\frac{2\pi\varepsilon}{\ln(b/a)}(\phi_b/E_0)\right]\approx\frac{2\pi\varepsilon}{\ln\left[(b+\phi_b/E_0)/a\right]} \tag{88}$$

where b is the mean or center line radius of the braid. We have set $x = x_0 = b - b_0$ in ΔC and we dropped the $C - C_0$ term in comparison to the final term.

Second Derivation An alternative derivation uses (approximating C_1 by C_0, where we assume that the auxiliary shield is within the braid radius $b_0 < b$ but that it is nearly the same as the braid radius $b_0 \approx b$)

$$\Delta C = -\frac{C_0 C_1}{P\varepsilon}\frac{1}{\ell P_0}\int_{S_0}(\phi_1/E_0)\,dS \approx -\frac{C_0^2}{P_0\varepsilon}\frac{1}{\ell P_0}\int_{S_0}(\phi_1/E_0)\,dS \tag{89}$$

Again the potential at a large interior distance from the shield wall (but near the shield compared to the distance $b - a$) for the $V_{sh} = 0$ part becomes $\phi_1 \sim E_0 x + \phi_b$ (note that x again changes positively when moving into the inner coaxial region from the braid; hence the effective braid surface corresponds to a shift in the solid shield geometry to $x = -\phi_b/E_0$, where the $x = 0$ position is at the braid center). Using this form of the potential, and noting that the integration cancels the factor ℓP_0, gives

$$\Delta C \sim -\frac{C_0^2}{P_0\varepsilon}x - \frac{C_0^2}{P_0\varepsilon}(\phi_b/E_0) \tag{90}$$

The first term in this case makes up for the original choice of boundary. Thus the Taylor expansion is (note that going beyond this first-order Taylor expansion may not be justified if we plan to use a planar solution for ϕ_b, since the cylindrical divergence of the coaxial fields are likely to come in at anything beyond the first-order correction)

$$C_1 = C_0 + \Delta C \sim C_0\left[1-\frac{C_0}{P_0\varepsilon}(b - b_0 + \phi_b/E_0)\right] \tag{91}$$

$$\approx \frac{2\pi\varepsilon}{\ln(b_0/a)}\left[1-\frac{1}{2\pi b_0\varepsilon}\frac{2\pi\varepsilon}{\ln(b_0/a)}(b - b_0 + \phi_b/E_0)\right] \tag{92}$$

where b is the mean radius of the braid and we have set $x = x_0 = b - b_0$. This result can be rewritten as

$$C(b+\phi_b/E_0) = C(b_0) + (b + \phi_b/E_0 - b_0)\frac{\partial C}{\partial b}(b_0) \tag{93}$$

and hence

$$C_1 = C(b+\phi_b/E_0) = \frac{2\pi\varepsilon}{\ln\left[(b+\phi_b/E_0)/a\right]} \tag{94}$$

where the functional form of C for a cylindrical coax is given by (51).

3.2.2 Transfer Capacitance

On the exterior, for simplicity we take the return conductor to be a cylinder of radius b_1, and set

$$C_{sh}V_{sh} = P\varepsilon E_\rho = P\varepsilon E_0^{sh} \tag{95}$$

where we use the capacitance per unit length of the exterior of the braid transmission line C_{sh} and

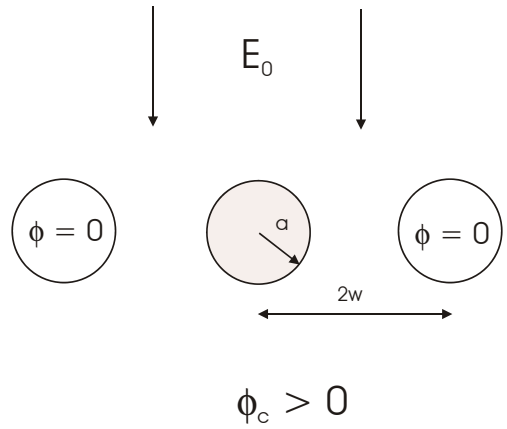

Figure 12. Illustration of the braid transfer potential ϕ_c with drive field E_0.

mean braid perimeter P to achieve the best correspondence of the planar uniform field excitation to the cylindrical coaxial geometry (the field E_0^{sh} is the exterior normal field at the braid, usually with the braid approximated by a solid shield at the mean braid location). The capacitance per unit length of a uniform cylindrical exterior braid transmission line is approximately the capacitance to the braid center line

$$C_{sh} \approx \frac{2\pi\varepsilon}{\ln\left(b_1/b\right)} \tag{96}$$

For a symmetrical shield (where the shift in position ϕ_b/E_0 is the same in both directions) we would expect that a more accurate description of this capacitance is $C_{sh} = 2\pi\varepsilon/\ln\left[b_1/\left(b-\phi_b/E_0^{sh}\right)\right]$. Then for the transfer capacitance defined in (78) we have

$$C_T = -\frac{C_0 C_{sh}}{P\varepsilon}\frac{1}{\ell P_0}\int_{S_0}\left(\phi_s/E_0^{sh}\right)dS \tag{97}$$

Note that at a large distance from the braid shield wall compared to the braid wire mesh structure (but near the shield compared to the distance $b-a$) the potential on the interior (which satisfies Laplace's equation) for the $V_0 = 0$ part, referring back to (88) or (94), is

$$\phi_s = -\phi_c + \frac{q_s}{2\pi\varepsilon}\ln\left(\frac{b+\phi_b/E_0}{\rho}\right) \tag{98}$$

where $\phi_s \to 0$ for $\rho \to a$. A physical meaning of the potential $-\phi_c$ is the potential penetrating the planar braid problem (it could also be defined in the cylindrical problem for an open circuit center conductor as will be done in a later section), a large distance behind the shield wall compared to the braid wire structure, with normal field E_0^{sh} directed away from the shield wall on the driven side and no field on the shadow side, with the braid wires held at zero potential; in the case where a plane field E_0 is directed toward the shield wall the potential $+\phi_c$ appears on the opposite side as shown in Figure 12 and hence the ratio ϕ_c/E_0, or ϕ_c/E_0^{sh}, is a constant dependent only on the geometry of the braid wires.

Note that to obtain a constant asymptotic form $\phi_s \sim -\phi_c$ in the coaxial region requires that the center conductor is also maintained at $-\phi_c$ instead of vanishing. If we wish to consider only the exterior drive with zero potential V_0 on the center conductor, then we need to include the potential variation due to charge on the center conductor as we just did with q_s. Notice that for $\rho = a$, and using the preceding braid capacitance per unit length C_1, we find

$$\phi_s(a_0) = -\phi_c + q_s/C_1 = 0 \tag{99}$$

as required. Proceeding with this potential (appropriate for $V_0 = 0$), but evaluating it near the braid $\rho = b - x$, gives

$$\begin{aligned}
\phi_s &= -\phi_c + \frac{q_s}{2\pi\varepsilon} \ln\left(\frac{b + \phi_b/E_0}{\rho}\right) \\
&= -\phi_c + \frac{q_s}{2\pi\varepsilon} \ln\left(\frac{b + \phi_b/E_0}{b - x}\right) \\
&\sim -\phi_c + \frac{q_s}{2\pi\varepsilon b}\left(x + \phi_b/E_0\right)
\end{aligned} \tag{100}$$

Noting that $V_{sh}C_T = q_s$ gives

$$\begin{aligned}
C_T &= -\frac{C_0 C_{sh}}{P\varepsilon}\frac{1}{\ell P_0}\int_{S_0}\left(\left[-\phi_c + \frac{V_{sh}C_T}{2\pi\varepsilon b}\left(x + \phi_b/E_0\right)\right]/E_0^{sh}\right) dS \\
&= \frac{C_0 C_{sh}}{P\varepsilon}\left[\phi_c/E_0^{sh} - \frac{\left(V_{sh}/E_0^{sh}\right)C_T}{2\pi\varepsilon b}\left(b - b_0 + \phi_b/E_0\right)\right]
\end{aligned} \tag{101}$$

where again the integration cancels the factor ℓP_0 ($x = b - b_0$ on S_0). Collecting terms

$$C_T\left[1 + \frac{C_0 C_{sh}}{P\varepsilon}\frac{\left(V_{sh}/E_0^{sh}\right)}{2\pi\varepsilon b}\left(b - b_0 + \phi_b/E_0\right)\right] = \frac{C_0 C_{sh}}{P\varepsilon}\left(\phi_c/E_0^{sh}\right) \tag{102}$$

or using $C_{sh}V_{sh} = P\varepsilon E_0^{sh}$

$$C_T\left[1 + \frac{C_0}{P\varepsilon}\left(b - b_0 + \phi_b/E_0\right)\right] = \frac{C_0 C_{sh}}{P\varepsilon}\left(\phi_c/E_0^{sh}\right) \tag{103}$$

and

$$C_T = \frac{C_0 C_{sh}}{P\varepsilon\left[1 + \frac{C_0}{P\varepsilon}\left(b - b_0 + \phi_b/E_0\right)\right]}\left(\phi_c/E_0^{sh}\right) \tag{104}$$

But using the fact that

$$C_1 \approx C_0\left[1 - \frac{C_0}{P\varepsilon}\left(b - b_0 + \phi_b/E_0\right)\right] \approx C_0\left[1 - \frac{C_0}{P_0\varepsilon}\left(b - b_0 + \phi_b/E_0\right)\right] \tag{105}$$

(again the replacement of P by P_0 in the first order term does not change the result) gives

$$C_T = \frac{C_1 C_{sh}}{P\varepsilon}\left(\phi_c/E_0^{sh}\right) \tag{106}$$

We note that C_{sh} is the actual outer capacitance per unit length, which for a symmetrical shield is $C_{sh} = 2\pi\varepsilon/\ln\left[b_1/\left(b - \phi_b/E_0^{sh}\right)\right]$. This agrees with the more physical derivation of the next section, although the present reciprocity approach can be generalized to noncircular geometries, which is particularly useful on the exterior (for example, a cable near a ground plane).

Note from the symmetry of Babinet's principle (for zero thickness shields) that the two constants ϕ_c/E_0 (where ϕ_c is the potential due to transmission of the field through the braid) and ϕ_b/E_0 have the same numerical value for geometrically thin shields. This symmetry does not necessarily hold for thick braids. This value can in general be arrived at by choosing the position to obtain the proper capacitance of a solid plate. But a better way to find it is to look at the asymptotic value of the potential on the drive side, with the uniform field potential $E_0 x$ subtracted out from the potential. This potential ϕ_b, from the drive side, becomes identical to the transfer potential ϕ_c, on the shadow side, when the braid becomes geometrically thin, but is in general different for geometrically thick braids. Note that ϕ_b can be either positive or negative, depending on braid geometry and thickness, whereas ϕ_c is always positive. (Also for reciprocal braids we expect ϕ_c to have the same value from either side.)

3.2.3 Alternative Physical Derivation of Electric Parameters

It is instructive to give a more physically motivated derivation of the electric parameters for the cylindrical coax using the potential behavior developed from the planar braid approximation.

Self Capacitance The alternative physical derivation is to first adjust an equivalent coax to have the same capacitance per unit length as the original braid structure using the planar excitation-side potential variation near the braid compared to the $x << b-a$ distance (but distant from the braid surface compared to the braid mesh size) $\phi_1 \sim E_0 x + \phi_b$, where $x = b - \rho$ and ϕ_b/E_0 is the excitation-side constant distance, determined from the planar problem. The field in the coax however is cylindrical

$$E_\rho(\rho) = \frac{q_1}{2\pi\varepsilon\rho} \tag{107}$$

where the charge per unit length on the center conductor is

$$q_1 = C_1 V \tag{108}$$

and C_1 is the coax capacitance per unit length (with a braided shield). Noting from $E_\rho = -\partial\phi_1/\partial\rho$, and the fact that we want the asymptotic potential to vanish at $x = b - \rho = -\phi_b/E_0$ (this is the braid location in the equivalent coax which has the same capacitance per unit length as the original braid structure), that

$$\phi_1 = \int_\rho^{b+\phi_b/E_0} E_\rho(\rho)\, d\rho = \frac{q_1}{2\pi\varepsilon} \ln\left(\frac{b+\phi_b/E_0}{\rho}\right) \tag{109}$$

Now the voltage is the potential on the center conductor

$$V = \phi_1(a) = \frac{q_1}{2\pi\varepsilon} \ln\left(\frac{b+\phi_b/E_0}{a}\right) \tag{110}$$

and hence the capacitance per unit length is

$$C_1 = q_1/V = \frac{2\pi\varepsilon}{\ln\left[\left(b+\phi_b/E_0\right)/a\right]} \tag{111}$$

and is shown in Figure 13.

Transfer Capacitance The next step is to excite the braid from the outside with a field $E_\rho = E_0^{sh}$ at $\rho = b$ giving rise to an interior potential receding from the braid toward the interior, which asymptotically is $\phi_s \sim -\phi_c$, where ϕ_c/E_0^{sh} is the shadow-side constant determined from the planar problem. The planar problem has no conductors in the unexcited half space, and thus the penetrant field originates (this term is

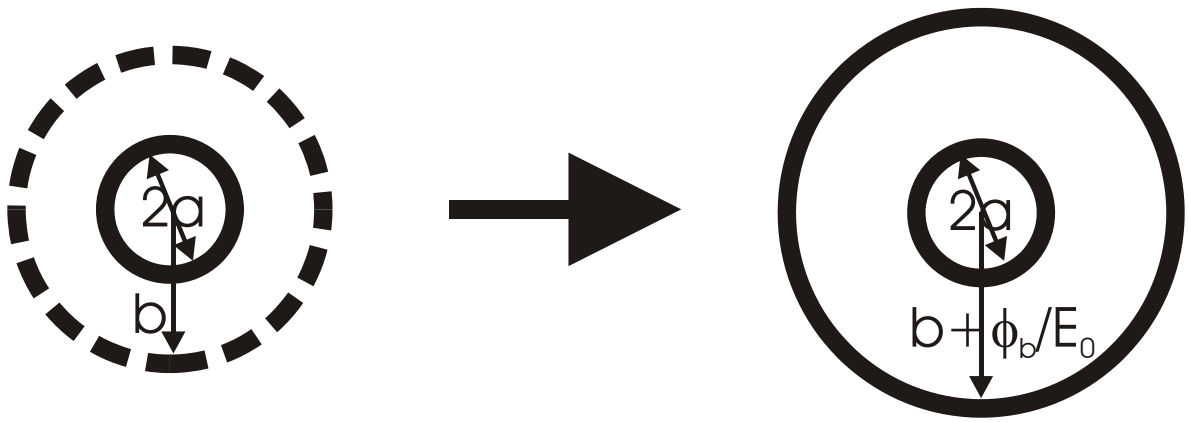

Figure 13. Replacement of the braid by an equivalent coax having the same capacitance.

used instead of terminates, since the exterior field is taken to be radially outward) on the braid wires.

If the unexcited half-space were terminated by a planar grounded conductor some distance from the braid wire shield, then some of the electric field lines would originate on its surface rather than on the braid. In the cylindrical problem, with a center conductor on the interior, some electric field lines do originate on the center conductor if it is also held at zero potential (grounded). The net electric flux per unit length from these field lines (originating on the center conductor) is the source charge per unit length q_s in the transmission line model. We can find this source charge per unit length by means of a compensation argument. If we place a potential $-\phi_c$ on the center conductor, then all field lines will again originate (or terminate if the exterior field is reversed) on the braid wires (at least in the limit of small braid wire spacing compared to the distance $b-a$), as in the case when there is no center conductor present (and consistent with the planar calculation, where no conductors were present in the unexcited half space). The center conductor charge per unit length $-q_s$ associated with this potential exactly compensates for the source charge per unit length from the exterior field, and therefore gives minus the source charge per unit length. However it is simple to find this charge through the braid cable self capacitance

$$-q_s = -\phi_c C_1 \tag{112}$$

where the capacitance is the preceding cable capacitance. The source current is then

$$i_s = -\frac{dq_s}{dt} = i\omega q_s \tag{113}$$

and thus the transmission line equation is

$$\frac{dI}{dz} = -Y_1 V_0 + i_s \tag{114}$$

We can also define a transfer capacitance by relating the exterior field E_0^{sh} to an exterior ground voltage $-V_{sh}$, supporting the exterior mode. In the case where the field on the shield is axisymmetric, we write

$$2\pi b\varepsilon E_0^{sh} = q_{sh} = V_{sh} C_{sh} \tag{115}$$

Denoting the source charge as

$$q_s = q_T = C_T V_{sh} \tag{116}$$

and inserting the preceding formulas for q_{sh} and $V_{sh}C_{sh}$ gives

$$\left(\phi_c / E_0^{sh}\right) \frac{C_1 C_{sh}}{2\pi b\varepsilon} = C_T \tag{117}$$

where the constant length ϕ_c/E_0^{sh} from the planar solution is present and it is now clear that the other multiplier is C_1, the capacitance of the braid coax, as we assumed in the preceding section. The transmission line equation is then

$$\frac{dI}{dz} = -Y_1 V_0 - Y_T V_{sh} \tag{118}$$

with

$$Y_T = -i\omega C_T \tag{119}$$

In situations where the exterior region is not strictly a transmission line, it is often convenient to avoid using an exterior transmission line voltage as the drive. The preceding current transmission line equation can thus also be written as

$$\frac{dI}{dz} = -Y_1 V_0 + i\omega \left(C_T / C_{sh}\right) q_{sh} \tag{120}$$

where we can use the continuity equation to connect the derivative of the exterior current to the exterior charge per unit length

$$\nabla \cdot \underline{J}_{sh} = i\omega \rho_{sh} \rightarrow \frac{dI_{sh}}{dz} = i\omega q_{sh} \tag{121}$$

3.2.4 Other Cable Cross Sections

The preceding integral forms (79) and (80) can be applied to nonuniform geometries. Here we consider two applications of the preceding transfer capacitance (79). First we explore the case where a circular coax is above a ground plane and second we consider the same situation when the coax itself is eccentric. The first case is the most common type of exterior arrangement. The second case represents a starting point for considering interior multiconductor arrangements.

Exterior Ground Plane Case When the exterior transmission line problem consists of a circular cable of radius b, with center height h above a ground plane, we write the exterior potential as (here we take the shield potential to vanish and the chassis ground plane $y = 0$ to be at potential $-V_{sh}$)

$$\phi_{sh} = \frac{q_{sh}}{2\pi\varepsilon} \ln \sqrt{\frac{x^2 + (y + h_e)^2}{x^2 + (y - h_e)^2}} - V_{sh} \tag{122}$$

where q_{sh} is the exterior charge per unit length of the cylinder and the effective line source is placed at height h_e above the ground plane where the charge centroid of the cylinder is located

$$h_e = \sqrt{h^2 - b^2} \tag{123}$$

Letting

$$x = \rho \cos \varphi \tag{124}$$

$$y = h + \rho \sin \varphi \tag{125}$$

$$\phi_{sh} = \frac{q_{sh}}{2\pi\varepsilon} \ln \sqrt{\frac{\rho^2 - b^2 + 2h\left(h + h_e\right) + 2\left(h + h_e\right)\rho \sin \varphi}{\rho^2 - b^2 + 2h\left(h - h_e\right) + 2\left(h - h_e\right)\rho \sin \varphi}} - V_{sh} \tag{126}$$

Note that the potential at the mean braid radius $\rho = b$ is constant and taken to vanish (this image representation is the well known exact image solution of a charged perfectly conducting cylinder above the chassis ground plane)

$$\phi_{sh}\left(b\right) = \frac{q_{sh}}{2\pi\varepsilon} \ln \sqrt{\frac{h + h_e}{h - h_e}} - V_{sh} = 0$$

and thus the exterior capacitance per unit length is

$$C_{sh} = q_{sh}/V_{sh} = 2\pi\varepsilon/\ln\sqrt{\frac{h+h_e}{h-h_e}} = 2\pi\varepsilon/\mathrm{Arccosh}\,(h/b) \tag{127}$$

We use the capacitance per unit length of the exterior of the braid transmission line with the mean radius b as an approximation to the exterior transmission line capacitance per unit length. The radial field is

$$E_\rho = -\frac{\partial\phi_{sh}}{\partial\rho}$$

$$= \frac{q_{sh}}{2\pi\varepsilon}\left[\frac{\rho+(h-h_e)\sin\varphi}{\rho^2-b^2+2h\,(h-h_e)+2\,(h-h_e)\,\rho\sin\varphi} - \frac{\rho+(h+h_e)\sin\varphi}{\rho^2-b^2+2h\,(h+h_e)+2\,(h+h_e)\,\rho\sin\varphi}\right] \tag{128}$$

and taking $\rho = b$ the radial field is

$$E_\rho\,(b,\varphi) = \frac{q_{sh}b}{4\pi\varepsilon}\left(\frac{1}{h-h_e} - \frac{1}{h+h_e}\right)\frac{1}{h+b\sin\varphi}$$

$$= \frac{q_{sh}}{2\pi\varepsilon b}\frac{h_e}{h+b\sin\varphi} = E_0^{sh} \tag{129}$$

Note that the integration of the displacement εE_0^{sh} around the perimeter is the charge which follows from

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{h_e}{h+b\sin\varphi}d\varphi = 1 \tag{130}$$

which can be shown by means of the identity [15]

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\cos\,(m\varphi)}{1+u^2-2u\cos\varphi}d\varphi = \frac{u^m}{1-u^2}\ ,\ u^2<1 \tag{131}$$

where we take

$$1-2uh/b+u^2 = 0 \tag{132}$$

or

$$u = (h\pm h_e)\,/b \tag{133}$$

and thus (choosing the minus sign)

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{h_e}{h+b\sin\varphi}d\varphi = \frac{1}{2\pi}\int_{-3\pi/2}^{\pi/2}\frac{h_e/b}{h/b+\sin\varphi}d\varphi = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{h_e/b}{h/b+\sin\,(\varphi-\pi/2)}d\varphi$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{h_e/b}{h/b-\cos\varphi}d\varphi = \frac{2\,(h-h_e)\,h_e}{b^2-(h-h_e)^2} = 1 \tag{134}$$

where we have used

$$h_e = \sqrt{h^2-b^2} \rightarrow b^2-(h-h_e)^2 = 2\,(h-h_e)\,h_e \tag{135}$$

It is useful in what follows to represent the radial field

$$E_\rho(b,\varphi) = \frac{q_{sh}}{2\pi\varepsilon b}\frac{h_e}{h - b\cos(\varphi+\pi/2)} = E_0^{sh} \tag{136}$$

in a Fourier series

$$E_\rho(b,\varphi) = \frac{q_{sh}}{2\pi\varepsilon b}\sum_{m=0}^{\infty} A_m \cos m(\varphi+\pi/2) \tag{137}$$

where the coefficients are (ε_m is the Neumann number which is unity for $m=0$ and equal to 2 otherwise)

$$A_m = \frac{\varepsilon_m}{2\pi}(h_e/b)\int_{-\pi}^{\pi}\cos(m\varphi)\frac{d\varphi}{h/b-\cos\varphi}d\varphi = \varepsilon_m\frac{2(h-h_e)^{m+1}h_e/b^m}{b^2-(h-h_e)^2}$$

$$= \varepsilon_m (h-h_e)^m/b^m \tag{138}$$

Now Laplace's equation

$$\nabla^2\phi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\phi}{\partial\varphi^2} = 0 \tag{139}$$

has solutions

$$\phi = A_m\rho^{\pm m}\cos m(\varphi+\pi/2) \tag{140}$$

and hence the potential which vanishes at $\rho=a$ and matches the value $-\left(\phi_c/E_0^{sh}\right)E_\rho$ at $\rho=b_0\approx b$, can be written as

$$\phi = -\left(\phi_c/E_0^{sh}\right)\frac{q_{sh}}{2\pi\varepsilon b}\sum_{m=0}^{\infty}A_m\left[\frac{(\rho/a)^m-(a/\rho)^m}{(b_0/a)^m-(a/b_0)^m}\right]\cos m(\varphi+\pi/2)$$

$$= -\left(\phi_c/E_0^{sh}\right)\frac{q_{sh}}{2\pi\varepsilon b}\left\{\frac{\ln(\rho/a)}{\ln(b_0/a)} + 2\sum_{m=1}^{\infty}\left(\frac{h-h_e}{b_0}\right)^m\left[\frac{(\rho/a)^m-(a/\rho)^m}{(b_0/a)^m-(a/b_0)^m}\right]\cos m(\varphi+\pi/2)\right\} \tag{141}$$

With this representation for the interior potential we can simply insert it into

$$C_T = -\frac{C_0}{\ell P_0}\int_{S_0}(\phi_s/V_{sh})\,dS \tag{142}$$

to find the source $i_s = i\omega q_s = -Y_T V_{sh} = i\omega C_T V_{sh}$ in the transmission line equation. The integration eliminates all modes but $m=0$ and we end up with

$$C_T = C_0\left(\phi_c/E_0^{sh}\right)\frac{q_{sh}/V_{sh}}{2\pi\varepsilon b} = \frac{C_0 C_{sh}}{\varepsilon P}\left(\phi_c/E_0^{sh}\right) \tag{143}$$

Again the exterior mode has voltage $-V_{sh}$ (this voltage is on the ground plane so that the shield wires can remain at zero potential).

Simpler Approach In a similar way to our preceding handling of the planar approximation we introduce the integrand decomposition in (78)

$$C_T = -\frac{C_0}{\ell P_0}\int_{S_0}\left(\phi_s/E_0^{sh}\right)\left(E_0^{sh}/V_{sh}\right)dS \tag{144}$$

where the ratio ϕ_s/E_0^{sh} is only dependent on the local braid geometry (not on the driving field strength) and we have shown that in the ground plane case that the exterior driving field on the braid surface is

$$E_0^{sh}/V_{sh} = \frac{q_{sh}/V_{sh}}{2\pi\varepsilon b}\frac{h_e}{h+b\sin\varphi} = \frac{C_{sh}}{2\pi\varepsilon b}\frac{h_e}{h+b\sin\varphi} \tag{145}$$

Then inserting the potential from (100) we obtain

$$C_T = -C_0\frac{1}{\ell P_0}\int_{S_0}\left(-\phi_c/E_0^{sh}\right)\left(E_0^{sh}/V_{sh}\right)dS$$

$$-C_0\frac{1}{\ell P_0}\int_{S_0}\left[\frac{(q_s/V_{sh})}{2\pi\varepsilon b}\left(x+\phi_b/E_0\right)\right]dS$$

$$= -C_0\left(-\phi_c/E_0^{sh}\right)\frac{1}{\ell P_0}\int_{S_0}\frac{C_{sh}}{2\pi\varepsilon b}\frac{h_e}{h+b\sin\varphi}dS - C_0\frac{1}{\ell P_0}\int_{S_0}\left[\frac{C_T}{2\pi\varepsilon b}\left(x+\phi_b/E_0\right)\right]dS$$

$$= -\frac{C_0C_{sh}}{\varepsilon P}\left(-\phi_c/E_0^{sh}\right)\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{h_e}{h+b\sin\varphi}d\varphi - C_0\frac{C_T}{2\pi\varepsilon b}\left(b-b_0+\phi_b/E_0\right)$$

$$= -\frac{C_0C_{sh}}{\varepsilon P}\left(-\phi_c/E_0^{sh}\right) - \frac{C_0C_T}{\varepsilon P}\left(b-b_0+\phi_b/E_0\right) \tag{146}$$

where we set $x = x_0 = b - b_0$, and

$$C_T\left[1+\frac{C_0}{\varepsilon P}\left(b-b_0+\phi_b/E_0\right)\right] = -\frac{C_0C_{sh}}{\varepsilon P}\left(-\phi_c/E_0^{sh}\right) \tag{147}$$

then as before

$$C_T = \frac{C_0C_{sh}}{P\varepsilon\left[1+\frac{C_0}{P\varepsilon}\left(b-b_0+\phi_b/E_0\right)\right]}\left(\phi_c/E_0^{sh}\right) \tag{148}$$

Note that in this simplified approach we have taken only the symmetric ($m = 0$) part of the interior potential of the coax ϕ_s resulting from the source charge q_s in the preceding expression. Although other azimuthal modes will exist, due to the asymmetric exterior excitation, the net source charge q_s should be determined by this symmetric component. Now using the fact that (again the replacement of P by P_0 in the first order term does not change the result) the capacitance per unit length of the inner coax with braid has essentially not changed when the exterior ground plane is present we can again write

$$C_1 \approx C_0\left[1-\frac{C_0}{P\varepsilon}\left(b-b_0+\phi_b/E_0\right)\right] \approx C_0\left[1-\frac{C_0}{P_0\varepsilon}\left(b-b_0+\phi_b/E_0\right)\right] \tag{149}$$

which then gives the same result as the more rigorous approach of the preceding section

$$C_T = \frac{C_1C_{sh}}{P\varepsilon}\left(\phi_c/E_0^{sh}\right) \tag{150}$$

An analog of this simplified approach is also applied in the next section.

Interior Eccentric Coax In addition to the ground plane exterior let us consider the situation when the interior is an eccentric coax [14]. Consider a complex potential being the superposition of two line charges $\pm q_0$

$$W = U + iV = -\frac{q_0}{2\pi\varepsilon}\ln\left(\frac{z - iy_c}{z + iy_c}\right) = \frac{q_0}{2\pi\varepsilon} i2\,\mathrm{arccot}\,(z/y_c) \tag{151}$$

where $z = x + iy$ and $y_c > 0$. The scalar potential is taken as

$$\phi_0 = \mathrm{Re}\,(W) = U = -\frac{q_0}{2\pi\varepsilon}\ln\left|\frac{z - iy_c}{z + iy_c}\right| \tag{152}$$

Inverting the transformation gives

$$z = x + iy = y_c \cot\left(\frac{U + iV}{2i\frac{q_0}{2\pi\varepsilon}}\right) \tag{153}$$

or

$$x = \frac{y_c \sin\left(V/\left(\frac{q_0}{2\pi\varepsilon}\right)\right)}{\cosh\left(U/\left(\frac{q_0}{2\pi\varepsilon}\right)\right) - \cos\left(V/\left(\frac{q_0}{2\pi\varepsilon}\right)\right)} \tag{154}$$

$$y = \frac{y_c \sinh\left(U/\left(\frac{q_0}{2\pi\varepsilon}\right)\right)}{\cosh\left(U/\left(\frac{q_0}{2\pi\varepsilon}\right)\right) - \cos\left(V/\left(\frac{q_0}{2\pi\varepsilon}\right)\right)} \tag{155}$$

or

$$\cosh\left(U/\left(\frac{q_0}{2\pi\varepsilon}\right)\right) - \cos\left(V/\left(\frac{q_0}{2\pi\varepsilon}\right)\right) = (y_c/x)\sin\left(V/\left(\frac{q_0}{2\pi\varepsilon}\right)\right) \tag{156}$$

$$\cosh\left(U/\left(\frac{q_0}{2\pi\varepsilon}\right)\right) - (y_c/y)\sinh\left(U/\left(\frac{q_0}{2\pi\varepsilon}\right)\right) = \cos\left(V/\left(\frac{q_0}{2\pi\varepsilon}\right)\right) \tag{157}$$

Subtracting these two gives

$$x\sinh\left(U/\left(\frac{q_0}{2\pi\varepsilon}\right)\right) = y\sin\left(V/\left(\frac{q_0}{2\pi\varepsilon}\right)\right) \tag{158}$$

and eliminating V

$$x^2\sinh^2\left(U/\left(\frac{q_0}{2\pi\varepsilon}\right)\right) = y^2 - \left[y\cosh\left(U/\left(\frac{q_0}{2\pi\varepsilon}\right)\right) - y_c\sinh\left(U/\left(\frac{q_0}{2\pi\varepsilon}\right)\right)\right]^2 \tag{159}$$

or

$$x^2 = y^2\mathrm{csch}^2\left(U/\left(\frac{q_0}{2\pi\varepsilon}\right)\right) - \left[y\coth\left(U/\left(\frac{q_0}{2\pi\varepsilon}\right)\right) - y_c\right]^2 \tag{160}$$

or

$$x^2 = -y^2 + 2yy_c\coth\left(U/\left(\frac{q_0}{2\pi\varepsilon}\right)\right) - y_c^2 \tag{161}$$

or

$$x^2 + y^2 - 2yy_c \coth\left(U/\left(\frac{q_0}{2\pi\varepsilon}\right)\right) + y_c^2 \coth^2\left(U/\left(\frac{q_0}{2\pi\varepsilon}\right)\right) = y_c^2\left[\coth^2\left(U/\left(\frac{q_0}{2\pi\varepsilon}\right)\right) - 1\right] \tag{162}$$

or

$$x^2 + \left[y - y_c \coth\left(U/\left(\frac{q_0}{2\pi\varepsilon}\right)\right)\right]^2 = y_c^2 \mathrm{csch}^2\left(U/\left(\frac{q_0}{2\pi\varepsilon}\right)\right) \tag{163}$$

Thus the equipotential contours are circles centered at $x = 0$ and

$$y = y_c \coth\left(U/\left(\frac{q_0}{2\pi\varepsilon}\right)\right) \tag{164}$$

with radius

$$r = y_c \mathrm{csch}\left(U/\left(\frac{q_0}{2\pi\varepsilon}\right)\right) \tag{165}$$

If we pick two potentials $U_2 > U_1$, then the associated radii satisfy $r_2 < r_1$ (the smaller is contained within the bigger), and the associated centers satisfy $y_2 < y_1$ (the center of the smaller is below the center of the larger). Now if we take the two centers and radii to be denoted by

$$y_1 = y_c \coth\left(U_1/\left(\frac{q_0}{2\pi\varepsilon}\right)\right) \tag{166}$$

$$y_2 = y_c \coth\left(U_2/\left(\frac{q_0}{2\pi\varepsilon}\right)\right) \tag{167}$$

$$a = y_c \mathrm{csch}\left(U_2/\left(\frac{q_0}{2\pi\varepsilon}\right)\right) \tag{168}$$

$$b = y_c \mathrm{csch}\left(U_1/\left(\frac{q_0}{2\pi\varepsilon}\right)\right) \tag{169}$$

we see that the difference of center positions is

$$d = y_1 - y_2$$

$$= y_c \coth\left(U_1/\left(\frac{q_0}{2\pi\varepsilon}\right)\right) - y_c \coth\left(U_2/\left(\frac{q_0}{2\pi\varepsilon}\right)\right) = \frac{y_c \sinh\left(\frac{U_2 - U_1}{\left(\frac{q_0}{2\pi\varepsilon}\right)}\right)}{\sinh\left(\frac{U_2}{\left(\frac{q_0}{2\pi\varepsilon}\right)}\right)\sinh\left(\frac{U_1}{\left(\frac{q_0}{2\pi\varepsilon}\right)}\right)} \tag{170}$$

We can show from the preceding three formulas that

$$\cosh\left(\frac{U_2 - U_1}{\left(\frac{q_0}{2\pi\varepsilon}\right)}\right) = \frac{a^2 + b^2 - d^2}{2ab} \tag{171}$$

Hence with the potential difference defined as

$$V_0 = U_2 - U_1 \tag{172}$$

we have

$$q_0 = V_0 2\pi\varepsilon/\text{Arccosh}\left(\frac{a^2+b^2-d^2}{2ab}\right) \tag{173}$$

From the preceding formula we can write

$$y_c d = ab\sinh\left(\frac{U_2-U_1}{\left(\frac{q_0}{2\pi\varepsilon}\right)}\right) \tag{174}$$

and then

$$y_c^2 d^2 = a^2b^2\left[\cosh^2\left(\frac{U_2-U_1}{\left(\frac{q_0}{2\pi\varepsilon}\right)}\right)-1\right] = \frac{1}{4}\left[\left(a^2+b^2-d^2\right)^2-4a^2b^2\right] \tag{175}$$

or

$$2y_c d = \sqrt{\left(a^2+b^2-d^2\right)^2-4a^2b^2} = \sqrt{\left[(b-a)^2-d^2\right]\left[(b+a)^2-d^2\right]} \tag{176}$$

Using the identity $\coth^2 x = 1+\text{csch}^2 x$ we find that

$$y_1 = y_c\sqrt{1+\text{csch}^2\left(U_1/\left(\frac{q_0}{2\pi\varepsilon}\right)\right)} = \sqrt{y_c^2+b^2} \tag{177}$$

or

$$2dy_1 = \sqrt{\left(a^2+b^2-d^2\right)^2-4a^2b^2+4d^2b^2} = \sqrt{\left(-a^2+b^2+d^2\right)^2} = -a^2+b^2+d^2 \tag{178}$$

Also from above

$$y_1/b = \sqrt{\left(y_c/b\right)^2+1} = \cosh\left(U_1/\left(\frac{q_0}{2\pi\varepsilon}\right)\right) \tag{179}$$

and

$$\frac{q_0}{2\pi\varepsilon}\text{Arccosh}\sqrt{\left(y_c/b\right)^2+1} = U_1 \tag{180}$$

Now we want to shift the coordinate system from $y = y_1$ to $y = h$ as the location of center of the b radius cylinder. We also want to shift the potential on the outer cylinder from U_1 to 0 (the potential at the inner cylinder of radius a is then V_0). Thus in summary

$$\phi_0 = -\frac{q_0}{2\pi\varepsilon}\ln\sqrt{\frac{x^2+\left(y+y_1-h-y_c\right)^2}{x^2+\left(y+y_1-h+y_c\right)^2}} - U_1 \tag{181}$$

$$q_0 = V_0 2\pi\varepsilon/\text{Arccosh}\left(\frac{a^2+b^2-d^2}{2ab}\right) \tag{182}$$

$$2y_c d = \sqrt{\left[(b-a)^2-d^2\right]\left[(b+a)^2-d^2\right]} \tag{183}$$

$$y_1 = \sqrt{y_c^2 + b^2} \tag{184}$$

$$\frac{q_0}{2\pi\varepsilon}\text{Arccosh}\left(y_1/b\right) = U_1 \tag{185}$$

The radial electric field at the outer shield boundary is then

$$E_{0\rho}\left(\rho,\varphi\right)/q_0 = -\frac{1}{q_0}\frac{\partial\phi_0}{\partial\rho} = \frac{1}{2\pi\varepsilon}\frac{\partial}{\partial\rho}\ln\sqrt{\frac{\rho^2\cos^2\varphi + \left(\rho\sin\varphi + y_1 - y_c\right)^2}{\rho^2\cos^2\varphi + \left(\rho\sin\varphi + y_1 + y_c\right)^2}}$$

$$= \frac{1}{2\pi\varepsilon}\left[\frac{\rho + \left(y_1 - y_c\right)\sin\varphi}{\rho^2\cos^2\varphi + \left(\rho\sin\varphi + y_1 - y_c\right)^2} - \frac{\rho + \left(y_1 + y_c\right)\sin\varphi}{\rho^2\cos^2\varphi + \left(\rho\sin\varphi + y_1 + y_c\right)^2}\right] \tag{186}$$

Note that in the vicinity of the shield (near the reference conductor) we can write the potential due to the exterior drive as

$$\phi_s \sim -\phi_c + E_\rho\left(b - 0,\varphi\right)\left(x + \phi_b/E_0\right)\ ,\ b - x = \rho \tag{187}$$

The transfer capacitance from (79) is thus

$$C_T = \frac{\varepsilon}{\ell}C_0\int_{S_0}\left(\phi_s/V_{sh}\right)\frac{1}{q_0}\frac{\partial\phi_0}{\partial\rho}dS$$

$$= \frac{\varepsilon}{\ell}C_0\int_{S_0}\left(\phi_s/E_0^{sh}\right)\left(E_0^{sh}/V_{sh}\right)\frac{1}{q_0}\frac{\partial\phi_0}{\partial\rho}dS \tag{188}$$

Therefore we finally obtain (note here that we use $E_\rho \to \left(q_s/q_0\right)E_{0\rho}$)

$$C_T = \varepsilon C_0\int_{-\pi}^{\pi}\left(\phi_s/E_0^{sh}\right)\left(E_0^{sh}/V_{sh}\right)\frac{1}{q_0}\frac{\partial\phi_0}{\partial\rho}\left(\rho = b\right)bd\varphi$$

$$\approx -\varepsilon C_0\left(-\phi_c/E_0^{sh}\right)\int_{-\pi}^{\pi}\left(E_0^{sh}/V_{sh}\right)\frac{1}{q_0}E_{0\rho}\left(b,\varphi\right)bd\varphi$$

$$-\varepsilon C_0\int_{-\pi}^{\pi}\left[\left(q_s/V_{sh}\right)\frac{1}{q_0}E_{0\rho}\left(b,\varphi\right)\left(x + \phi_b/E_0\right)\right]\frac{1}{q_0}E_{0\rho}\left(b,\varphi\right)bd\varphi$$

$$\approx -\varepsilon C_0\left(-\phi_c/E_0^{sh}\right)\int_{-\pi}^{\pi}\left(E_0^{sh}/V_{sh}\right)\frac{1}{q_0}E_{0\rho}\left(b,\varphi\right)bd\varphi$$

$$-\varepsilon PC_0C_T\left(b - b_0 + \phi_b/E_0\right)\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{1}{q_0}E_{0\rho}\left(b,\varphi\right)\right]^2 d\varphi \tag{189}$$

The interior field can be simplified as

$$\frac{1}{q_0}E_{0\rho}\left(b,\varphi\right) = -\frac{1}{q_0}\frac{\partial\phi_0}{\partial\rho}\left(\rho = b\right) = \frac{1}{2\pi\varepsilon}\left[\frac{b + \left(y_1 - y_c\right)\sin\varphi}{b^2\cos^2\varphi + \left(b\sin\varphi + y_1 - y_c\right)^2} - \frac{b + \left(y_1 + y_c\right)\sin\varphi}{b^2\cos^2\varphi + \left(b\sin\varphi + y_1 + y_c\right)^2}\right]$$

$$= \frac{2y_c b}{\pi\varepsilon}\left[\frac{y_1 + b\sin\varphi}{\left\{b^2 + 2b\left(y_1 - y_c\right)\sin\varphi + \left(y_1 - y_c\right)^2\right\}\left\{b^2 + 2b\left(y_1 + y_c\right)\sin\varphi + \left(y_1 + y_c\right)^2\right\}}\right]$$

$$= \frac{1}{2\pi\varepsilon b}\frac{y_c}{y_1 + b\sin\varphi} \tag{190}$$

where we used

$$y_1 = \sqrt{y_c^2 + b^2} = \left(-a^2 + b^2 + d^2\right)/\left(2d\right) \tag{191}$$

and we note that

$$2y_c d = \sqrt{\left[\left(b-a\right)^2 - d^2\right]\left[\left(b+a\right)^2 - d^2\right]} \tag{192}$$

$$\left(y_1 - y_c\right)\left(y_1 + y_c\right) = \left(y_1^2 - y_c^2\right) = b^2 \tag{193}$$

Note that we can rotate the angle φ in this formula (versus in the outer short circuit field) to rotate the displacement d in the eccentric coax versus the outer short circuit field; such a rotation by $\pi/2$ is done as an extra example below. The outer short circuit field for the ground plane case is (129)

$$E_0^{sh}/V_{sh} = \frac{C_{sh}}{2\pi\varepsilon b}\frac{h_e}{h + b\sin\varphi} \tag{194}$$

$$h_e = \sqrt{h^2 - b^2} \tag{195}$$

Using the approximate approach of the preceding section the transfer capacitance becomes

$$C_T \approx \frac{C_0 C_{sh}}{\varepsilon P}\left(\phi_c/E_0^{sh}\right)\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{h_e}{h + b\sin\varphi}\frac{y_c}{y_1 + b\sin\varphi}d\varphi$$

$$-\frac{C_T}{\varepsilon P}C_0\left(b - b_0 + \phi_b/E_0\right)\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{y_c}{y_1 + b\sin\varphi}\right)^2 d\varphi \tag{196}$$

Noting that $y_c = \sqrt{y_1^2 - b^2}$ is the same form as $h_e = \sqrt{h^2 - b^2}$ from (130) we can write

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{h_e}{h - b\cos\varphi}d\varphi = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{y_c}{y_1 - b\cos\varphi}d\varphi = 1 \tag{197}$$

and thus we see that the "average" is unity if either of the factors in the integrand of the first term of the transfer capacitance is unity (as in an inner or outer cylindrical coaxial arrangement). However if both factors are present, due to the eccentric inner and outer (the ground plane is a form of eccentric coax) arrangements, then the transfer capacitance is an average of the product of the two factors. Using partial fractions we can write

$$\left(\frac{y_1 - h}{h_e y_c}\right)\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{h_e}{h + b\sin\varphi}\frac{y_c}{y_1 + b\sin\varphi}d\varphi = \left(1/h_e\right)\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{h_e}{h + b\sin\varphi}d\varphi - \left(1/y_c\right)\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{y_c}{y_1 + b\sin\varphi}d\varphi$$

$$= 1/h_e - 1/y_c \tag{198}$$

Taking the limit $h \to y_1$ gives the identity

$$\lim_{y_1 \to h} \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{h_e}{h + b\sin\varphi} \frac{y_c}{y_1 + b\sin\varphi} d\varphi = \lim_{h \to y_1} \frac{\sqrt{y_1^2 - b^2} - \sqrt{h^2 - b^2}}{y_1 - h}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{y_c}{y_1 + b\sin\varphi} \right)^2 d\varphi = y_1/y_c \tag{199}$$

The final result for the transfer capacitance is thus

$$C_T \approx \frac{C_0 C_{sh}}{\varepsilon P} \left(\phi_c / E_0^{sh} \right) \frac{y_c - h_e}{y_1 - h}$$

$$- \frac{C_T}{\varepsilon P} C_0 \left(b - b_0 + \phi_b / E_0 \right) y_1 / y_c \tag{200}$$

or

$$C_T \left[1 + \frac{1}{\varepsilon P} C_0 \left(b - b_0 + \phi_b / E_0 \right) y_1 / y_c \right] \approx \frac{C_0 C_{sh}}{\varepsilon P} \left(\phi_c / E_0^{sh} \right) \frac{y_c - h_e}{y_1 - h} \tag{201}$$

or

$$C_T \approx \frac{C_{sh}}{\varepsilon P} \frac{C_0}{\left[1 + \frac{1}{\varepsilon P} C_0 \left(b - b_0 + \phi_b / E_0 \right) y_1 / y_c \right]} \left(\phi_c / E_0^{sh} \right) \frac{y_c - h_e}{y_1 - h} \tag{202}$$

Noting that in the case of the eccentric coax we have

$$C_1 = C \left(b + \phi_b / E_0 \right) \approx C \left(b_0 \right) + \left(b + \phi_b / E_0 - b_0 \right) \frac{\partial C}{\partial b} \left(b_0 \right)$$

$$\approx C_0 \left[1 + \left(b + \phi_b / E_0 - b_0 \right) \frac{1}{C_0} \frac{\partial C}{\partial b} \left(b_0 \right) \right] \tag{203}$$

where

$$C \left(b \right) = 2\pi\varepsilon / \mathrm{Arccosh} \left(\frac{a^2 + b^2 - d^2}{2ab} \right) \tag{204}$$

$$C_0 = C \left(b_0 \right) \tag{205}$$

and

$$\mathrm{Arccosh} \left(x \right) = \ln \left(x + \sqrt{x^2 - 1} \right) \tag{206}$$

$$\frac{d}{dx} \mathrm{Arccosh} \left(x \right) = 1 / \sqrt{x^2 - 1} \tag{207}$$

we find that

$$C_0\left[1-\frac{1}{\varepsilon P}C\left(b-b_0+\phi_b/E_0\right)y_1/y_c\right]\approx C_1 \tag{208}$$

$$\frac{1}{C_0}\frac{\partial C}{\partial b}\left(b_0\right)=-\frac{C_0}{\varepsilon P_0}\left(-a^2+b_0^2+d^2\right)/\sqrt{\left(a^2+b_0^2-d^2\right)^2-4a^2b_0^2}$$

$$\approx-\frac{C_0}{\varepsilon P}\left(-a^2+b^2+d^2\right)/\sqrt{\left(a^2+b^2-d^2\right)^2-4a^2b^2}$$

$$=-\frac{C_0}{\varepsilon P}y_1/y_c \tag{209}$$

Hence

$$C_1\approx C_0\left[1-\left(b-b_0+\phi_b/E_0\right)\frac{C_0}{\varepsilon P}y_1/y_c\right] \tag{210}$$

and therefore we obtain the transfer capacitance

$$C_T\approx\frac{C_1C_{sh}}{P\varepsilon}\left(\phi_c/E_0^{sh}\right)\frac{y_c-h_e}{y_1-h} \tag{211}$$

versus our previous result from the last section

$$C_T=\frac{C_1C_{sh}}{P\varepsilon}\left(\phi_c/E_0^{sh}\right) \tag{212}$$

Note that the extra factor associated with the cable eccentricity can be written as

$$\frac{y_c-h_e}{y_1-h}=\frac{y_c-\sqrt{h^2-b^2}}{y_1-h}=\frac{\sqrt{\left[\left(b-a\right)^2-d^2\right]\left[\left(b+a\right)^2-d^2\right]}-2d\sqrt{h^2-b^2}}{-a^2+b^2+d^2-2dh} \tag{213}$$

This ratio is unity if $d=0$ (which makes the field of the interior problem uniform in azimuth) or if $h\to\infty$ (which makes the field of the exterior problem uniform in azimuth).

Note that if we rotate the azimuth of the eccentric coax by $\pi/2$ relative to the outer ground plane arrangement we have (as in the preceding example we combine the two terms to form C_1 in the coefficient of the integral)

$$C_T\approx C_1\left(\phi_c/E_0^{sh}\right)\frac{C_{sh}}{2\pi\varepsilon b}\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{h_e}{h+b\sin\varphi}\frac{y_c}{y_1+b\cos\varphi}d\varphi \tag{214}$$

Using

$$\left(\frac{b}{h_ey_c}\right)\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{h_e}{h+b\sin\varphi}\frac{y_c}{y_1+b\cos\varphi}d\varphi$$

$$=\left(1/h_e\right)\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{h_e\cos\varphi}{h+b\sin\varphi}d\varphi+\left(1/y_c\right)\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{y_c\sin\varphi}{y_1+b\cos\varphi}d\varphi$$

$$-\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{y_1\cos\varphi+h\sin\varphi}{(h+b\sin\varphi)(y_1+b\cos\varphi)}d\varphi \tag{215}$$

and

$$b\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{y_1\cos\varphi+h\sin\varphi}{(h+b\sin\varphi)(y_1+b\cos\varphi)}d\varphi=(y_1/h_e)\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{h_e}{h+b\sin\varphi}d\varphi+(h/y_c)\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{y_c}{y_1+b\cos\varphi}d\varphi$$

$$-\left(\frac{y_1^2+h^2}{h_ey_c}\right)\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{h_e}{h+b\sin\varphi}\frac{y_c}{y_1+b\cos\varphi}d\varphi \tag{216}$$

we can write

$$\left\{\frac{b}{h_ey_c}-\frac{1}{b}\left(\frac{y_1^2+h^2}{h_ey_c}\right)\right\}\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{h_e}{h+b\sin\varphi}\frac{y_c}{y_1+b\cos\varphi}d\varphi$$

$$=(1/h_e)\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{h_e\cos\varphi}{h+b\sin\varphi}d\varphi+(1/y_c)\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{y_c\sin\varphi}{y_1+b\cos\varphi}d\varphi$$

$$-\frac{1}{b}(y_1/h_e)\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{h_e}{h+b\sin\varphi}d\varphi-\frac{1}{b}(h/y_c)\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{y_c}{y_1+b\cos\varphi}d\varphi$$

$$=(1/h_e)\frac{1}{2\pi}\left(\int_0^{\pi/2}+\int_{\pi/2}^{\pi}+\int_{-\pi}^{-\pi/2}+\int_{-\pi/2}^{0}\right)\frac{h_e\cos\varphi}{h+b\sin\varphi}d\varphi+(1/y_c)\frac{1}{2\pi}\left(\int_0^{\pi/2}+\int_{\pi/2}^{\pi}+\int_{-\pi}^{-\pi/2}+\int_{-\pi/2}^{0}\right)\frac{y_c\sin\varphi}{y_1+b\cos\varphi}d\varphi$$

$$-\frac{1}{b}(y_1/h_e)-\frac{1}{b}(h/y_c) \tag{217}$$

In the first and second sequences of integrals we let (in the second integral $\varphi=\pi-\varphi'$, in the third integral $\varphi=-\pi+\varphi'$, and in the fourth integral $\varphi=-\varphi'$). This shows that these all cancel out and thus

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{h_e}{h+b\sin\varphi}\frac{y_c}{y_1+b\cos\varphi}d\varphi=\frac{y_1y_c+hh_e}{y_1^2+h_e^2}=\frac{\sqrt{y_c^2+b^2}y_c+h\sqrt{h^2-b^2}}{y_c^2+h^2} \tag{218}$$

or

$$C_T\approx C_1\left(\phi_c/E_0^{sh}\right)\frac{C_{sh}}{2\pi\varepsilon b}\frac{y_1y_c+hh_e}{y_1^2+h_e^2} \tag{219}$$

3.2.5 Evaluation Of Energy Formulas Using Approximate Planar Form Of Potentials

It is instructive to show how the preceding perturbation results can be directly derived from approximations of the energy formulas for capacitance rather than using reciprocity. This is done by applying the preceding energy formulas stood off from the surface of the braid using an approximate planar evaluation of the potential near the braid to evaluate the self and transfer capacitances. It turns out to be convenient to initially use the elastances defined by

$$V_1=S_{11}Q_1+S_{12}Q_2 \tag{220}$$

$$V_2 = S_{21}Q_1 + S_{22}Q_2 \tag{221}$$

These are convenient because, unlike the capacitance matrix where a particular diagonal element is determined with other conductors short circuited (because it will then typically have a small induced charge contributing to the potential variations), a particular diagonal element of the elastance matrix is determined with other conductors open circuited (no net charge). The relation between the elastances and capacitances (with the braid as a reference)

$$Q_1 = C_{11}V_1 + C_{12}V_2 \tag{222}$$

$$Q_2 = C_{21}V_1 + C_{22}V_2 \tag{223}$$

can be written as

$$C_{11} = S_{22}/D$$

$$C_{22} = S_{11}/D$$

$$C_{12} = -S_{12}/D$$

$$C_{21} = -S_{21}/D \tag{224}$$

where

$$D = S_{11}S_{22} - S_{12}S_{21} \tag{225}$$

In the case where $S_{11}S_{22} >> S_{12}S_{21}$ we find

$$C_{11} \approx 1/S_{11} \tag{226}$$

$$C_{22} \approx 1/S_{22} \tag{227}$$

and in a reciprocal media

$$C_{12} = C_{21} = C_m \approx -S_m/\left(S_{11}S_{22}\right) \approx -C_{11}C_{22}S_m \tag{228}$$

where

$$S_m = S_{12} = S_{21} \tag{229}$$

The continuity equation is

$$\nabla \cdot \underline{J} = -\frac{\partial}{\partial t}\rho_v \tag{230}$$

or in integral form

$$\oint_S \underline{J} \cdot \underline{n} dS = -\frac{\partial}{\partial t} \int_V \rho_v dV \tag{231}$$

This yields

$$\frac{\partial I}{\partial z} = -\frac{\partial q}{\partial t} \tag{232}$$

where I is the net current on a conductor and q is the charge per unit length on a conductor. Hence, from the continuity equation we can determine the current changes over a short periodic section along the line as

$$I_1(z+\ell) - I_1(z) = -\frac{dQ_1}{dt} = -\frac{d}{dt}(\ell q_1) \tag{233}$$

$$I_2(z+\ell) - I_2(z) = -\frac{dQ_2}{dt} = -\frac{d}{dt}(\ell q_2) \tag{234}$$

The power removed from a periodic section of line is minus the derivative of the stored electric energy

$$V_1 [I_1(z+\ell) - I_1(z)] + V_2 [I_2(z+\ell) - I_2(z)] = -V_1 \frac{dQ_1}{dt} - V_2 \frac{dQ_2}{dt}$$

$$= -(S_{11}Q_1 + S_{12}Q_2)\frac{dQ_1}{dt} - (S_{21}Q_1 + S_{22}Q_2)\frac{dQ_2}{dt}$$

$$= -\frac{d}{dt}\left(\frac{1}{2}S_{11}Q_1^2 + S_m Q_1 Q_2 + \frac{1}{2}S_{22}Q_2^2\right) \tag{235}$$

If we take two sources and denote the resulting field due to these by subscripts 1 and 2 then by superposition we can write the total field as

$$\underline{E} = \underline{E}_1 + \underline{E}_2 \tag{236}$$

Equating electric energies in a region gives

$$\frac{1}{2}S_{11}Q_1^2 + S_m Q_1 Q_2 + \frac{1}{2}S_{22}Q_2^2 = \frac{1}{2}\int_V \underline{D} \cdot \underline{E} dV = \frac{1}{2}\int_V DE dV$$

$$= \frac{1}{2}\int_V D_1 E_1 dV + \frac{1}{2}\int_V \underline{D}_1 \cdot \underline{E}_2 dV + \frac{1}{2}\int_V \underline{E}_1 \cdot \underline{D}_2 dV + \frac{1}{2}\int_V D_2 E_2 dV$$

$$= \int_V \frac{1}{2}\varepsilon E_1^2 dV + \int_V \varepsilon \underline{E}_1 \cdot \underline{E}_2 dV + \int_V \frac{1}{2}\varepsilon E_2^2 dV \tag{237}$$

The 1 problem using the elastances has a charge on conductor 1 (the center conductor) with an equal and opposite charge on the braid shield and no charge on the 2 conductor (the chassis) (see Figure 10). The 2 problem has a charge Q_2 on the chassis with equal and opposite charge on the braid shield and no charge on the 1 conductor (the center conductor). We can identify the self elastances as

$$S_{11}Q_1^2 = \int_V \varepsilon E_1^2 dV = -\int_V \nabla\phi_1 \cdot \underline{D}_1 dV \tag{238}$$

where we can use

$$\nabla \cdot (\phi_1 \underline{D}_1) = \nabla\phi_1 \cdot \underline{D}_1 + \phi_1 \nabla \cdot \underline{D}_1 = \nabla\phi_1 \cdot \underline{D}_1 + \phi_1 \rho_{v1} \tag{239}$$

to obtain

$$S_{11}Q_1^2 = -\oint_S \underline{n} \cdot (\phi_1 \underline{D}_1)\, dS = V_1 Q_1 \tag{240}$$

or

$$S_{11} = V_1/Q_1 \text{ with } Q_2 = 0 \tag{241}$$

and

$$S_{22}Q_2^2 = \int_V \varepsilon E_2^2 dV = -\int_V \nabla\phi_2 \cdot \underline{D}_2 dV \tag{242}$$

where we can use

$$\nabla \cdot (\phi_2 \underline{D}_2) = \nabla\phi_2 \cdot \underline{D}_2 + \phi_2 \nabla \cdot \underline{D}_2 = \nabla\phi_2 \cdot \underline{D}_2 + \phi_2 \rho_{v2} \tag{243}$$

to obtain

$$S_{22}Q_2^2 = -\oint_S \underline{n} \cdot (\phi_2 \underline{D}_2)\, dS = V_2 Q_2 \tag{244}$$

or

$$S_{22} = V_2/Q_2 \text{ with } Q_1 = 0 \tag{245}$$

In the first case the closed surface integral is focused on the region 1 because the field $\underline{D}_1$ is generated by charge on the center conductor. In the second case the closed surface integral is focused on the region 2 because the field $\underline{D}_2$ is generated by charge on the chassis.

The mutual elastance is identified as

$$Q_1 Q_2 S_m = \frac{1}{2}\int_V \underline{D}_1 \cdot \underline{E}_2 dV + \frac{1}{2}\int_V \underline{E}_1 \cdot \underline{D}_2 dV \tag{246}$$

and voltage charge relations

$$V_1 = S_{11}Q_1 + S_m Q_2 \tag{247}$$

$$V_2 = S_m Q_1 + S_{22}Q_2 \tag{248}$$

Figure 14 shows the field $\underline{E}_1$ generated by a positive charge Q_1 on the center conductor with no net charge on the chassis, as well as the field $\underline{E}_{sh} = \underline{E}_2$ generated by a positive charge $Q_{sh} = -Q_2$ on the braided shield with no net charge on the chassis. In this case the voltage charge relations with $V_2 = -V_{sh}$ are

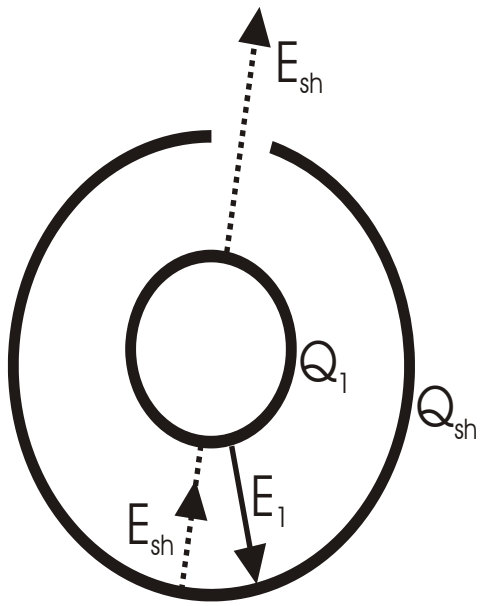

Figure 14. Field E_1 generated by charge Q_1 and field E_{sh} generated by charge Q_{sh}.

$$V_1 = S_{11}Q_1 - S_m Q_{sh} \tag{249}$$

$$V_{sh} = -S_m Q_1 + S_{22}Q_{sh} \tag{250}$$

We see from Figure 14 that the potential on the center conductor (V_1 in the preceding equation) will be negative with respect to the braided shield (taken to have zero potential) when the center conductor is uncharged ($Q_1 = 0$) and the braided shield is positively charged ($Q_{sh} > 0$) with respect to the chassis; hence we expect $S_m > 0$. The inverse equations become

$$Q_1 = C_{11}V_1 - C_m V_{sh} = C_{11}V_1 + \ell C_T V_{sh} \tag{251}$$

$$Q_{sh} = -C_m V_1 + C_{22}V_{sh} = \ell C_T V_1 + C_{22}V_{sh} \tag{252}$$

Using potentials we can rewrite the mutual elastance as

$$Q_1 Q_2 S_m = \int_V \underline{E}_1 \cdot \underline{D}_2 dV = -\int_V \nabla\phi_1 \cdot \underline{D}_2 dV \tag{253}$$

where using the identity

$$\nabla \cdot (\phi_1 \underline{D}_2) = \nabla\phi_1 \cdot \underline{D}_2 + \phi_1 \nabla \cdot \underline{D}_2 = \nabla\phi_1 \cdot \underline{D}_2 + \phi_1 \rho_{v2} \tag{254}$$

we find

$$\int_V \underline{E}_1 \cdot \underline{D}_2 dV = -\oint_S \phi_1 \underline{D}_2 \cdot \underline{n} dS$$

$$= -\int_{S_c} \phi_1 \underline{D}_2 \cdot \underline{n} dS - \int_{S_w} \phi_1 \underline{D}_2 \cdot \underline{n} dS - \int_{S_{chassis}} \phi_1 \underline{D}_2 \cdot \underline{n} dS = -\int_{S_{chassis}} \phi_1 \underline{D}_2 \cdot \underline{n} dS = V_{21} Q_2 \tag{255}$$

or (V_{21} is actually V_2 excited by a charge on conductor 1)

$$S_m = V_2/Q_1 \text{ with } Q_2 = 0 \tag{256}$$

Alternatively

$$Q_1 Q_2 S_m = \int_V \underline{E}_2 \cdot \underline{D}_1 dV = -\int_V \nabla \phi_2 \cdot \underline{D}_1 dV \tag{257}$$

where using the identity

$$\nabla \cdot (\phi_2 \underline{D}_1) = \nabla \phi_2 \cdot \underline{D}_1 + \phi_2 \nabla \cdot \underline{D}_1 = \nabla \phi_2 \cdot \underline{D}_1 + \phi_2 \rho_{v1} \tag{258}$$

we find

$$\int_V \underline{E}_2 \cdot \underline{D}_1 dV = -\oint_S \phi_2 \underline{D}_1 \cdot \underline{n} dS$$

$$= -\int_{S_c} \phi_2 \underline{D}_1 \cdot \underline{n} dS - \int_{S_w} \phi_2 \underline{D}_1 \cdot \underline{n} dS - \int_{S_{chassis}} \phi_2 \underline{D}_1 \cdot \underline{n} dS = -\int_{S_c} \phi_2 \underline{D}_1 \cdot \underline{n} dS = V_{12} Q_1 \tag{259}$$

or (V_{12} is actually V_1 excited by a charge on conductor 2)

$$S_m = V_1/Q_2 \text{ with } Q_1 = 0 \tag{260}$$

Depending on which field we choose to represent with the potential we will end up with a different surface integral in the end. In the first case the closed surface integral is focused on the 2 region outside the braid because the field $\underline{D}_2$ is generated by charge on the chassis. In the second case the closed surface integral is focused on the 1 region inside the braid because the field $\underline{D}_1$ is generated by charge on the center conductor.

Self Capacitance Approximate Evaluation The self capacitance for nonuniform geometries is now estimated. Hence from the preceding expression the self terms S_{11} and S_{22} are

$$S_{11} Q_1^2 = \int_V D_1 E_1 dV \tag{261}$$

$$S_{22} Q_2^2 = \int_V D_2 E_2 dV \tag{262}$$

We will focus on the elastance for the 1 problem and select the auxiliary volume V_0 to extend from the center conductor to a distance out near the braid. Now we break up the volume into two parts

$$S_{11} Q_1^2 = \int_{V_0} D_1 E_1 dV + \int_{\Delta V} D_1 E_1 dV \approx \int_{V_0} D_1 E_0 dV + \int_{\Delta V} D_1 E_1 dV \tag{263}$$

where we approximate the electric field in the V_0 region by $\underline{E}_0$ the field for a solid shield at the boundaries of V_0, and $\Delta V = V - V_0$. The auxiliary problem has charge $Q_0 = Q_1$ but the center conductor potential

with a solid shield at the boundary S_0 (of volume V_0) is slightly different than with the braid $V_0 \neq V_1$. Using the scalar potentials

$$\underline{E}_0 = -\nabla\phi_0 \tag{264}$$

we can write

$$\int_{V_0} D_1 E_0 dV \approx \int_{V_0} \underline{D}_1 \cdot \underline{E}_0 dV = -\int_{V_0} \underline{D}_1 \cdot \nabla\phi_0 dV \tag{265}$$

with

$$\nabla \cdot (\phi_0 \underline{D}_1) = \underline{D}_1 \cdot \nabla\phi_0 + \phi_0 \nabla \cdot \underline{D}_1 \tag{266}$$

and by virtue of Gauss's law

$$\nabla \cdot \underline{D}_1 = \rho_{1v} \tag{267}$$

we obtain (dropping the volume charge density outside the conductors)

$$\int_{V_0} D_1 E_0 dV \approx -\oint_{S_0} \phi_0 \underline{D}_1 \cdot \underline{n} dS = -\int_{S_c} \phi_0 \underline{D}_1 \cdot \underline{n} dS - \int_{S_0} \phi_0 \underline{D}_1 \cdot \underline{n} dS \tag{268}$$

If we note that ϕ_0 should be constructed to vanish on the solid auxiliary shield S_0 then the final term vanishes. Also using

$$\underline{E}_1 = -\nabla\phi_1 \tag{269}$$

we have

$$\int_{\Delta V} D_1 E_1 dV = -\int_{\Delta V} \underline{D}_1 \cdot \nabla\phi_1 dV \tag{270}$$

Using

$$\nabla \cdot (\phi_1 \underline{D}_1) = \underline{D}_1 \cdot \nabla\phi_1 + \phi_1 \nabla \cdot \underline{D}_1 \tag{271}$$

and dropping the volume charge density outside the conductors, gives

$$\int_{\Delta V} D_1 E_1 dV = -\oint_{\Delta S} \phi_1 \underline{D}_1 \cdot \underline{n} dS = -\int_{S_0} \phi_1 \underline{D}_1 \cdot \underline{n} dS - \int_{S_w} \phi_1 \underline{D}_1 \cdot \underline{n} dS - \int_{S_{chassis}} \phi_1 \underline{D}_1 \cdot \underline{n} dS \tag{272}$$

where we have used periodicity of the fields and potential to drop the surface integrals on the ends of the periodic region in z. Noting that ϕ_1 should be constructed to vanish on the braid wire surface and hence the integral on S_w vanishes. Furthermore, in this section the chassis is open circuited in the 1 problem and hence $\int_{S_{chassis}} \underline{D}_1 \cdot \underline{n} dS = 0$; with ϕ_1 equal to a constant on $S_{chassis}$ the final integral also vanishes. Hence we finally have

$$S_{11} Q_1^2 \approx -\int_{S_c} \phi_0 \underline{D}_1 \cdot \underline{n} dS - \int_{S_0} \phi_1 \underline{D}_1 \cdot \underline{n} dS$$

$$\approx V_0 Q_1 - \int_{S_0} \phi_1 \underline{D}_1 \cdot \underline{n} dS \tag{273}$$

where $\phi_0 = V_0$ (not to be confused with the auxiliary volume V_0 on the integral limit) on S_c and the integral yields minus the center conductor charge.

Now near the braid but still far from the individual braid wires we have the local behavior

$$\phi_1 \sim E_0 x + \phi_b \ , \ x = b - \rho \to 0 \tag{274}$$

and thus

$$S_{11} Q_1^2 \approx V_0 Q_1 - \int_{S_0} (E_0 x + \phi_b) \underline{D}_1 \cdot \underline{n} dS \tag{275}$$

For a circular shield, like in the eccentric coax, we can take $x = x_0 = b - b_0$

$$S_{11} Q_1^2 \approx V_0 Q_1 - \int_{S_0} [E_0 (b - b_0) + \phi_b] \underline{D}_1 \cdot \underline{n} dS \tag{276}$$

Note that the electric field E_0 and the potential ϕ_b are in general varying around the circumference. We factor the potential term by writing (here we take $E_0 = E_\rho$)

$$\phi_b = (\phi_b / E_0) E_0 \tag{277}$$

where ϕ_b / E_0 is a constant dependent only on the braid wire geometry in the planar approximation to the braid. Then (noting that $\underline{n} = -\underline{e}_\rho$ on S_0 from the ΔV region)

$$S_{11} Q_1^2 \approx V_0 Q_1 - [(b - b_0) + \phi_b / E_0] \int_{S_0} E_0 \underline{D}_1 \cdot \underline{n} dS$$

$$\approx V_0 Q_0 (Q_1 / Q_0) + [(b - b_0) + \phi_b / E_0] (Q_1 / Q_0) \int_{S_0} \varepsilon E_0^2 dS \tag{278}$$

Noting that the charge in the auxiliary problem is taken to be the same $Q_0 = Q_1$ and taking $Q_0 = C_{00} V_0$ we find

$$C_{00} / C_{11} \approx C_{00} S_{11} \approx 1 + [(b - b_0) + \phi_b / E_0] \frac{1}{C_{00} V_0^2} \int_{S_0} \varepsilon E_0^2 dS \tag{279}$$

where we have used the approximation $S_{11} \approx 1 / C_{11}$. Now changing to the capacitances per unit length

$$C_{00} = \ell C_0 \tag{280}$$

$$C_{11} = C_1 \ell \tag{281}$$

we have

$$C_1 \approx C_0 - [(b - b_0) + \phi_b / E_0] \int_{S_0} \varepsilon (E_0 / V_0)^2 dS / \ell \tag{282}$$

We can write a similar expression for the exterior capacitance per unit length $C_2 = C_{22} / \ell$.

Transfer Capacitance Approximate Evaluation The mutual or transfer capacitance is now estimated for nonuniform geometries from the cross terms

$$S_m Q_1 Q_2 = \frac{1}{2}\int_V \underline{D}_1 \cdot \underline{E}_2 dV + \frac{1}{2}\int_V \underline{E}_1 \cdot \underline{D}_2 dV = \int_V \varepsilon \underline{E}_1 \cdot \underline{E}_2 dV$$

$$= \int_V \underline{D}_1 \cdot \underline{E}_2 dV = \int_V \underline{E}_1 \cdot \underline{D}_2 dV \tag{283}$$

Now we break up the volume into two parts

$$S_m Q_1 Q_2 = \int_V \varepsilon \underline{E}_1 \cdot \underline{E}_2 dV = \int_{V_0} \varepsilon \underline{E}_2 \cdot \underline{E}_1 dV + \int_{\Delta V} \underline{E}_2 \cdot \varepsilon \underline{E}_1 dV$$

$$= \int_{V_0} \underline{D}_2 \cdot \underline{E}_1 dV + \int_{\Delta V} \underline{E}_2 \cdot \underline{D}_1 dV$$

$$\approx \int_{V_0} \underline{D}_2 \cdot \underline{E}_0 dV + \int_{\Delta V} \underline{E}_2 \cdot \underline{D}_1 dV \tag{284}$$

where the auxiliary volume is V_0, we have approximated $E_1 \approx E_0$ in V_0, and the remaining volume is $\Delta V = V - V_0$. Now noting that we can write

$$\int_{V_0} \underline{D}_2 \cdot \underline{E}_0 dV = -\int_{V_0} \underline{D}_2 \cdot \nabla \phi_0 dV \tag{285}$$

with

$$\nabla \cdot (\phi_0 \underline{D}_2) = \underline{D}_2 \cdot \nabla \phi_0 + \phi_0 \nabla \cdot \underline{D}_2 \tag{286}$$

and by virtue of Gauss's law

$$\nabla \cdot \underline{D}_2 = \rho_{2v} \tag{287}$$

we obtain

$$\int_{V_0} \underline{D}_2 \cdot \underline{E}_0 dV = -\oint_{S_0} \phi_0 \underline{D}_2 \cdot \underline{n} dS = -\int_{S_c} \phi_0 \underline{D}_2 \cdot \underline{n} dS - \int_{S_0} \phi_0 \underline{D}_2 \cdot \underline{n} dS \tag{288}$$

where here $\underline{n}$ points out of V_0. If we note that ϕ_0 should be constructed to vanish on the auxiliary shield S_0 then the final term vanishes. Also

$$\int_{\Delta V} \underline{E}_2 \cdot \underline{D}_1 dV = -\int_{S_0} \phi_2 \underline{D}_1 \cdot \underline{n} dS - \int_{S_w} \phi_2 \underline{D}_1 \cdot \underline{n} dS - \int_{S_{chassis}} \phi_2 \underline{D}_1 \cdot \underline{n} dS \tag{289}$$

where here $\underline{n}$ points out of ΔV. Now ϕ_1 as well as ϕ_2 are constructed to vanish on the braid wire surface and hence the integral over S_w of the preceding equation vanishes. In addition in this section, ϕ_1 is constructed with an open circuited chassis surface in the 1 problem, so there is no net charge on $S_{chassis}$ giving $\int_{S_{chassis}} \underline{D}_1 \cdot \underline{n} dS = 0$ (where ϕ_2 is constant on $S_{chassis}$), and ϕ_2 is constructed with an open circuited center conductor surface in the 2 problem, so there is no net charge on S_c giving $\int_{S_c} \underline{D}_2 \cdot \underline{n} dS = 0$ (where ϕ_1 and ϕ_0 are constant on S_c). Hence the V_0 integration vanishes; in the final line of the preceding ΔV integration the final term also vanishes. Therefore we can write the mutual elastance as

$$S_m Q_1 Q_2 = \int_V \varepsilon \underline{E}_1 \cdot \underline{E}_2 dV \approx \int_{\Delta V} \underline{E}_2 \cdot \underline{D}_1 dV = -\int_{S_0} \phi_2 \underline{D}_1 \cdot \underline{n} dS \tag{290}$$

Now we note that near S_0 we can write the potential in the 2 problem as (since there is no net charge on the center conductor in the 2 problem)

$$\phi_2 \sim -\phi_c \approx (-\phi_c/E_0)\, E_0^{sh} \tag{291}$$

where (we usually evaluate the outward exterior field E_0^{sh} at the mean braid wire location of S_w for a solid shield rather than on S_0 since this field exists exterior to the braided shield) the ratio $-\phi_c/E_0$ is a constant for a given braid geometry and the normal field E_0^{sh} in general varies around the braided shield (which can be determined from the exterior potential ϕ_{sh} for a solid shield at the braid center line)

$$-E_0^{sh} = -\frac{\partial \phi_{sh}}{\partial n} \tag{292}$$

where $\underline{n}$ here is taken to point inward from the exterior region (is also consistent with the sign from the ΔV region). Then the mutual elastance is

$$S_m \approx -(-\phi_c/E_0)\, \frac{1}{Q_1 Q_2} \int_{S_0} E_0^{sh} \underline{D}_1 \cdot \underline{n} dS \tag{293}$$

where the center conductor charge Q_1 normalizes the integration of the normal component of the interior displacement in the 1 problem and the chassis charge Q_2 normalizes the normal component of the exterior field level at the braided shield location. Now we approximate the field $\underline{D}_1 \approx \underline{D}_0$ on S_0 and replace $Q_1 = Q_0$ and $Q_2 = -Q_{sh}$, where $Q_{sh} = C_{00}^{sh} V_{sh}$, $C_{00}^{sh} = \ell C_{sh}$ and V_{sh} is the exterior voltage from the approximate solid shield to the chassis

$$S_m \approx (-\phi_c/E_0)\, \frac{1}{Q_0 Q_{sh}} \int_{S_0} E_0^{sh} \underline{D}_0 \cdot \underline{n} dS$$

$$\approx (-\phi_c/E_0)\, \frac{1}{Q_0 C_{00}^{sh}} \int_{S_0} \left(E_0^{sh}/V_{sh}\right) \underline{D}_0 \cdot \underline{n} dS \tag{294}$$

Now setting

$$Q_0 = \ell q_0 \tag{295}$$

and

$$D_0 = -\underline{D}_0 \cdot \underline{n} \tag{296}$$

and using

$$C_m \approx -S_m/\left(S_{11} S_{22}\right) \approx -C_{11} C_{22} S_m \tag{297}$$

gives

$$C_m \approx (-\phi_c/E_0)\, \frac{C_{11} C_{22}}{C_{00}^{sh}} \int_{S_0} \left(E_0^{sh}/V_{sh}\right) \left(D_0/q_0\right) dS/\ell$$

$$\approx (-\phi_c/E_0)\, C_{11} \int_{S_0} \left(E_0^{sh}/V_{sh}\right) (D_0/q_0)\, dS/\ell \tag{298}$$

where we have approximated $C_{22} \approx C_{00}^{sh}$. Finally setting

$$C_{11} = \ell C_1 \tag{299}$$

$$C_m = -\ell C_T \tag{300}$$

we arrive at

$$C_T \approx (\phi_c/E_0)\, C_1 \int_{S_0} \left(E_0^{sh}/V_{sh}\right) (D_0/q_0)\, dS/\ell \tag{301}$$

We include a brief repeat of the application of these results to the uniform and nonuniform coaxial geometries below in order to illustrate a slightly different set of integral identities in the evaluation.

Uniform Cylindrical Geometry Applying these to the simple case where the geometry is a uniform cylinder both inside and outside

$$\varepsilon E_0/\left(C_0 V_0\right) = D_0/q_0 = 1/\left(2\pi\rho\right) = \varepsilon E_0^{sh}/\left(C_{sh} V_{sh}\right) \tag{302}$$

$$C_0 = \frac{2\pi\varepsilon}{\ln\left(b_0/a\right)} \tag{303}$$

and

$$C_1 \approx C_0 - \left[(b - b_0) + \phi_b/E_0\right] \int_{S_0} \varepsilon \left(E_0/V_0\right)^2 dS/\ell$$

$$\approx C_0 \left[1 - \left\{(b - b_0) + \phi_b/E_0\right\} \frac{C_0}{\varepsilon} \left(\frac{1}{2\pi b_0}\right)\right]$$

$$\approx C_0 \left[1 - \left\{(b - b_0) + \phi_b/E_0\right\} C_0 \frac{\partial\left(1/C_0\right)}{\partial b_0}\right]$$

$$\approx C_0 + \left\{(b - b_0) + \phi_b/E_0\right\} \frac{\partial C_0}{\partial b_0}$$

$$\approx C_0\left(b_0 \to b + \phi_b/E_0\right) = \frac{2\pi\varepsilon}{\ln\left[\left(b + \phi_b/E_0\right)/a\right]} \tag{304}$$

$$C_T \approx C_1 \left(\phi_c/E_0\right) \int_{S_0} \left(E_0^{sh}/V_{sh}\right) \left(D_0/q_0\right) dS/\ell$$

$$\approx \left(\phi_c/E_0^{sh}\right) \frac{C_1 C_{sh}}{2\pi b\varepsilon} \tag{305}$$

where the preceding arrow means "replaced by".

Exterior Ground Plane Case When the exterior transmission line problem consists of a circular cable of radius b, with center height h above a ground plane, we write the exterior potential as (here we take the shield potential to vanish and the ground plane $y = 0$ to be at potential $-V_{sh}$)

$$\phi_{sh} = \frac{q_{sh}}{2\pi\varepsilon} \ln \sqrt{\frac{x^2 + (y + h_e)^2}{x^2 + (y - h_e)^2}} - V_{sh} \tag{306}$$

where q_{sh} is the exterior charge per unit length and the effective line source is placed at height h_e above the ground plane where the charge centroid of the cylinder is

$$h_e = \sqrt{h^2 - b^2} \tag{307}$$

Letting

$$x = \rho \cos \varphi \tag{308}$$

$$y = h + \rho \sin \varphi \tag{309}$$

$$\phi_{sh} = \frac{q_{sh}}{2\pi\varepsilon} \ln \sqrt{\frac{\rho^2 - b^2 + 2h(h + h_e) + 2(h + h_e)\rho \sin \varphi}{\rho^2 - b^2 + 2h(h - h_e) + 2(h - h_e)\rho \sin \varphi}} - V_{sh} \tag{310}$$

Note that the potential at the mean braid radius $\rho = b$ is constant and taken to vanish (this image representation is the well known exact image solution of a charged perfectly conducting cylinder above the chassis ground plane). The exterior capacitance per unit length is

$$C_{sh} = q_{sh}/V_{sh} = 2\pi\varepsilon / \ln \sqrt{\frac{h + h_e}{h - h_e}} = 2\pi\varepsilon / \mathrm{Arccosh}\,(h/b) \tag{311}$$

and we use the capacitance per unit length of the exterior of the braid transmission line with the mean radius b as an approximation to the exterior transmission line capacitance per unit length. The exterior radial field is

$$E_\rho(b, \varphi) = -\frac{\partial \phi_{sh}}{\partial \rho}(b, \varphi) = \frac{q_{sh}}{2\pi\varepsilon b} \frac{h_e}{h + b \sin \varphi} = E_0^{sh} \tag{312}$$

$$q_{sh} = C_{sh} V_{sh} \tag{313}$$

where we note that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{h_e}{h + b \sin \varphi} d\varphi = 1 \tag{314}$$

The transfer capacitance when the interior is a uniform cylinder is then

$$C_T \approx C_1 C_{sh} (\phi_c / E_0) \int_{S_0} \left(E_0^{sh} / q_{sh}\right) (D_0 / q_0)\, dS/\ell \tag{315}$$

$$\approx \frac{C_1 C_{sh}}{2\pi\varepsilon b}\left(\phi_c/E_0\right)\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{h_e}{h+b\sin\varphi}d\varphi$$

$$\approx \frac{C_1 C_{sh}}{2\pi\varepsilon b}\left(\phi_c/E_0\right) \tag{316}$$

The only difference is that C_{sh} has a different form.

Interior Eccentric Coax In addition to an exterior ground plane let us consider the situation when the interior is an eccentric coax [14]. The outer cylinder again has radius b_0 and the inner cylinder has radius a. We place the center of the outer cylinder at $x = 0$ and $y = h$. The potential of the outer cylinder is taken to vanish and the potential of the inner cylinder is V_0. Thus in summary

$$\phi_0 = -\frac{q_0}{2\pi\varepsilon}\ln\sqrt{\frac{x^2+\left(y+y_1-h-y_c\right)^2}{x^2+\left(y+y_1-h+y_c\right)^2}} - U_1$$

$$q_0 = C_0 V_0$$

$$C_0 = 2\pi\varepsilon/\mathrm{Arccosh}\left(\frac{a^2+b_0^2-d^2}{2ab_0}\right)$$

$$2y_c d = \sqrt{\left[\left(b_0-a\right)^2-d^2\right]\left[\left(b_0+a\right)^2-d^2\right]}$$

$$y_1 = \sqrt{y_c^2+b_0^2} = \left(-a^2+b_0^2+d^2\right)/\left(2d\right)$$

$$\frac{q_0}{2\pi\varepsilon}\mathrm{Arccosh}\left(y_1/b_0\right) = U_1 \tag{317}$$

with interior electric field

$$\frac{1}{q_0}E_\rho\left(b_0,\varphi\right) = -\frac{1}{q_0}\frac{\partial\phi_0}{\partial\rho}\left(\rho=b_0\right) = \frac{1}{2\pi\varepsilon b_0}\frac{y_c}{y_1+b_0\sin\varphi} = E_0/q_0 \tag{318}$$

Note that we can rotate the angle φ in this formula (versus in the outer short circuit field) to rotate the displacement d in the eccentric coax versus the outer short circuit field; such a rotation by $\pi/2$ is done as an extra example below.

The interior self capacitance per unit length is then

$$C_1 \approx C_0 - C_0^2\left\{\left(b-b_0\right)+\phi_b/E_0\right\}\int_{S_0}\varepsilon\left(E_0/q_0\right)^2 dS/\ell \tag{319}$$

$$\approx C_0\left[1-\frac{C_0}{2\pi\varepsilon b}\left\{\left(b-b_0\right)+\phi_b/E_0\right\}\left(y_1/y_c\right)\right]$$

$$\approx C_0\left[1-\left\{\left(b-b_0\right)+\phi_b/E_0\right\}C_0\frac{\partial\left(1/C_0\right)}{\partial b_0}\right]$$

$$\approx C_0 + \{(b - b_0) + \phi_b/E_0\} \frac{\partial C_0}{\partial b_0}$$

$$\approx C_0 \left(b_0 \to b + \phi_b/E_0\right)$$

where we used

$$\frac{\partial \left(1/C_0\right)}{\partial b_0} = -\frac{1}{C_0^2} \frac{\partial C_0}{\partial b_0} = \frac{(y_1/y_c)}{2\pi\varepsilon b_0}$$

and the identity

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{y_c}{y_1 + b_0 \sin\varphi}\right)^2 d\varphi = y_1/y_c \tag{320}$$

Taking the outer short circuit field for the chassis ground plane case (312) the transfer capacitance is

$$C_T \approx C_1 \left(\phi_c/E_0\right) \int_{S_0} \left(E_0^{sh}/V_{sh}\right) \left(D_0/q_0\right) dS/\ell$$

$$\approx \frac{C_{sh} C_1}{2\pi\varepsilon b} \left(\phi_c/E_0\right) \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{h_e}{h + b\sin\varphi}\right) \left(\frac{y_c}{y_1 + b_0 \sin\varphi}\right) d\varphi$$

$$\approx \frac{C_{sh} C_1}{2\pi\varepsilon b} \left(\phi_c/E_0\right) \frac{b y_c - b_0 h_e}{b y_1 - b_0 h} \tag{321}$$

where we used the identity (the individual integrals divided by 2π are each unity)

$$\left(\frac{b y_1 - b_0 h}{h_e y_c}\right) \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{h_e}{h + b\sin\varphi} \frac{y_c}{y_1 + b_0 \sin\varphi} d\varphi$$

$$= \left(b/h_e\right) \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{h_e}{h + b\sin\varphi} d\varphi - \left(b_0/y_c\right) \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{y_c}{y_1 + b_0 \sin\varphi} d\varphi$$

$$= b/h_e - b_0/y_c \tag{322}$$

To avoid having a somewhat arbitrary $b_0 \sim b$ it is better to simplify the final expression by letting $b_0 \to b$ in y_c and in y_1 with

$$C_T \approx \frac{C_{sh} C_1}{2\pi\varepsilon b} \left(\phi_c/E_0\right) \frac{y_c - h_e}{y_1 - h}$$

We see that the "average" in (321) is unity if one or the other of the factors in the integrand is unity (as in an inner or outer cylindrical coaxial arrangement) and we return to the expression (305). The extra factor in the final result from this integral average is $\left(y_c - h_e\right) / \left(y_1 - h\right)$ which is unity if $d = 0$ (which makes the field of the interior problem uniform in azimuth) or if $h \to \infty$ (which makes the field of the exterior problem uniform in azimuth). However if both factors are present, due to the eccentric inner and outer (the ground plane is a form of eccentric coax) arrangements, then the transfer capacitance is an average of the product of the two factors.

Note that if we rotate the azimuth of the eccentric coax by $\pi/2$ relative to the outer ground plane arrangement we have

$$C_T \approx C_1 \left(\phi_c/E_0^{sh}\right) \frac{C_{sh}}{2\pi\varepsilon b} \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{h_e}{h + b\sin\varphi} \frac{y_c}{y_1 + b_0\cos\varphi} d\varphi \tag{323}$$

Using

$$\left(\frac{bb_0}{h_e y_c}\right) \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{h_e}{h + b\sin\varphi} \frac{y_c}{y_1 + b_0\cos\varphi} d\varphi$$

$$= (b/h_e) \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{h_e\cos\varphi}{h + b\sin\varphi} d\varphi + (b_0/y_c) \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{y_c\sin\varphi}{y_1 + b_0\cos\varphi} d\varphi$$

$$-\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{by_1\cos\varphi + b_0 h\sin\varphi}{(h + b\sin\varphi)(y_1 + b_0\cos\varphi)} d\varphi \tag{324}$$

and

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{by_1\cos\varphi + b_0 h\sin\varphi}{(h + b\sin\varphi)(y_1 + b_0\cos\varphi)} d\varphi$$

$$= \frac{b}{b_0} (y_1/h_e) \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{h_e}{h + b\sin\varphi} d\varphi + \frac{b_0}{b} (h/y_c) \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{y_c}{y_1 + b_0\cos\varphi} d\varphi$$

$$-\left(\frac{y_1^2 b^2 + h^2 b_0^2}{h_e y_c b b_0}\right) \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{h_e}{h + b\sin\varphi} \frac{y_c}{y_1 + b_0\cos\varphi} d\varphi \tag{325}$$

we can write

$$\left\{\frac{bb_0}{h_e y_c} - \frac{1}{bb_0}\left(\frac{y_1^2 b^2 + h^2 b_0^2}{h_e y_c}\right)\right\} \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{h_e}{h + b\sin\varphi} \frac{y_c}{y_1 + b_0\cos\varphi} d\varphi$$

$$= (b/h_e) \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{h_e\cos\varphi}{h + b\sin\varphi} d\varphi + (b_0/y_c) \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{y_c\sin\varphi}{y_1 + b_0\cos\varphi} d\varphi$$

$$-\frac{b}{b_0} (y_1/h_e) \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{h_e}{h + b\sin\varphi} d\varphi - \frac{b_0}{b} (h/y_c) \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{y_c}{y_1 + b_0\cos\varphi} d\varphi$$

$$= (b/h_e) \frac{1}{2\pi} \left(\int_0^{\pi/2} + \int_{\pi/2}^{\pi} + \int_{-\pi}^{-\pi/2} + \int_{-\pi/2}^{0}\right) \frac{h_e\cos\varphi}{h + b\sin\varphi} d\varphi$$

$$+ (b_0/y_c) \frac{1}{2\pi} \left(\int_0^{\pi/2} + \int_{\pi/2}^{\pi} + \int_{-\pi}^{-\pi/2} + \int_{-\pi/2}^{0}\right) \frac{y_c\sin\varphi}{y_1 + b_0\cos\varphi} d\varphi - \frac{b}{b_0} (y_1/h_e) - \frac{b_0}{b} (h/y_c) \tag{326}$$

In the first and second sequences of integrals we let (in the second integral $\varphi = \pi - \varphi'$, in the third integral $\varphi = -\pi + \varphi'$, and in the fourth integral $\varphi = -\varphi'$). This shows that these all cancel out and thus

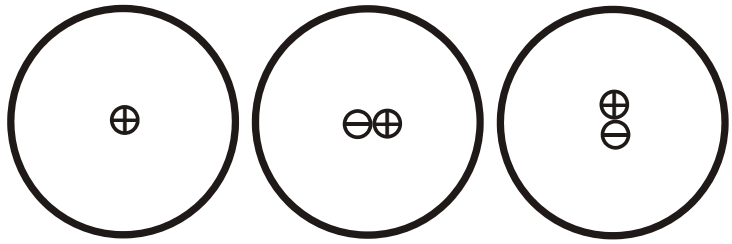

Figure 15. A sequence of line multipole charges to represent the transverse variations of the electric field.

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{h_e}{h+b\sin\varphi}\frac{y_c}{y_1+b_0\cos\varphi}d\varphi=\frac{b^2y_1y_c+b_0^2hh_e}{y_1^2b^2+h_e^2b_0^2} \tag{327}$$

or

$$C_T\approx C_1\left(\phi_c/E_0^{sh}\right)\frac{C_{sh}}{2\pi\varepsilon b}\frac{b^2y_1y_c+b_0^2hh_e}{y_1^2b^2+h_e^2b_0^2} \tag{328}$$

Again to avoid having a somewhat arbitrary $b_0\sim b$ it is better to simplify the final expression by letting $b_0\to b$ in y_c and in y_1 with

$$C_T\approx C_1\left(\phi_c/E_0^{sh}\right)\frac{C_{sh}}{2\pi\varepsilon b}\frac{y_1y_c+hh_e}{y_1^2+h_e^2}$$

3.3 Electric Multipole Representation

In the preceding sections, we have assumed that the values of ϕ_c/E_0 and of ϕ_b/E_0 have been determined. The way we will actually determine these quantities is by solving for the potential surrounding a periodic cell of the braid. This could be done in the actual cylindrical braid, but as an approximation, and because the planar shield is of interest in its own right, we will concentrate at present on the planar problem as depicted in Figure 12. The drive potential in the planar problem will be taken as

$$\phi_{inc}=E_0y \tag{329}$$

where $y=0$ is at the braid center. The representation of the potential resulting from the surface charges on the perfectly conducting wires is now discussed.

It is efficient to represent the electric scalar potential by an electric multipole summation as shown in Figure 15 to capture the transverse field behavior. The potential for an axially varying line charge $q(z)$ is

$$\phi_{scatt}=\frac{1}{4\pi\varepsilon}\int q(z')\frac{dz'}{|\underline{r}-\underline{r}'|} \tag{330}$$

If the charge is discretized as pulses of strength q_n, length Δ_n, centered at z_n', we can write

$$\phi_{scatt}=\frac{q_n}{4\pi\varepsilon}\int_{z_n'-\Delta_n/2}^{z_n'+\Delta_n/2}\frac{dz'}{\sqrt{\rho^2+(z-z')^2}}=\frac{q_n}{4\pi\varepsilon}\int_{(z-z_n'-\Delta_n/2)/\rho}^{\left(z-z_n'+\Delta_n/2\right)/\rho}\frac{du}{\sqrt{1+u^2}}$$

$$= \frac{q_n}{4\pi\varepsilon}\left[\text{Arcsinh}\left(\frac{z - z'_n + \Delta_n/2}{\rho}\right) - \text{Arcsinh}\left(\frac{z - z'_n - \Delta_n/2}{\rho}\right)\right]$$

$$= \frac{q_n}{4\pi\varepsilon}\ln\left[\frac{(z - z'_n + \Delta_n/2) + \sqrt{\rho^2 + (z - z'_n + \Delta_n/2)^2}}{(z - z'_n - \Delta_n/2) + \sqrt{\rho^2 + (z - z'_n - \Delta_n/2)^2}}\right] \tag{331}$$

Now to put this into a framework for general orientations we take the end positions of the N segments to be denoted by

$$\underline{r} = \underline{r}_n^{\pm}\ ,\ n = 1, ..., N \tag{332}$$

The vector along the axis of a segment is

$$\underline{s}_n = \underline{r}_n^{+} - \underline{r}_n^{-} \tag{333}$$

The unit vector along the axis of a segment is

$$\underline{e}_{sn} = \underline{s}_n / s_n \tag{334}$$

The center point of a segment is

$$\underline{r}_n^c = \frac{1}{2}\left(\underline{r}_n^{+} + \underline{r}_n^{-}\right) \tag{335}$$

The projected distance along a segment is denoted by

$$s = \underline{e}_{sn} \cdot (\underline{r} - \underline{r}_n^c) \tag{336}$$

The vector distance perpendicular to the segment is

$$\underline{\rho} = -\underline{e}_{sn} \times [\underline{e}_{sn} \times (\underline{r} - \underline{r}_n^c)] \tag{337}$$

The potential is then

$$\phi_{scatt} = -\frac{q_n}{4\pi\varepsilon}\ln\left[\frac{(s - s_n/2) + \sqrt{\rho^2 + (s - s_n/2)^2}}{(s + s_n/2) + \sqrt{\rho^2 + (s + s_n/2)^2}}\right]$$

$$= -\frac{q_n}{4\pi\varepsilon}\ln\left[\frac{(s - s_n/2) + |\underline{r} - \underline{r}_n^{+}|}{(s + s_n/2) + |\underline{r} - \underline{r}_n^{-}|}\right] \tag{338}$$

The lattice parameters are now used to image this potential contribution over the periodic cells of the planar braid model. The two periodic lattice vectors (associated with the two cross braid directions) are taken as $\underline{u}$ and $\underline{v}$. The components of the lattice vectors along and perpendicular to the direction of a particular braid segment n are taken as

$$\underline{u} = u_{sn}\underline{e}_{sn} + \underline{u}_{\rho n} \tag{339}$$

$$\underline{v} = v_{sn}\underline{e}_{sn} + \underline{v}_{\rho n} \tag{340}$$

Thus we have the total potential

$$\phi_{scatt}^{tot} = \sum_{n=1}^{N} \frac{q_n}{4\pi\varepsilon} \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \ln \left[\frac{(s - s_n/2 - ju_{sn} - kv_{sn}) + \sqrt{\left|\underline{\rho} - j\underline{u}_{\rho n} - k\underline{v}_{\rho n}\right|^2 + (s - s_n/2 - ju_{sn} - kv_{sn})^2}}{(s + s_n/2 - ju_{sn} - kv_{sn}) + \sqrt{\left|\underline{\rho} - j\underline{u}_{\rho n} - k\underline{v}_{\rho n}\right|^2 + (s + s_n/2 - ju_{sn} - kv_{sn})^2}} \right]$$

$$= \sum_{n=1}^{N} \frac{q_n}{4\pi\varepsilon} \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \ln \left[\frac{(s - s_n/2 - ju_{sn} - kv_{sn}) + \left|\underline{r} - \underline{r}_n^+ - j\underline{u} - k\underline{v}\right|}{(s + s_n/2 - ju_{sn} - kv_{sn}) + \left|\underline{r} - \underline{r}_n^- - j\underline{u} - k\underline{v}\right|} \right] \tag{341}$$

The constant potential condition on each n segment $\phi_{tot} + \phi_{inc} = V_{n'}$ around the braid wires uses the evaluation

$$\phi_{scatt}^{tot} = \sum_{n=1}^{N} \frac{q_n}{4\pi\varepsilon} \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \ln \left[\frac{(s_{n'} - s_n/2 - ju_{sn} - kv_{sn}) + \sqrt{\left|\underline{\rho}_{n'} - j\underline{u}_{\rho n} - k\underline{v}_{\rho n}\right|^2 + (s_{n'} - s_n/2 - ju_{sn} - kv_{sn})^2}}{(s_{n'} + s_n/2 - ju_{sn} - kv_{sn}) + \sqrt{\left|\underline{\rho}_{n'} - j\underline{u}_{\rho n} - k\underline{v}_{\rho n}\right|^2 + (s_{n'} + s_n/2 - ju_{sn} - kv_{sn})^2}} \right]$$

$$= \sum_{n=1}^{N} \frac{q_n}{4\pi\varepsilon} \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \ln \left[\frac{(s_{n'} - s_n/2 - ju_{sn} - kv_{sn}) + \left|\underline{r}_{n'} - \underline{r}_n^+ - j\underline{u} - k\underline{v}\right|}{(s_{n'} + s_n/2 - ju_{sn} - kv_{sn}) + \left|\underline{r}_{n'} - \underline{r}_n^- - j\underline{u} - k\underline{v}\right|} \right] \tag{342}$$

where the observation or match points are (a is the braid wire radius)

$$s_{n'} = \underline{e}_{sn} \cdot (\underline{r}_{n'} - \underline{r}_n^c) \tag{343}$$

$$\underline{\rho}_{n'} = -\underline{e}_{sn} \times \underline{e}_{sn} \times (\underline{r}_{n'}^c - \underline{r}_n^c) + a\underline{e}_{\rho n'} \tag{344}$$

$$\underline{r}_{n'} = \underline{r}_{n'}^c + a\underline{e}_{\rho n'} \tag{345}$$

To construct the unit vector perpendicular to the n' wire $\underline{e}_{\rho n'}$ we need a vector $\underline{r}_0$ linearly independent of $\underline{s}_{n'}$. Then we can take

$$\underline{e}_{\rho n'} = -\underline{e}_{sn'} \times \underline{e}_{sn'} \times (\underline{r}_0 - \underline{r}_{n'}^c) \,/\, |\underline{e}_{sn'} \times \underline{e}_{sn'} \times (\underline{r}_0 - \underline{r}_{n'}^c)| \tag{346}$$

The choice is obviously not unique. The unit vectors used in the code are

$$\underline{e}_{\rho 1 n'} = \underline{e}_{sn'} \times (\underline{r}_0 - \underline{r}_{n'}^c) \,/\, |\underline{e}_{sn'} \times (\underline{r}_0 - \underline{r}_{n'}^c)| \tag{347}$$

$$\underline{e}_{\rho 2 n'} = \underline{e}_{sn'} \times \underline{e}_{sn'} \times (\underline{r}_0 - \underline{r}_{n'}^c) \,/\, |\underline{e}_{sn'} \times \underline{e}_{sn'} \times (\underline{r}_0 - \underline{r}_{n'}^c)| \tag{348}$$

We can rotate the vector to obtain other observation points around the wire (see the Appendix)

$$\underline{e}_{\rho n'}^{m'} = \cos(m'\pi/M)\, \underline{e}_{\rho n'} + \sin(m'\pi/M)\, \underline{e}_{sn'} \times \underline{e}_{\rho n'}\,,\ m' = 0, 1, ..., 2M - 1 \tag{349}$$

where $\underline{\rho}_{n'}^{(m')}$ and $\underline{r}_{n'}^{(m')}$ go with these. So far we only have the line monopole moment of each segment. Hence to match the wire boundary condition we can at present only match the average of the potential condition around the wire (we would also need the average of the incident potential $\langle\phi_{inc}\rangle$)

$$\left\langle\phi_{scatt}^{tot}\right\rangle = \sum_{n=1}^{N}\frac{q_n}{4\pi\varepsilon}\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\frac{1}{2M}\sum_{m'=0}^{2M-1}\ln\left[\frac{(s_{n'}-s_n/2-ju_{sn}-kv_{sn})+\left|\underline{r}_{n'}^{(m)}-\underline{r}_n^{+}-j\underline{u}-k\underline{v}\right|}{(s_{n'}+s_n/2-ju_{sn}-kv_{sn})+\left|\underline{r}_{n'}^{(m)}-\underline{r}_n^{-}-j\underline{u}-k\underline{v}\right|}\right] \tag{350}$$

The monopole moments are not sufficient to match the potential condition at many points around the wire. To improve the calculation we include a series of line multipole moments. Thus the combination of multipoles in the potential, for a given position n, is written as (where $q_n = \underline{p}^{(0)}$)

$$\phi_{scatt}^{n} = \sum_{m=0}^{M}\phi_{scatt}^{nm} = -\frac{1}{4\pi\varepsilon}\sum_{m=0}^{M}\underline{p}^{(0)}\underline{p}^{(1)}...\underline{p}^{(m)}\cdot\nabla_t^m\ln\left[\frac{(s-s_n/2)+\sqrt{\rho^2+(s-s_n/2)^2}}{(s+s_n/2)+\sqrt{\rho^2+(s+s_n/2)^2}}\right] \tag{351}$$

where ∇_t is the "del" operator transverse to the particular wire segment and the meaning of the "dot" product notation will be made clear on the following pages. Hence the total potential is

$$\phi_{scatt}^{tot} = \sum_{n=1}^{N}\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\phi_{scatt}^{n} = \sum_{n=1}^{N}\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\sum_{m=0}^{M}\phi_{scatt}^{nm} = \sum_{n=1}^{N}\frac{1}{4\pi\varepsilon}\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}$$

$$\sum_{m=0}^{M}\underline{p}^{(0)}...\underline{p}^{(m)}\cdot\nabla_t^m\ln\left[\frac{(s-s_n/2-ju_{sn}-kv_{sn})+\sqrt{\left|\underline{\rho}-j\underline{u}_{\rho n}-k\underline{v}_{\rho n}\right|^2+(s-s_n/2-ju_{sn}-kv_{sn})^2}}{(s+s_n/2-ju_{sn}-kv_{sn})+\sqrt{\left|\underline{\rho}-j\underline{u}_{\rho n}-k\underline{v}_{\rho n}\right|^2+(s+s_n/2-ju_{sn}-kv_{sn})^2}}\right]$$

$$= \sum_{n=1}^{N}\frac{1}{4\pi\varepsilon}\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}$$

$$\sum_{m=0}^{M}\underline{p}^{(0)}...\underline{p}^{(m)}\cdot\nabla_t^m\ln\left[\frac{(s-s_j/2-ju_{sj}-kv_{sj})+\left|\underline{r}-\underline{r}_j^{+}-j\underline{u}-k\underline{v}\right|}{(s+s_j/2-ju_{sj}-kv_{sj})+\left|\underline{r}-\underline{r}_j^{-}-j\underline{u}-k\underline{v}\right|}\right] \tag{352}$$

Now the final matching equation to determine the $2M$ multipole moments on each of N segments imposes the constant

$$V_{n'} = \phi = \phi_{scatt}^{tot} + \phi_{inc} \tag{353}$$

with $s = s_{n'}$ and $\underline{r} = \underline{r}_{n'}^{(m')}$ with $n' = 1, ..., N$ and $m' = 0, 1, ..., 2M-1$. Note from above, (343), (344) and (349), that the matching positions $\left(\underline{\rho}_{n'}, s_{n'}\right)$ depend on the the source point n, m as well as the observation point n', m'.

Once the potential ϕ is found, with the potential on the braid taken, say, to vanish $V_{n'} = 0$, we can then proceed to find the potential constant behaviors of interest. For $y < 0$ (the shadow side of the shield) we evaluate the total potential far from the braid to find

$$\phi \to \phi_c\ ,\ y \to -\infty \tag{354}$$

For $y > 0$ (the illuminated side of the shield) we evaluate the potential to find

$$\phi \to E_0 y + \phi_b \ , \ y \to +\infty \tag{355}$$

Normalizing by the drive field E_0 we find the desired constants ϕ_c/E_0 and ϕ_b/E_0.

3.3.1 Multipole Evaluation

The transverse derivatives of

$$\phi_n^{(0)} = -\frac{p^{(0)}}{4\pi\varepsilon} \ln \left[\frac{(s - s_n/2) + \sqrt{\rho^2 + (s - s_n/2)^2}}{(s + s_n/2) + \sqrt{\rho^2 + (s + s_n/2)^2}} \right]$$

$$= -\frac{p^{(0)}}{4\pi\varepsilon} \ln \left[\frac{(s - s_n/2) + |\underline{r} - \underline{r}_n^+|}{(s + s_n/2) + |\underline{r} - \underline{r}_n^-|} \right] \tag{356}$$

needed to obtain the line dipole moments can be taken as

$$\phi_n^{(1)} = \underline{p}^{(1)} \cdot \nabla_t \phi_n^{(0)} / p^{(0)} = -\frac{\underline{p}^{(1)} \cdot \nabla_t \rho}{4\pi\varepsilon} \left[\frac{\rho / \sqrt{\rho^2 + (s - s_n/2)^2}}{(s - s_n/2) + \sqrt{\rho^2 + (s - s_n/2)^2}} \right.$$

$$\left. - \frac{\rho / \sqrt{\rho^2 + (s + s_n/2)^2}}{(s + s_n/2) + \sqrt{\rho^2 + (s + s_n/2)^2}} \right] \tag{357}$$

and the quadrupole moment contribution

$$\phi_n^{(2)} = \underline{p}^{(1)} \underline{p}^{(2)} \cdot \nabla_t \nabla_t \phi_n^{(0)} / p^{(0)} =$$

$$-\frac{\underline{p}^{(1)} \underline{p}^{(2)} \cdot}{4\pi\varepsilon} \left[\nabla_t \nabla_t \rho \frac{\rho / \sqrt{\rho^2 + (s - s_n/2)^2}}{(s - s_n/2) + \sqrt{\rho^2 + (s - s_n/2)^2}} + (\nabla_t \rho)^2 \frac{1 / \sqrt{\rho^2 + (s - s_n/2)^2}}{(s - s_n/2) + \sqrt{\rho^2 + (s - s_n/2)^2}} \right.$$

$$- (\nabla_t \rho)^2 \frac{\rho^2 / \left\{ \sqrt{\rho^2 + (s - s_n/2)^2} \right\}^3}{(s - s_n/2) + \sqrt{\rho^2 + (s - s_n/2)^2}} - (\nabla_t \rho)^2 \frac{\rho^2 / \left\{ \sqrt{\rho^2 + (s - s_n/2)^2} \right\}^2}{\left\{ (s - s_n/2) + \sqrt{\rho^2 + (s - s_n/2)^2} \right\}^2}$$

$$+ (\nabla_t \rho)^2 \frac{\rho^2 / \left\{ \sqrt{\rho^2 + (s + s_n/2)^2} \right\}^3}{(s + s_n/2) + \sqrt{\rho^2 + (s + s_n/2)^2}} + (\nabla_t \rho)^2 \frac{\rho^2 / \left\{ \sqrt{\rho^2 + (s + s_n/2)^2} \right\}^2}{\left\{ (s + s_n/2) + \sqrt{\rho^2 + (s + s_n/2)^2} \right\}^2}$$

$$-\nabla_t\nabla_t\rho\frac{\rho/\sqrt{\rho^2+(s+s_n/2)^2}}{(s+s_n/2)+\sqrt{\rho^2+(s+s_n/2)^2}}-(\nabla_t\rho)^2\frac{1/\sqrt{\rho^2+(s+s_n/2)^2}}{(s+s_n/2)+\sqrt{\rho^2+(s+s_n/2)^2}}\Bigg] \tag{358}$$

In general we can write this as

$$\phi_n=\sum_{m=0}^{M}\phi_n^{(m)}=-\frac{1}{4\pi\varepsilon}\sum_{m=0}^{M}\underline{p}^{(0)}\underline{p}^{(1)}...\underline{p}^{(m)}\cdot\nabla_t^m\ln\left[\frac{(s-s_n/2)+\sqrt{\rho^2+(s-s_n/2)^2}}{(s+s_n/2)+\sqrt{\rho^2+(s+s_n/2)^2}}\right] \tag{359}$$

where the transverse "del" operator is

$$\nabla_t=\underline{e}_x\frac{\partial}{\partial x}+\underline{e}_y\frac{\partial}{\partial y}=\underline{e}_\rho\frac{\partial}{\partial\rho}+\underline{e}_\varphi\frac{1}{\rho}\frac{\partial}{\partial\varphi}$$

$$=\underline{e}_x\left(\cos\varphi\frac{\partial}{\partial\rho}-\sin\varphi\frac{1}{\rho}\frac{\partial}{\partial\varphi}\right)+\underline{e}_y\left(\sin\varphi\frac{\partial}{\partial\rho}+\cos\varphi\frac{1}{\rho}\frac{\partial}{\partial\varphi}\right) \tag{360}$$

The dipole term is

$$\underline{p}^{(1)}\cdot\nabla_t=p_x^{(1)}\frac{\partial}{\partial x}+p_y^{(1)}\frac{\partial}{\partial y}=p_x^{(1)}\left(\cos\varphi\frac{\partial}{\partial\rho}-\sin\varphi\frac{1}{\rho}\frac{\partial}{\partial\varphi}\right)+p_y^{(1)}\left(\sin\varphi\frac{\partial}{\partial\rho}+\cos\varphi\frac{1}{\rho}\frac{\partial}{\partial\varphi}\right) \tag{361}$$

Now if $\frac{\partial}{\partial\varphi}=0$ (as in the monopole term) then

$$\underline{p}^{(1)}\cdot\nabla_t=\left(p_x^{(1)}\cos\varphi+p_y^{(1)}\sin\varphi\right)\frac{\partial}{\partial\rho} \tag{362}$$

There are two independent terms here: $\sin\varphi$ and $\cos\varphi$; there is only a single independent function of ρ.

Next the quadrupole term is

$$\underline{p}^{(1)}\underline{p}^{(2)}\cdot\nabla_t\nabla_t=p_x^{(1)}p_x^{(2)}\frac{\partial^2}{\partial x^2}+\left[p_x^{(1)}p_y^{(2)}+p_y^{(1)}p_x^{(2)}\right]\frac{\partial^2}{\partial x\partial y}+p_y^{(1)}p_y^{(2)}\frac{\partial^2}{\partial y^2}$$

$$=p_x^{(1)}p_x^{(2)}\left(\cos\varphi\frac{\partial}{\partial\rho}-\sin\varphi\frac{1}{\rho}\frac{\partial}{\partial\varphi}\right)^2$$

$$+\left[p_x^{(1)}p_y^{(2)}+p_y^{(1)}p_x^{(2)}\right]\left(\cos\varphi\frac{\partial}{\partial\rho}-\sin\varphi\frac{1}{\rho}\frac{\partial}{\partial\varphi}\right)\left(\sin\varphi\frac{\partial}{\partial\rho}+\cos\varphi\frac{1}{\rho}\frac{\partial}{\partial\varphi}\right)$$

$$+p_y^{(1)}p_y^{(2)}\left(\sin\varphi\frac{\partial}{\partial\rho}+\cos\varphi\frac{1}{\rho}\frac{\partial}{\partial\varphi}\right)^2 \tag{363}$$

or

$$\underline{p}^{(1)}\underline{p}^{(2)}\cdot\nabla_t\nabla_t=p_x^{(1)}p_x^{(2)}\left(\cos\varphi\frac{\partial}{\partial\rho}-\sin\varphi\frac{1}{\rho}\frac{\partial}{\partial\varphi}\right)\left(\cos\varphi\frac{\partial}{\partial\rho}-\sin\varphi\frac{1}{\rho}\frac{\partial}{\partial\varphi}\right)$$

$$+\left[p_x^{(1)}p_y^{(2)}+p_y^{(1)}p_x^{(2)}\right]\left(\cos\varphi\frac{\partial}{\partial\rho}-\sin\varphi\frac{1}{\rho}\frac{\partial}{\partial\varphi}\right)\left(\sin\varphi\frac{\partial}{\partial\rho}+\cos\varphi\frac{1}{\rho}\frac{\partial}{\partial\varphi}\right)$$

$$+p_y^{(1)}p_y^{(2)}\left(\sin\varphi\frac{\partial}{\partial\rho}+\cos\varphi\frac{1}{\rho}\frac{\partial}{\partial\varphi}\right)\left(\sin\varphi\frac{\partial}{\partial\rho}+\cos\varphi\frac{1}{\rho}\frac{\partial}{\partial\varphi}\right) \tag{364}$$

Now if $\frac{\partial}{\partial\varphi}=0$

$$\underline{p}^{(1)}\underline{p}^{(2)}\cdot\nabla_t\nabla_t=p_x^{(1)}p_x^{(2)}\left(\cos^2\varphi\frac{\partial}{\partial\rho}+\sin^2\varphi\frac{1}{\rho}\right)\frac{\partial}{\partial\rho}$$

$$+\left[p_x^{(1)}p_y^{(2)}+p_y^{(1)}p_x^{(2)}\right]\cos\varphi\sin\varphi\left(\frac{\partial}{\partial\rho}-\frac{1}{\rho}\right)\frac{\partial}{\partial\rho}$$

$$+p_y^{(1)}p_y^{(2)}\left(\sin^2\varphi\frac{\partial}{\partial\rho}+\cos^2\varphi\frac{1}{\rho}\right)\frac{\partial}{\partial\rho}$$

$$=\frac{1}{2}\left[p_x^{(1)}p_x^{(2)}+p_y^{(1)}p_y^{(2)}\right]\left(\frac{\partial}{\partial\rho}+\frac{1}{\rho}\right)\frac{\partial}{\partial\rho}$$

$$+\frac{1}{2}\left[p_x^{(1)}p_x^{(2)}-p_y^{(1)}p_y^{(2)}\right]\cos(2\varphi)\left(\frac{\partial}{\partial\rho}-\frac{1}{\rho}\right)\frac{\partial}{\partial\rho}$$

$$+\frac{1}{2}\left[p_x^{(1)}p_y^{(2)}+p_x^{(2)}p_y^{(1)}\right]\sin(2\varphi)\left(\frac{\partial}{\partial\rho}-\frac{1}{\rho}\right)\frac{\partial}{\partial\rho} \tag{365}$$

There are only three independent terms here: a constant 1, $\sin 2\varphi$, and $\cos 2\varphi$; there are only two independent functions of ρ. This follows from having $\partial^2/\partial x^2$, $\partial^2/\partial y^2$, and $\partial^2/\partial x\partial y$ derivatives taken.

The octupole term in the series is

$$\underline{p}^{(1)}\underline{p}^{(2)}\underline{p}^{(3)}\cdot\nabla_t\nabla_t\nabla_t=$$

$$=p_x^{(1)}p_x^{(2)}p_x^{(3)}\frac{\partial^3}{\partial x^3}$$

$$+\left[p_x^{(1)}p_x^{(2)}p_y^{(3)}+p_x^{(1)}p_y^{(2)}p_x^{(3)}+p_y^{(1)}p_x^{(2)}p_x^{(3)}\right]\frac{\partial^3}{\partial x^2\partial y}$$

$$+\left[p_x^{(1)}p_y^{(2)}p_y^{(3)}+p_y^{(1)}p_x^{(2)}p_y^{(3)}+p_y^{(1)}p_y^{(2)}p_x^{(3)}\right]\frac{\partial^3}{\partial x\partial y^2}$$

$$+p_y^{(1)}p_y^{(2)}p_y^{(3)}\frac{\partial^3}{\partial y^3}$$

$$=p_x^{(1)}p_x^{(2)}p_x^{(3)}\left(\cos\varphi\frac{\partial}{\partial\rho}-\sin\varphi\frac{1}{\rho}\frac{\partial}{\partial\varphi}\right)^3$$

$$+\left[p_x^{(1)}p_x^{(2)}p_y^{(3)}+p_x^{(1)}p_y^{(2)}p_x^{(3)}+p_y^{(1)}p_x^{(2)}p_x^{(3)}\right]\left(\cos\varphi\frac{\partial}{\partial\rho}-\sin\varphi\frac{1}{\rho}\frac{\partial}{\partial\varphi}\right)^2\left(\sin\varphi\frac{\partial}{\partial\rho}+\cos\varphi\frac{1}{\rho}\frac{\partial}{\partial\varphi}\right)$$

$$+\left[p_x^{(1)}p_y^{(2)}p_y^{(3)}+p_y^{(1)}p_x^{(2)}p_y^{(3)}+p_y^{(1)}p_y^{(2)}p_x^{(3)}\right]\left(\cos\varphi\frac{\partial}{\partial\rho}-\sin\varphi\frac{1}{\rho}\frac{\partial}{\partial\varphi}\right)\left(\sin\varphi\frac{\partial}{\partial\rho}+\cos\varphi\frac{1}{\rho}\frac{\partial}{\partial\varphi}\right)^2$$

$$+p_y^{(1)}p_y^{(2)}p_y^{(3)}\left(\sin\varphi\frac{\partial}{\partial\rho}+\cos\varphi\frac{1}{\rho}\frac{\partial}{\partial\varphi}\right)^3$$

$$=p_x^{(1)}p_x^{(2)}p_x^{(3)}\left(\cos\varphi\frac{\partial}{\partial\rho}-\sin\varphi\frac{1}{\rho}\frac{\partial}{\partial\varphi}\right)\left(\cos^2\varphi\frac{\partial}{\partial\rho}+\sin^2\varphi\frac{1}{\rho}\right)\left(\frac{\partial}{\partial\rho}\right)$$

$$+\left[p_x^{(1)}p_x^{(2)}p_y^{(3)}+p_x^{(1)}p_y^{(2)}p_x^{(3)}+p_y^{(1)}p_x^{(2)}p_x^{(3)}\right]$$

$$\left(\cos\varphi\frac{\partial}{\partial\rho}-\sin\varphi\frac{1}{\rho}\frac{\partial}{\partial\varphi}\right)\left(\cos\varphi\sin\varphi\frac{\partial}{\partial\rho}-\sin\varphi\cos\varphi\frac{1}{\rho}\right)\left(\frac{\partial}{\partial\rho}\right)$$

$$+\left[p_x^{(1)}p_y^{(2)}p_y^{(3)}+p_y^{(1)}p_x^{(2)}p_y^{(3)}+p_y^{(1)}p_y^{(2)}p_x^{(3)}\right]$$

$$\left(\cos\varphi\frac{\partial}{\partial\rho}-\sin\varphi\frac{1}{\rho}\frac{\partial}{\partial\varphi}\right)\left(\sin^2\varphi\frac{\partial}{\partial\rho}+\cos^2\varphi\frac{1}{\rho}\right)\left(\frac{\partial}{\partial\rho}\right)$$

$$+p_y^{(1)}p_y^{(2)}p_y^{(3)}\left(\sin\varphi\frac{\partial}{\partial\rho}+\cos\varphi\frac{1}{\rho}\frac{\partial}{\partial\varphi}\right)\left(\sin^2\varphi\frac{\partial}{\partial\rho}+\cos^2\varphi\frac{1}{\rho}\right)\left(\frac{\partial}{\partial\rho}\right)$$

$$=p_x^{(1)}p_x^{(2)}p_x^{(3)}\left[\cos^3\varphi\frac{\partial^2}{\partial\rho^2}+\sin^2\varphi\cos\varphi\frac{3}{\rho}\left(\frac{\partial}{\partial\rho}-\frac{1}{\rho}\right)\right]\left(\frac{\partial}{\partial\rho}\right)$$

$$+\left[p_x^{(1)}p_x^{(2)}p_y^{(3)}+p_x^{(1)}p_y^{(2)}p_x^{(3)}+p_y^{(1)}p_x^{(2)}p_x^{(3)}\right]$$

$$\left[\cos^2\varphi\sin\varphi\left(\frac{\partial^2}{\partial\rho^2}-\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\right)-\sin\varphi\left(\cos^2\varphi-\sin^2\varphi\right)\frac{1}{\rho}\left(\frac{\partial}{\partial\rho}-\frac{1}{\rho}\right)\right]\left(\frac{\partial}{\partial\rho}\right)$$

$$+\left[p_x^{(1)}p_y^{(2)}p_y^{(3)}+p_y^{(1)}p_x^{(2)}p_y^{(3)}+p_y^{(1)}p_y^{(2)}p_x^{(3)}\right]$$

$$\left[\cos\varphi\sin^2\varphi\left(\frac{\partial^2}{\partial\rho^2}-\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\right)+\cos\varphi\left(\cos^2\varphi-\sin^2\varphi\right)\frac{1}{\rho}\left(\frac{\partial}{\partial\rho}-\frac{1}{\rho}\right)\right]\left(\frac{\partial}{\partial\rho}\right)$$

$$+p_y^{(1)}p_y^{(2)}p_y^{(3)}\left[\sin^3\varphi\frac{\partial^2}{\partial\rho^2}+\cos^2\varphi\sin\varphi\frac{3}{\rho}\left(\frac{\partial}{\partial\rho}-\frac{1}{\rho}\right)\right]\left(\frac{\partial}{\partial\rho}\right)$$

$$= \frac{1}{4} p_x^{(1)} p_x^{(2)} p_x^{(3)} \left[(3\cos\varphi + \cos 3\varphi) \frac{\partial^2}{\partial\rho^2} + (\cos\varphi - \cos 3\varphi) \frac{3}{\rho} \left(\frac{\partial}{\partial\rho} - \frac{1}{\rho} \right) \right] \left(\frac{\partial}{\partial\rho} \right)$$

$$+ \frac{1}{4} \left[p_x^{(1)} p_x^{(2)} p_y^{(3)} + p_x^{(1)} p_y^{(2)} p_x^{(3)} + p_y^{(1)} p_x^{(2)} p_x^{(3)} \right]$$

$$\left[(\sin\varphi + \sin 3\varphi) \left(\frac{\partial^2}{\partial\rho^2} - \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2} \right) - 2(-\sin\varphi + \sin 3\varphi) \frac{1}{\rho} \left(\frac{\partial}{\partial\rho} - \frac{1}{\rho} \right) \right] \left(\frac{\partial}{\partial\rho} \right)$$

$$+ \frac{1}{4} \left[p_x^{(1)} p_y^{(2)} p_y^{(3)} + p_y^{(1)} p_x^{(2)} p_y^{(3)} + p_y^{(1)} p_y^{(2)} p_x^{(3)} \right]$$

$$\left[(\cos\varphi - \cos 3\varphi) \left(\frac{\partial^2}{\partial\rho^2} - \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2} \right) + 2(\cos\varphi + \cos 3\varphi) \frac{1}{\rho} \left(\frac{\partial}{\partial\rho} - \frac{1}{\rho} \right) \right] \left(\frac{\partial}{\partial\rho} \right)$$

$$+ \frac{1}{4} p_y^{(1)} p_y^{(2)} p_y^{(3)} \left[(3\sin\varphi - \sin 3\varphi) \frac{\partial^2}{\partial\rho^2} + (\sin\varphi + \sin 3\varphi) \frac{3}{\rho} \left(\frac{\partial}{\partial\rho} - \frac{1}{\rho} \right) \right] \left(\frac{\partial}{\partial\rho} \right)$$

$$= \frac{1}{4} \left[-p_x^{(1)} p_x^{(2)} p_x^{(3)} + p_x^{(1)} p_y^{(2)} p_y^{(3)} + p_y^{(1)} p_x^{(2)} p_y^{(3)} + p_y^{(1)} p_y^{(2)} p_x^{(3)} \right]$$

$$\cos 3\varphi \left[-\frac{\partial^2}{\partial\rho^2} + \frac{3}{\rho} \left(\frac{\partial}{\partial\rho} - \frac{1}{\rho} \right) \right] \left(\frac{\partial}{\partial\rho} \right)$$

$$- \frac{1}{4} \left[p_x^{(1)} p_x^{(2)} p_y^{(3)} + p_x^{(1)} p_y^{(2)} p_x^{(3)} + p_y^{(1)} p_x^{(2)} p_x^{(3)} - p_y^{(1)} p_y^{(2)} p_y^{(3)} \right]$$

$$\sin 3\varphi \left[-\frac{\partial^2}{\partial\rho^2} + \frac{3}{\rho} \left(\frac{\partial}{\partial\rho} - \frac{1}{\rho} \right) \right] \left(\frac{\partial}{\partial\rho} \right)$$

$$+ \frac{1}{4} \left[3p_x^{(1)} p_x^{(2)} p_x^{(3)} + p_x^{(1)} p_y^{(2)} p_y^{(3)} + p_y^{(1)} p_x^{(2)} p_y^{(3)} + p_y^{(1)} p_y^{(2)} p_x^{(3)} \right]$$

$$\cos\varphi \left[\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho} \left(\frac{\partial}{\partial\rho} - \frac{1}{\rho} \right) \right] \left(\frac{\partial}{\partial\rho} \right)$$

$$+ \frac{1}{4} \left[p_x^{(1)} p_x^{(2)} p_y^{(3)} + p_x^{(1)} p_y^{(2)} p_x^{(3)} + p_y^{(1)} p_x^{(2)} p_x^{(3)} + 3p_y^{(1)} p_y^{(2)} p_y^{(3)} \right]$$

$$\sin\varphi \left[\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho} \left(\frac{\partial}{\partial\rho} - \frac{1}{\rho} \right) \right] \left(\frac{\partial}{\partial\rho} \right) \tag{366}$$

There are four independent terms here: $\sin\varphi$, $\cos\varphi$, $\sin 3\varphi$, and $\cos 3\varphi$; there are also two independent functions of ρ. This follows from from having $\partial^3/\partial x^3$, $\partial^3/\partial y^3$, $\partial^3/\partial x \partial y^2$, and $\partial^3/\partial x^2 \partial y$ derivatives taken. Note that the functions $\sin(m\varphi)$ and $\cos(m\varphi)$ have the same functional form of ρ for each value of m.

For long segments compared to their radius we could also introduce a two-dimensional redundancy between double x and double y derivatives by virtue of the two-dimensional Laplace equation. This would imply only two independent quantities for each value of m because $\partial^2/\partial x^2 = -\partial^2/\partial y^2$! One can

see from the preceding results that the constant Fourier term in the quadrupole cancels out because $p_x^{(1)}p_x^{(2)}+p_y^{(1)}p_y^{(2)}=2\widetilde{p}^{(2)}=0$ (and we can select $\left(p_x^{(1)}p_x^{(2)}-p_y^{(1)}p_y^{(2)}\right)/2=\widetilde{p}^{(0)}=\left(p_x^{(1)}p_y^{(2)}+p_x^{(2)}p_y^{(1)}\right)/2=1$). In addition the first Fourier modes in the octupole term cancel out because $3p_x^{(1)}p_x^{(2)}p_x^{(3)}+p_x^{(1)}p_y^{(2)}p_y^{(3)}+p_y^{(1)}p_x^{(2)}p_y^{(3)}+p_y^{(1)}p_y^{(2)}p_x^{(3)}=4\widetilde{p}^{(3)}=0$ and $p_x^{(1)}p_x^{(2)}p_y^{(3)}+p_x^{(1)}p_y^{(2)}p_x^{(3)}+p_y^{(1)}p_x^{(2)}p_x^{(3)}+3p_y^{(1)}p_y^{(2)}p_y^{(3)}=4\widetilde{p}^{(3)}=0$. We are then left with only the $\cos(m\varphi)$ and $\sin(m\varphi)$ terms for each m.

Radial Derivatives Because the logarithm is only a function of ρ our starting point for these terms is the radial derivative of the monopole; for example, the dipole term is

$$\underline{p}^{(1)}\cdot\nabla_t\phi_n^{(0)}/p^{(0)}=-\frac{1}{4\pi\varepsilon}\left(p_x^{(1)}\cos\varphi+p_y^{(1)}\sin\varphi\right)\frac{\partial}{\partial\rho}\ln\left[\frac{(s-s_n/2)+\sqrt{\rho^2+(s-s_n/2)^2}}{(s+s_n/2)+\sqrt{\rho^2+(s+s_n/2)^2}}\right] \tag{367}$$

In general we could collect all the $\cos(m\varphi)$ and $\sin(m\varphi)$ terms and write

$$\phi_n=-\frac{1}{4\pi\varepsilon}\sum_{m=0}^{M}\left[p_e^{(m)}\cos(m\varphi)+p_o^{(m)}\sin(m\varphi)\right]L_m\left(\rho,\frac{\partial}{\partial\rho}\right)\ln\left[\frac{(s-s_n/2)+\sqrt{\rho^2+(s-s_n/2)^2}}{(s+s_n/2)+\sqrt{\rho^2+(s+s_n/2)^2}}\right] \tag{368}$$

where the operator L_m includes all the preceding differential forms in ρ (in general this operator also involves arbitrary constants). For $M=3$

$$L_0=\widetilde{p}^{(2)}\left(\frac{\partial}{\partial\rho}+\frac{1}{\rho}\right)\frac{\partial}{\partial\rho}+\widetilde{p}^{(0)}\to 1$$

$$L_1=\left[\widetilde{p}^{(3)}\left\{\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\left(\frac{\partial}{\partial\rho}-\frac{1}{\rho}\right)\right\}+\widetilde{p}^{(1)}\right]\left(\frac{\partial}{\partial\rho}\right)\to\left(\frac{\partial}{\partial\rho}\right)$$

$$L_2=\left(\frac{\partial}{\partial\rho}-\frac{1}{\rho}\right)\frac{\partial}{\partial\rho}$$

$$L_3=\left[\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\left(\frac{\partial}{\partial\rho}-\frac{1}{\rho}\right)\right]\left(\frac{\partial}{\partial\rho}\right)$$

where the constants $\widetilde{p}^{(2)}/\widetilde{p}^{(0)}$ and $\widetilde{p}^{(3)}/\widetilde{p}^{(1)}$ can be chosen arbitrarily (for a selected segment with $\rho=a$ these extra behaviors can be absorbed in the arbitrary coefficients $p_e^{(m)}$ and $p_o^{(m)}$ and hence do not change the variation around the selected segment, which should form a complete set of Fourier modes, but they would effect the levels produced by the selected segment at other segment locations). We take them to vanish based on the two dimensional Laplacian relation between transverse derivatives (other choices could be also made but this is a simple one). Note that there are $2M$ unknown amplitudes for the multipole coefficients; note that there are $M+1$ of the $p_e^{(m)}$ unknowns but $M-1$ of the $p_o^{(m)}$ unknowns (the $\sin(m\varphi_{m'})$ term with $\varphi_{m'}=m'\pi/M$ does not contribute for $m'=0$ and $m'=M$) which must be determined from point matching at $m'=0,...,2M-1$ or $2M$ points around the circumference of each of the N braid wire segments.

Derivatives of the logarithm are: to first order

$$\frac{\partial}{\partial\rho}\ln\left[\frac{(s-s_n/2)+\sqrt{\rho^2+(s-s_n/2)^2}}{(s+s_n/2)+\sqrt{\rho^2+(s+s_n/2)^2}}\right]$$

$$=\frac{\rho/\sqrt{\rho^2+(s-s_n/2)^2}}{(s-s_n/2)+\sqrt{\rho^2+(s-s_n/2)^2}}-\frac{\rho/\sqrt{\rho^2+(s+s_n/2)^2}}{(s+s_n/2)+\sqrt{\rho^2+(s+s_n/2)^2}} \tag{369}$$

to second order

$$\frac{\partial^2}{\partial\rho^2}\ln\left[\frac{(s-s_n/2)+\sqrt{\rho^2+(s-s_n/2)^2}}{(s+s_n/2)+\sqrt{\rho^2+(s+s_n/2)^2}}\right]$$

$$=\frac{1/\sqrt{\rho^2+(s-s_n/2)^2}}{(s-s_n/2)+\sqrt{\rho^2+(s-s_n/2)^2}}$$

$$-\frac{\rho^2/\left\{\sqrt{\rho^2+(s-s_n/2)^2}\right\}^3}{(s-s_n/2)+\sqrt{\rho^2+(s-s_n/2)^2}}-\frac{\rho^2/\left\{\sqrt{\rho^2+(s-s_n/2)^2}\right\}^2}{\left\{(s-s_n/2)+\sqrt{\rho^2+(s-s_n/2)^2}\right\}^2}$$

$$-\frac{1/\sqrt{\rho^2+(s+s_n/2)^2}}{(s+s_n/2)+\sqrt{\rho^2+(s+s_n/2)^2}}$$

$$+\frac{\rho^2/\left\{\sqrt{\rho^2+(s+s_n/2)^2}\right\}^3}{(s+s_n/2)+\sqrt{\rho^2+(s+s_n/2)^2}}+\frac{\rho^2/\left\{\sqrt{\rho^2+(s+s_n/2)^2}\right\}^2}{\left\{(s+s_n/2)+\sqrt{\rho^2+(s+s_n/2)^2}\right\}^2} \tag{370}$$

and to third order

$$\frac{\partial^3}{\partial\rho^3}\ln\left[\frac{(s-s_n/2)+\sqrt{\rho^2+(s-s_n/2)^2}}{(s+s_n/2)+\sqrt{\rho^2+(s+s_n/2)^2}}\right]$$

$$=-\frac{\rho/\left\{\sqrt{\rho^2+(s-s_n/2)^2}\right\}^3}{(s-s_n/2)+\sqrt{\rho^2+(s-s_n/2)^2}}-\frac{\rho/\left\{\sqrt{\rho^2+(s-s_n/2)^2}\right\}^2}{\left\{(s-s_n/2)+\sqrt{\rho^2+(s-s_n/2)^2}\right\}^2}$$

$$-\frac{2\rho/\left\{\sqrt{\rho^2+(s-s_n/2)^2}\right\}^3}{(s-s_n/2)+\sqrt{\rho^2+(s-s_n/2)^2}}+3\frac{\rho^3/\left\{\sqrt{\rho^2+(s-s_n/2)^2}\right\}^5}{(s-s_n/2)+\sqrt{\rho^2+(s-s_n/2)^2}}+\frac{\rho^3/\left\{\sqrt{\rho^2+(s-s_n/2)^2}\right\}^4}{\left\{(s-s_n/2)+\sqrt{\rho^2+(s-s_n/2)^2}\right\}^2}$$

$$-\frac{2\rho/\left\{\sqrt{\rho^2+(s-s_n/2)^2}\right\}^2}{\left\{(s-s_n/2)+\sqrt{\rho^2+(s-s_n/2)^2}\right\}^2}+2\frac{\rho^3/\left\{\sqrt{\rho^2+(s-s_n/2)^2}\right\}^4}{\left\{(s-s_n/2)+\sqrt{\rho^2+(s-s_n/2)^2}\right\}^2}+2\frac{\rho^3/\left\{\sqrt{\rho^2+(s-s_n/2)^2}\right\}^3}{\left\{(s-s_n/2)+\sqrt{\rho^2+(s-s_n/2)^2}\right\}^3}$$

$$+\frac{\rho/\left\{\sqrt{\rho^2+(s+s_n/2)^2}\right\}^3}{(s+s_n/2)+\sqrt{\rho^2+(s+s_n/2)^2}}+\frac{\rho/\left\{\sqrt{\rho^2+(s+s_n/2)^2}\right\}^2}{\left\{(s+s_n/2)+\sqrt{\rho^2+(s+s_n/2)^2}\right\}^2}$$

$$+\frac{2\rho/\left\{\sqrt{\rho^2+(s+s_n/2)^2}\right\}^3}{(s+s_n/2)+\sqrt{\rho^2+(s+s_n/2)^2}}-3\frac{\rho^3/\left\{\sqrt{\rho^2+(s+s_n/2)^2}\right\}^5}{(s+s_n/2)+\sqrt{\rho^2+(s+s_n/2)^2}}-\frac{\rho^3/\left\{\sqrt{\rho^2+(s+s_n/2)^2}\right\}^4}{\left\{(s+s_n/2)+\sqrt{\rho^2+(s+s_n/2)^2}\right\}^2}$$

$$+\frac{2\rho/\left\{\sqrt{\rho^2+(s+s_n/2)^2}\right\}^2}{\left\{(s+s_n/2)+\sqrt{\rho^2+(s+s_n/2)^2}\right\}^2}-2\frac{\rho^3/\left\{\sqrt{\rho^2+(s+s_n/2)^2}\right\}^4}{\left\{(s+s_n/2)+\sqrt{\rho^2+(s+s_n/2)^2}\right\}^2}-2\frac{\rho^3/\left\{\sqrt{\rho^2+(s+s_n/2)^2}\right\}^3}{\left\{(s+s_n/2)+\sqrt{\rho^2+(s+s_n/2)^2}\right\}^3}$$

$$=-\frac{3\rho\,(s-s_n/2)^2\,/\left\{\sqrt{\rho^2+(s-s_n/2)^2}\right\}^5}{(s-s_n/2)+\sqrt{\rho^2+(s-s_n/2)^2}}-\frac{3\rho\,(s-s_n/2)^2\,/\left\{\sqrt{\rho^2+(s-s_n/2)^2}\right\}^4}{\left\{(s-s_n/2)+\sqrt{\rho^2+(s-s_n/2)^2}\right\}^2}$$

$$+\frac{2\rho^3/\left\{\sqrt{\rho^2+(s-s_n/2)^2}\right\}^3}{\left\{(s-s_n/2)+\sqrt{\rho^2+(s-s_n/2)^2}\right\}^3}$$

$$+\frac{3\rho\,(s+s_n/2)^2\,/\left\{\sqrt{\rho^2+(s+s_n/2)^2}\right\}^5}{(s+s_n/2)+\sqrt{\rho^2+(s+s_n/2)^2}}+\frac{3\rho\,(s+s_n/2)^2\,/\left\{\sqrt{\rho^2+(s+s_n/2)^2}\right\}^4}{\left\{(s+s_n/2)+\sqrt{\rho^2+(s+s_n/2)^2}\right\}^2}$$

$$-\frac{2\rho^3/\left\{\sqrt{\rho^2+(s+s_n/2)^2}\right\}^3}{\left\{(s+s_n/2)+\sqrt{\rho^2+(s+s_n/2)^2}\right\}^3} \tag{371}$$

3.4 Electric Multipoles With Dielectric Materials

Now we consider the electric problem when dielectric materials are present. With the approximate local planar model we take two dielectric half spaces (the exterior 2 half space could also be truncated into a finite thickness layer to model the outer jacket) about the braid as shown in Figure 16. Because we are representing the braid wires by line multipole moment segments, these charge multipoles can be imaged in the dielectric interfaces. We first decompose the total field into a uniform electric displacement in the y direction (where ε is equal to ε_j with $j = 0, 1, 2$ depending on which region the observation is made)

$$D_0 = \varepsilon E_0 \tag{372}$$

in addition to a field generated by the multipolar charges. These charges can be imaged in the dielectric interfaces to represent the potential in the various regions. Let us consider a charge q in the center region at $y = 0$ and the interfaces are at $y = h_2$ and $y = -h_1$ with respect to this charge position. The incident uniform field potential can be written as

$$\phi_{inc} = yD_0/\varepsilon_0 \ , \ \ -h_1 < y < h_2$$

$$= (y - h_2)\, D_0/\varepsilon_2 + h_2 D_0/\varepsilon_0 \ , \ y > h_2$$

$$= (y + h_1)\, D_0/\varepsilon_1 - h_1 D_0/\varepsilon_0 \ , \ y < -h_1 \tag{373}$$

If driven from above (transfer capacitance problem) we would have $D_0 = \varepsilon_2 E_0$ whereas when driven from below (self capacitance of inner coax problem) we would have $D_0 = \varepsilon_1 E_0$.

When viewed from the central braid region the potential due to this charge can be written as

$$\left(\frac{4\pi\varepsilon_0}{q}\right)\phi = \frac{1}{\sqrt{x^2+y^2+z^2}}$$

$$+\frac{\varepsilon_0-\varepsilon_2}{\varepsilon_0+\varepsilon_2}\frac{1}{\sqrt{x^2+(y-2h_2)^2+z^2}} + \frac{\varepsilon_0-\varepsilon_1}{\varepsilon_0+\varepsilon_1}\frac{1}{\sqrt{x^2+(y+2h_1)^2+z^2}}$$

$$+\frac{\varepsilon_0-\varepsilon_2}{\varepsilon_0+\varepsilon_2}\frac{\varepsilon_0-\varepsilon_1}{\varepsilon_0+\varepsilon_1}\frac{1}{\sqrt{x^2+(y+2h_2+2h_1)^2+z^2}} + \frac{\varepsilon_0-\varepsilon_1}{\varepsilon_0+\varepsilon_1}\frac{\varepsilon_0-\varepsilon_2}{\varepsilon_0+\varepsilon_2}\frac{1}{\sqrt{x^2+(y-2h_1-2h_2)^2+z^2}}$$

$$+\left(\frac{\varepsilon_0-\varepsilon_2}{\varepsilon_0+\varepsilon_2}\right)^2\frac{\varepsilon_0-\varepsilon_1}{\varepsilon_0+\varepsilon_1}\frac{1}{\sqrt{x^2+(y-4h_2-2h_1)^2+z^2}} + \left(\frac{\varepsilon_0-\varepsilon_1}{\varepsilon_0+\varepsilon_1}\right)^2\frac{\varepsilon_0-\varepsilon_2}{\varepsilon_0+\varepsilon_2}\frac{1}{\sqrt{x^2+(y+4h_1+2h_2)^2+z^2}}$$

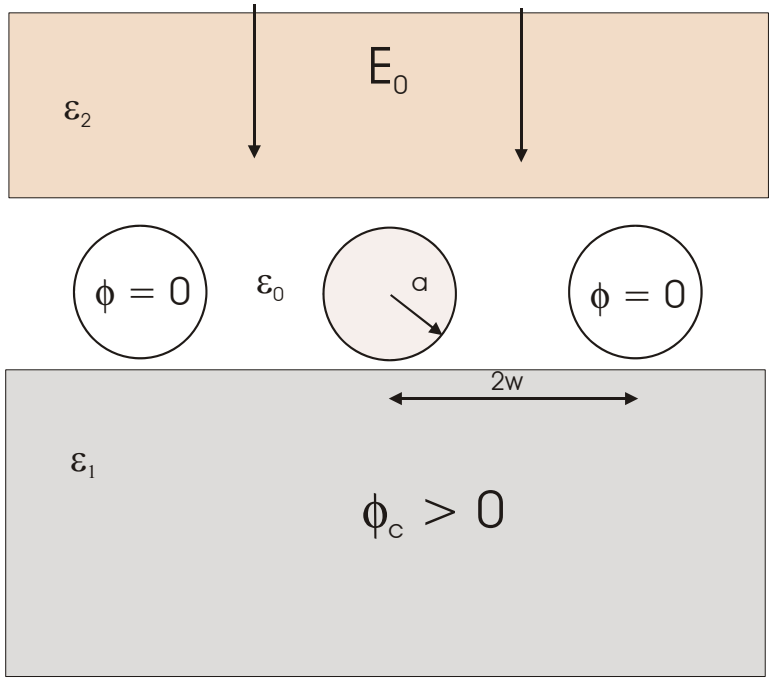

Figure 16. Dielectric materials surround the planar braid layer.

$$+\left(\frac{\varepsilon_0-\varepsilon_2}{\varepsilon_0+\varepsilon_2}\right)^2\left(\frac{\varepsilon_0-\varepsilon_1}{\varepsilon_0+\varepsilon_1}\right)^2\frac{1}{\sqrt{x^2+(y+4h_2+4h_1)^2+z^2}}+\left(\frac{\varepsilon_0-\varepsilon_1}{\varepsilon_0+\varepsilon_1}\right)^2\left(\frac{\varepsilon_0-\varepsilon_2}{\varepsilon_0+\varepsilon_2}\right)^2\frac{1}{\sqrt{x^2+(y-4h_1-4h_2)^2+z^2}}$$

$$+\left(\frac{\varepsilon_0-\varepsilon_2}{\varepsilon_0+\varepsilon_2}\right)^3\left(\frac{\varepsilon_0-\varepsilon_1}{\varepsilon_0+\varepsilon_1}\right)^2\frac{1}{\sqrt{x^2+(y-6h_2-4h_1)^2+z^2}}+\left(\frac{\varepsilon_0-\varepsilon_1}{\varepsilon_0+\varepsilon_1}\right)^3\left(\frac{\varepsilon_0-\varepsilon_2}{\varepsilon_0+\varepsilon_2}\right)^2\frac{1}{\sqrt{x^2+(y+6h_1+4h_2)^2+z^2}}$$

$$+\cdots \tag{374}$$

Notice if there is no outer dielectric jacket material $\varepsilon_2=\varepsilon_0$ we end up with only two terms (the source and one image in ε_1).

This same structure of images applies to the multipolar charges of the braid wires (the image placement is the same but the rate of decay increases with the order of the multipole). Hence we can modify the preceding potential distributions to include the image charges in order to construct the potential near the braid wires with the dielectric interfaces present. We can further approximate the sum of images generated by the multipole dielectric interfaces since the higher-order multipole moments have increasing rates of fall-off with distance. Such an image potential construction then allows us to match the equipotential boundary condition on the braid wires to determine the appropriate multipole moments in the presence of adjacent locally planar dielectric regions.

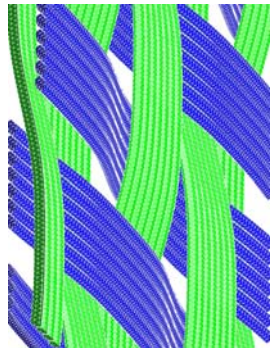

Figure 17. A planar rendition of a braid (in this case a numerical mesh representation is shown) highlighting the individual wires making up the strip carriers.

4 MAGNETOSTATIC COUPLING FORMULATION

The goal of this section is to formulate the transfer impedance per unit length Z_T of the braid penetration as well as the inner coaxial impedance per unit length Z_1. In this section we begin with the simplification of a perfectly conducting braid where these impedances are inductive reactances. The self impedance is then

$$Z_1 = -i\omega L_1 \tag{375}$$

and the transfer impedance is

$$Z_T = -i\omega L_T \tag{376}$$

We carry out the formulation first using an energy argument and Faraday's law. We then use reciprocity to approximately treat arbitrary coaxial geometries, first with the magnetic scalar potential and then with the magnetic vector potential to examine advantages and disadvantages associated with the two representations. A typical example of a planar approximation of a braid is shown in Figure 17. We expect some form of symmetry with the electric problem but there may be new effects due to the known contributions of the braid interweave (and in the next section the diffusion resulting from the finite conductivity of the wires).

4.1 Magnetic Flux Boundary Conditions & Braid Wire Currents

We discussed issues with the electric potential boundary conditions associated with the collection of braid wires in a preceding section. Now we want to examine these issues in the magnetic problem. The currents on the braid wires within a strip carrier could be taken to be fixed along the wires if they are insulated from each other, or they could be allowed to vary due to contact between wires if the contact impedance were known. In the perfectly conducting case, contact between wires in a carrier strip would mean that the electric field or magnetic flux between wires vanishes and thus the net magnetic flux between wires in a carrier strip vanishes. (In the finitely conducting case the wire internal impedance also plays a role in selecting the current distribution of the wires in a carrier strip to prevent the combination of inductive and resistive voltage drops between wires in the carrier strip.) The magnetic flux is defined by

$$\Phi = \int_S \underline{B} \cdot \underline{n} dS = \oint_C \underline{A} \cdot \underline{d\ell} \tag{377}$$

where the magnetic induction is found from the vector potential as

$$\underline{B} = \nabla \times \underline{A} \tag{378}$$

In addition, we could ask about the total current on a carrier strip. Incidental contact between carrier strips could allow this total current to vary. However, this contact may involve a substantial impedance, and periodicity between strips in a cylindrically symmetric geometry might indicate that there is no preferred carrier. If the current is confined to the carriers, then the total wire current in a strip carrier is fixed, and for the cylindrically symmetric case would be taken to be the total braid current divided by the number of carriers; the individual wire currents could then be taken to conform with an arrangement that is dictated by no potential difference between wires in a carrier (at least over a braid period). Alternatively, if there is no contact between wires in a carrier, we could use the same current division between wires to assure no net magnetic flux over a period to prevent accumulation of voltage differences between individual wires along the cable, assuming they are connected together at the drive point. If we take the contour (377) to be along the surface of two of the braid wires over an axial period ℓ we can use periodicity to eliminate the ends of the contour and equate the integral of the vector potential along the braid wires

$$\int_{C_{pj}} \underline{A} \cdot \underline{d\ell} = \text{constant for the braid wires} \tag{379}$$

where the contour C_{pj} extends over an axial period on the surface of the jth braid wire. Because the currents are assumed to be periodic over a short axial distance ℓ we construct the vector potential $\underline{A}$ to be periodic also.

4.2 Inductance Per Unit Length Of Perfectly Conducting Braid

The self inductance can found from the stored magnetic energy by means of

$$\frac{1}{2}LI^2 = \int_V \frac{1}{2}\mu_0 H^2 dV \tag{380}$$

Taking the constitutive relation to be that of free space

$$\underline{B} = \mu_0 \underline{H} \tag{381}$$

where μ_0 is the magnetic permeability of free space, we find

$$\frac{1}{2}LI^2 = \frac{1}{2}\int_V (\nabla \times \underline{A}) \cdot \underline{H} dV \tag{382}$$

Now using the identity

$$\nabla \cdot (\underline{A} \times \underline{H}) = (\nabla \times \underline{A}) \cdot \underline{H} - \underline{A} \cdot (\nabla \times \underline{H}) \tag{383}$$

then by means of Ampere's law

$$\nabla \times \underline{H} = \underline{J} \tag{384}$$

we have

$$(\nabla \times \underline{A}) \cdot \underline{H} = \underline{A} \cdot \underline{J} + \nabla \cdot (\underline{A} \times \underline{H}) \tag{385}$$

Using the divergence theorem

$$LI^2 = \int_V \underline{A} \cdot \underline{J} dV + \oint_S (\underline{A} \times \underline{H}) \cdot \underline{n} dS \tag{386}$$

or

$$LI^2 = \int_V \underline{A} \cdot \underline{J} dV - \oint_S \underline{A} \cdot (\underline{n} \times \underline{H}) \, dS \tag{387}$$

If we take the volume V to consist of the region surrounding the center conductor, excluding the braid wires, and including the outer space to the return chassis, then the volume integral vanishes since no current is present in the volume. The surface integral can be broken into the various parts

$$LI^2 = -\int_{S_c} \underline{A} \cdot (\underline{n} \times \underline{H}) \, dS - \int_{S_w} \underline{A} \cdot (\underline{n} \times \underline{H}) \, dS - \int_{S_{chassis}} \underline{A} \cdot (\underline{n} \times \underline{H}) \, dS \tag{388}$$

where we again drop the end surfaces when this is applied over a short axial period ℓ; because the currents and fields are periodic over a short axial distance ℓ we assume that the vector potential $\underline{A}$ is constructed to be periodic also. If we assume that $\underline{n} \times \underline{H}$ is predominantly z directed on the perfect center conductor and that A_z is approximately independent of position around the perimeter over a short axial distance, then

$$LI^2 = I_1 A_z (S_c) - \int_{S_w} \underline{A} \cdot (\underline{n} \times \underline{H}) \, dS - \int_{S_{chassis}} \underline{A} \cdot (\underline{n} \times \underline{H}) \, dS \tag{389}$$

The electric field is

$$\underline{E} = -\nabla \phi + i\omega \underline{A} \tag{390}$$

with axial component

$$E_z = -\frac{\partial \phi}{\partial z} + i\omega A_z = -i\Gamma\phi + i\omega A_z \tag{391}$$

where the propagation constant of the interior cable transmission line

$$\Gamma = \omega\sqrt{L_1 C_1} \tag{392}$$

is assumed to have a small value such that $\Gamma b << 1$. Note that on the center conductor, taking $\phi = V_1$, over a short axial distance we have

$$0 = E_z = -i\Gamma V_1 + i\omega A_z (S_c) \tag{393}$$

4.2.1 Self Inductance Per Unit Length

For the moment let us take the outer loop to be open circuited, and assume the magnetic field near the chassis is negligible, so that we can drop the final integral (the current I in this case is identified as the current on the center conductor I_1)

$$\ell L_1 I_1^2 = \ell I_1 A_z\left(S_c\right) - \int_{S_w} \underline{A} \cdot \left(\underline{n} \times \underline{H}\right) dS \tag{394}$$

On the surface of the braid wires we note that the surface current density $\underline{K}$ satisfies (note here that the unit normal $\underline{n}$ points into the conductor)

$$\underline{K} = -\underline{n} \times \underline{H} \tag{395}$$

and then

$$L_1 I_1^2 = I_1 A_z\left(S_c\right) + \int_{S_w} \underline{A} \cdot \underline{K} dS/\ell \tag{396}$$

Now in the cylindrical case by symmetry we assume that the contribution from each of the N_s strip carriers is the same (only true on cylindrical conductor geometry, not on asymmetric geometries)

$$L_1 I_1^2 = I_1 A_z\left(S_c\right) + N_s \int_{S_s} \underline{A} \cdot \underline{K} dS/\ell \tag{397}$$

or

$$L_1 I_1 = A_z\left(S_c\right) + \int_{S_s} \underline{A} \cdot \left(\underline{K} \frac{N_s}{I_1}\right) dS/\ell \tag{398}$$

These results, (396) and (398), define the inductance per unit length L_1 in terms of the vector potential solution of the coax. The surface integral is over the wire conductors in the braid carrier strips for an axial braid period. In the perfectly conducting case the normal component of the magnetic induction vanishes on the surface of the braid wires

$$\underline{n} \cdot \underline{B} = 0 = \underline{n} \cdot \underline{H} \tag{399}$$

and hence the normal magnetic flux on any wire surface S_w' vanishes

$$\Phi_w = \int_{S_w'} \underline{B} \cdot \underline{n} dS = \oint_{C_w'} \underline{A} \cdot \underline{d\ell} = 0 \tag{400}$$

This means that contour integrals on the conductor surface are independent of the integration path and only depend on the end points because we can always add the vanishing contour integral to change to another path. Furthermore, on the braid wire surfaces we take the scalar potential to vanish, and the tangential components of the electric field to vanish also

$$0 = \underline{n} \times \underline{E} = i\omega \underline{n} \times \underline{A} \tag{401}$$

Hence the tangential components of the vector potential vanish and (396) becomes

$$L_1 I_1 = A_z\left(S_c\right) \tag{402}$$

Thus the self inductance is determined from the magnetic flux passing between the center conductor and the braid wires. If the scalar potential is selected not to vanish on the braid then neither does the tangential vector potential and the inductance reverts back to (396).

4.2.2 Transfer Inductance Per Unit Length

The magnetic flux current relations in a two port can be written as

$$\Phi_1 = L_{11} I_1 + M_{12} I_2 \tag{403}$$

$$\Phi_2 = M_{21} I_1 + L_{22} I_2 \tag{404}$$

From Faraday's law

$$\nabla \times \underline{E} = -\frac{\partial}{\partial t} \underline{B} \tag{405}$$

or in integral form

$$\oint_C \underline{E} \cdot \underline{d\ell} = -\int_S \frac{\partial}{\partial t} \underline{B} dS \tag{406}$$

we can write

$$\frac{\partial V}{\partial z} = -\frac{\partial \Phi}{\partial t} \tag{407}$$

The voltage current relations in a two port inductive circuit can thus be written as

$$V_1(z+\ell) - V_1(z) = -\frac{\partial \Phi_1}{\partial t} = -L_{11}\frac{dI_1}{dt} - M_{12}\frac{dI_2}{dt} \tag{408}$$

$$V_2(z+\ell) - V_2(z) = -\frac{\partial \Phi_2}{\partial t} = -M_{21}\frac{dI_1}{dt} - L_{22}\frac{dI_2}{dt} \tag{409}$$

In a reciprocal media the cross terms are equal $M_{12} = M_{21} = M$. The power removed from a periodic section is minus the derivative of the magnetic energy

$$\left[V_1(z+\ell) - V_1(z)\right] I_1 + \left[V_2(z+\ell) - V_2(z)\right] I_2 = -\frac{d}{dt}\left(\frac{1}{2}L_{11}I_1^2 + M I_1 I_2 + \frac{1}{2}L_{22}I_2^2\right) \tag{410}$$

Taking two sources we write the total magnetic field as

$$\underline{H} = \underline{H}_1 + \underline{H}_2 \tag{411}$$

The 1 problem is defined to have a current I_1 on the center conductor and z directed with return on the braid and the chassis open circuited (no current); the 2 problem is defined to have a current I_2 on the chassis and z directed with return on the braid and the center conductor open circuited (no current). Equating the energies from the circuit and field gives

$$\frac{1}{2}L_{11}I_1^2 + M I_1 I_2 + \frac{1}{2}L_{22}I_2^2 = \int_V \frac{1}{2}\mu_0 H^2 dV$$

$$= \int_V \frac{1}{2}\mu_0 H_1^2 dV + \int_V \mu_0 \underline{H}_1 \cdot \underline{H}_2 dV + \int_V \frac{1}{2}\mu_0 H_2^2 dV \tag{412}$$

The self inductances $L_{11} = \ell L_1$ and $L_{22} = \ell L_{sh}$ (where L_{sh} is the inductance per unit length in the outer transmission line) are the first and final terms

$$L_{11} I_1^2 = \ell L_1 I_1^2 = \int_V \frac{1}{2}\mu_0 H_1^2 dV \tag{413}$$

$$L_{22} I_2^2 = \ell L_{sh} I_2^2 = \int_V \frac{1}{2}\mu_0 H_2^2 dV \tag{414}$$

The mutual inductance M is the cross term

$$M = \frac{1}{I_1 I_2} \int_V \mu_0 \underline{H}_1 \cdot \underline{H}_2 dV \tag{415}$$

We intend to take conductor 1 as the center conductor and conductor 2 as the chassis. The sign of the mutual inductance in the preceding expressions depends on whether the magnetic flux generated in circuit 1 by the current I_2 has the same sign as the flux generated by current I_1 in circuit 1; referring to Figure 18 we see that with $I_2 = -I_{sh}$ positive the field $\underline{H}_2 = \underline{H}_{sh}$ will reinforce $\underline{H}_1$ and the sign of M is positive. However, setting the interior to be open circuited and taking the current $I_2 = I_{chassis} = -I_{sh}$ to be negative (where I_{sh} is the positive z directed braid current) with $\Phi_2 = -\Phi_{sh}$, defining the transfer inductance by means of $M = L_T \ell$, and setting $L_{11} = \ell L_1$ and $L_{22} = \ell L_{sh}$, we find

$$\Phi_1 = L_{11} I_1 - M I_{sh} = \ell L_1 I_1 - L_T \ell I_{sh} \tag{416}$$

$$\Phi_{sh} = -M I_1 + L_{22} I_{sh} = -L_T \ell I_1 + \ell L_{sh} I_{sh} \tag{417}$$

The voltage along the inner transmission line from Faraday's law applied over a period is found by taking $I_1 = 0$ in (416)

$$V_1(\ell + z) - V_1(z) = v_s \ell = -\frac{\partial}{\partial t}\Phi_1 = i\omega \Phi_1 \tag{418}$$

$$v_s = -i\omega L_T I_{sh} = Z_T I_{sh} \tag{419}$$

and the transfer inductance per unit length, replacing I_2 by $-I_{sh}$ and $\underline{H}_2$ by $\underline{H}_{sh}$ (also in the magnetic problem we do not reverse the sign of the field even though we have reversed the sign of the current relative to the 2 problem) is

$$L_T = -\frac{1}{\ell I_1 I_{sh}} \int_V \mu_0 \underline{H}_1 \cdot \underline{H}_{sh} dV \tag{420}$$

Now as shown in Figure 18 if we suppose the braid has a large aperture we see that the penetrant magnetic field $\underline{H}_{sh}$, for positive shield current I_{sh}, produces a magnetic flux around the center conductor of the inner line with the opposite sign as that produced by the center conductor current I_1 and thus the minus sign in (416) introduced in the mutual magnetic flux to make L_T positive for low optical coverage cables (the negative porpoising contribution will result in negative values of L_T for high optical coverage cables); the sign of the voltage source in (418) and (419) is positive for increasing current I_{sh}. The positive reference of this source voltage is on the positive z side of the elements since the inner transmission line

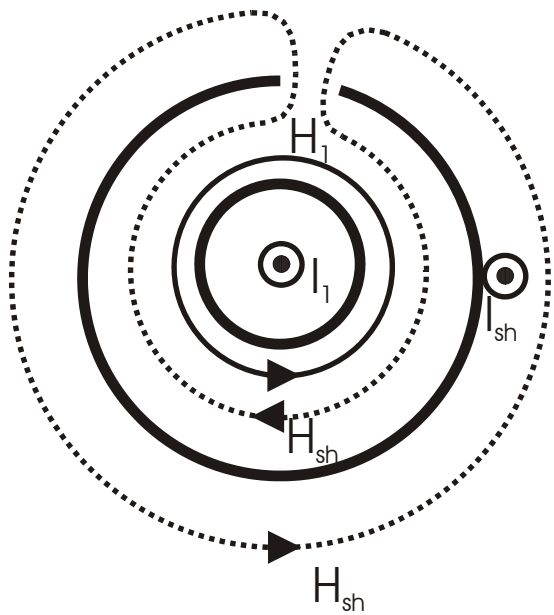

Figure 18. Field H_1 generated by I_1 and field H_{sh} generated by I_{sh}.

equation is written as

$$\frac{dV_1}{dz} + Z_1 I_1 = v_s \tag{421}$$

Taking the constitutive relation to be that of free space

$$\underline{B} = \mu_0 \underline{H} \tag{422}$$

and using the vector potential

$$\underline{B} = \nabla \times \underline{A} \tag{423}$$

gives

$$L_T = \frac{1}{\ell I_1 I_2} \int_V (\nabla \times \underline{A}_1) \cdot \underline{H}_2 dV \tag{424}$$

or

$$L_T = \frac{1}{\ell I_1 I_2} \int_V \underline{H}_1 \cdot (\nabla \times \underline{A}_2)\, dV \tag{425}$$

Now using the identity

$$\nabla \cdot (\underline{A}_1 \times \underline{H}_2) = (\nabla \times \underline{A}_1) \cdot \underline{H}_2 - \underline{A}_1 \cdot (\nabla \times \underline{H}_2) \tag{426}$$

or

$$\nabla \cdot (\underline{A}_2 \times \underline{H}_1) = \underline{H}_1 \cdot (\nabla \times \underline{A}_2) - \underline{A}_2 \cdot (\nabla \times \underline{H}_1)$$

and by means of Ampere's law

$$\nabla \times \underline{H} = \underline{J} \tag{427}$$

we can write

$$\nabla \cdot (\underline{A}_1 \times \underline{H}_2) = (\nabla \times \underline{A}_1) \cdot \underline{H}_2 - \underline{A}_1 \cdot \underline{J}_2 \tag{428}$$

or

$$\nabla \cdot (\underline{A}_2 \times \underline{H}_1) = \underline{H}_1 \cdot (\nabla \times \underline{A}_2) - \underline{A}_2 \cdot \underline{J}_1 \tag{429}$$

and

$$L_T = \frac{1}{\ell I_1 I_2} \oint_S (\underline{A}_1 \times \underline{H}_2) \cdot \underline{n} dS + \frac{1}{\ell I_1 I_2} \int_V \underline{A}_1 \cdot \underline{J}_2 dV \tag{430}$$

$$= -\frac{1}{\ell I_1 I_2} \oint_S \underline{A}_1 \cdot (\underline{n} \times \underline{H}_2)\, dS + \frac{1}{\ell I_1 I_2} \int_V \underline{A}_1 \cdot \underline{J}_2 dV$$

or

$$L_T = \frac{1}{\ell I_1 I_2} \oint_S (\underline{A}_2 \times \underline{H}_1) \cdot \underline{n} dS + \frac{1}{\ell I_1 I_2} \int_V \underline{A}_2 \cdot \underline{J}_1 dV \tag{431}$$

$$= -\frac{1}{\ell I_1 I_2} \oint_S \underline{A}_2 \cdot (\underline{n} \times \underline{H}_1) \, dS + \frac{1}{\ell I_1 I_2} \int_V \underline{A}_2 \cdot \underline{J}_1 dV \tag{432}$$

In the perfectly conducting case let us take the volume outside the conductors and then the volume integral vanishes

$$L_T = -\frac{1}{\ell I_1 I_2} \oint_S \underline{A}_1 \cdot (\underline{n} \times \underline{H}_2) \, dS \tag{433}$$

or

$$L_T = -\frac{1}{\ell I_1 I_2} \oint_S \underline{A}_2 \cdot (\underline{n} \times \underline{H}_1) \, dS \tag{434}$$

Let us use the second expression and set $I_2 = -I_{sh}$ and $\underline{H}_2 = \underline{H}_{sh}$ with $\underline{A}_2 = \underline{A}_{sh}$ to give

$$L_T = -\frac{1}{\ell I_1 I_{sh}} \int_V \mu_0 \underline{H}_1 \cdot \underline{H}_{sh} dV = \frac{1}{\ell I_1 I_{sh}} \oint_S \underline{A}_{sh} \cdot (\underline{n} \times \underline{H}_1) \, dS$$

$$= -\frac{1}{\ell I_1 I_{sh}} \int_{S_c} \underline{A}_{sh} \cdot \underline{K}_1 dS - \frac{1}{\ell I_1 I_{sh}} \int_{S_w} \underline{A}_{sh} \cdot \underline{K}_1 dS \tag{435}$$

where periodicity again eliminates the end surfaces and the surface integral on the chassis is not present with the choice (434) because $\underline{A}_{sh} \sim A_{shz} \underline{e}_z$, with A_{shz} asymptotically constant on this perfect conductor (since there is no normal magnetic field), and there is no net current on the chassis in the 1 problem (the chassis integral would be present if we had selected the first integral form (433), in which case the center conductor integral would be absent). On the center conductor S_c we can approximate the integral and find

$$L_T \approx -\frac{1}{I_{sh}} A_{shz} (S_c) - \frac{1}{I_1 I_{sh}} \int_{S_w} \underline{A}_{sh} \cdot \underline{K}_1 dS / \ell \tag{436}$$

The current $\underline{K}_1$ is negative z directed on the braid wires. Hence the final term is the appropriate subtraction (relative to the first term) of the potential $\underline{A}_{sh}$ on the braid wires weighted by the current density on the braid wires with a distribution appropriate to the interior coaxial mode.

Again on the braid wire surfaces we take the scalar potential to vanish, and the tangential components of the electric field to vanish also

$$0 = \underline{n} \times \underline{E} = i\omega \underline{n} \times \underline{A} \tag{437}$$

Hence the tangential components of the vector potential vanish and (436) becomes

$$L_T I_{sh} \approx -A_{shz} (S_c) \tag{438}$$

Thus the transfer inductance is determined from the magnetic flux passing between the center conductor and the braid wires due to the drive from the current on the outer transmission line; the negative sign results from the fact that this flux has the opposite orientation from the flux associated with the inner transmission line with positive current I_1.

4.2.3 Braid Wire Current Distribution

The above argument of zero net flux between braid wires, for determining the braid wire current distribution can be replaced by an energy argument based on (388). We determine the braid wire current distribution by minimizing the preceding magnetic energy

$$0 = \frac{\partial}{\partial I_j}\left(LI^2\right) = -\frac{\partial}{\partial I_j}\oint_S \underline{A}\cdot\left(\underline{n}\times\underline{H}\right)dS\ ,\ j = 2, ..., N_s N_w \tag{439}$$

with the total braid wire current given

$$\sum_{j=1}^{N_w N_s} I_j = I \tag{440}$$

and N_w is the number of wires in a carrier strip and N_s is the number of carrier strips. This constraint would have to be applied to determine the braid wire current distribution both for the problem where the inner coax is driven $I = I_c$ with $I_{sh} = 0$ and for the problem where the outer coax is driven $I = I_{sh}$ with $I_c = 0$. This approach results in a system of equations where the coefficients only involve integration on the conductor surfaces S. This is more useful in the mixed potential approach to follow.

4.3 Integral Forms For Perfectly Conducting Braid Transfer And Coaxial Inductances Per Unit Length Using Magnetic Scalar Potential

For completeness we also illustrate the approach taken in [12] for cable shields with apertures using a scalar magnetic potential. Following the scalar reciprocity derivation in [12] we consider an auxiliary magnetostatic potential of the coaxial structure ϕ_{m0} without shield penetrations, satisfying Laplace's equation

$$\nabla^2\phi_{m0} = 0 \tag{441}$$

where the field is

$$\underline{H}_0 = -\nabla\phi_{m0} \tag{442}$$

and since there are no magnetic charges

$$\nabla\cdot\underline{B}_0 = \nabla\cdot\left(\mu_0\underline{H}_0\right) = 0 \tag{443}$$

This potential is associated with a current I_0 on the center conductor and a current $-I_0$ on the surface of the auxiliary shield S_0. The boundary conditions on the two conductors are

$$\frac{\partial\phi_{m0}}{\partial n} = 0\,,\ \rho = a, b_0 \tag{444}$$

There is a branch surface associated with the scalar magnetic potential from the inner conductor out to the outer conductor on which the potential jumps by I_0 [12].

The actual problem has potential ϕ_m, satisfying Laplace's equation

$$\nabla^2 \phi_m = 0 \tag{445}$$

with net current I_s in the shield and I in the center conductor. The boundary conditions for this potential are

$$\frac{\partial \phi_m}{\partial n} = 0\,,\ \rho = a \text{ and on shield wires} \tag{446}$$

The difference potential can be defined as

$$\phi_{md} = \phi_m - \phi_{m0} \tag{447}$$

satisfying Laplace's equation

$$\nabla^2 \phi_{md} = 0 \tag{448}$$

with no jump surface in the interior coaxial region since the difference current vanishes on the center conductor. Then from these equations we can write

$$\phi_{m0}\nabla^2\phi_{md} - \phi_{md}\nabla^2\phi_{m0} = \nabla\cdot(\phi_{m0}\nabla\phi_{md} - \phi_{md}\nabla\phi_{m0}) = 0 \tag{449}$$

Integrating over the auxiliary volume of the coaxial region and using the divergence theorem we can write

$$\oint_{S_0}\left(\phi_{m0}\frac{\partial\phi_{md}}{\partial n} - \phi_{md}\frac{\partial\phi_{m0}}{\partial n}\right)dS = 0 \tag{450}$$

where the unit normal $\underline{n}$ points out of the region. Now breaking up the surface integral into several components

$$\int_{J_+}\phi_{m0}\frac{\partial\phi_{md}}{\partial n}dS + \int_{J_-}\phi_{m0}\frac{\partial\phi_{md}}{\partial n}dS = -\int_{S_0}\phi_{m0}\frac{\partial\phi_{md}}{\partial n}dS$$

$$= \int_{J_+}\left(\phi^+_{m0} - \phi^-_{m0}\right)\frac{\partial\phi_{md}}{\partial n}dS \tag{451}$$

where the jump surface consists of the two faces $J_\pm$ (where the plus sign corresponds, say, to an azimuth value φ which is 2π plus the azimuth value of the J_- surface, $\underline{n}$ on J_+ points in the $+\varphi$ direction and on J_- points in the $-\varphi$ direction), which extends from the center conductor to the auxiliary shield S_0 surface, the normal derivative of ϕ_{m0} vanishes on S_0, and $\underline{n}$ points into S_0. Note that (where the contour C is directed in φ around the line current)

$$\phi^+_{m0} - \phi^-_{m0} = -\int_C \underline{H}_0\cdot\underline{d\ell} = -I_0 \text{ on } J_\pm \tag{452}$$

The magnetic difference flux directed around the center conductor in the φ direction is

$$\Phi_d = -\mu_0\int_{J_+}\frac{\partial\phi_{md}}{\partial n}dS + \mu_0\int_{J_0}H_n dS \tag{453}$$

where the magnetic induction in the region is $\underline{B} = \mu_0\underline{H}$, surface J_0 extends from the edge of the jump surface J_+ at auxiliary surface S_0 down to the braid wires, and again $\underline{n}$ points in the φ direction. From the preceding expression (451) we can write this difference flux as

$$\Phi_d = -\frac{\mu_0}{I_0}\int_{S_0}\phi_{m0}\frac{\partial\phi_{md}}{\partial n}dS + \mu_0\int_{J_0}H_n dS \tag{454}$$

Letting

$$\frac{\partial\phi_{md}}{\partial n} = \frac{\partial\phi_m}{\partial n} = -H_n \text{ on } S_0 \tag{455}$$

the difference voltage source is thus

$$V_d = i\omega\Phi_d = i\omega\mu_0\int_{S_0}\left(\frac{\phi_{m0}}{I_0}\right)H_n dS + i\omega\mu_0\int_{J_0}H_n dS \tag{456}$$

Again there are two sources for the penetrant field H_n. The first is the current I_0 on the center conductor, the second is an exterior current on a return conductor $-I_{sh}$. We can write these as the per unit length voltage

$$v_d = V_d/\ell = Z_T I_0 - \Delta Z I_{sh} \tag{457}$$

in the transmission line equation

$$\frac{dV}{dz} + Z_0 I = v_d \tag{458}$$

where in this perfectly conducting case the impedance correction is also inductive

$$\Delta Z = -i\omega\Delta L \tag{459}$$

$$Z_0 = -i\omega L_0 \tag{460}$$

$$Z_T = -i\omega L_T \tag{461}$$

The inductance per unit length of the coaxial region is then

$$L_1 = L_0 + \Delta L \tag{462}$$

where the inductance of the auxiliary shield is

$$L_0 = \frac{\mu_0}{2\pi}\ln(b_0/a) \tag{463}$$

Thus

$$L_T = -\mu_0\frac{1}{\ell}\int_{S_0}\left(\frac{\phi_{m0}}{I_0}\right)\frac{H_n}{I_{sh}}dS - \mu_0\frac{1}{\ell}\int_{J_0}\frac{H_n}{I_{sh}}dS \tag{464}$$

$$\Delta L = \mu_0\frac{1}{\ell}\int_{S_0}\left(\frac{\phi_{m0}}{I_0}\right)\frac{H_n}{I_0}dS + \mu_0\frac{1}{\ell}\int_{J_0}\frac{H_n}{I_0}dS \tag{465}$$

4.3.1 Planar Braid Approximation

To simplify the braid calculation we can treat the planar braid as we did in the electric case. This can be accomplished by noting that the behavior of the magnetic potential in the coaxial auxiliary problem near the braid is

$$\phi_{m0}/I_0 \sim \frac{-s}{P_0} \tag{466}$$

where $s = b_0\varphi \approx b\varphi$ and $P_0 = 2\pi b_0 \approx P$ is again the perimeter. We can then use a plane incident or drive potential

$$\phi_m^{inc} = -xH_0 \tag{467}$$

where

$$H_0 = I_0/P \tag{468}$$

to excite the planar braid.

We defer further consideration of the detailed setup of the planar braid problem until after we discuss the vector potential approach. The vector potential integral formulation exhibits analogs with the preceding electric braid problem and is therefore instructive to examine.

4.3.2 Scalar Potential Construction For Single Braid Wire

To illustrate the discontinuous form of the magnetic scalar potential with a single current filament I_0 at the origin of a two-dimensional coordinate system we write the complex potential W as

$$W = -\frac{I_0}{2\pi}\ln(z) = -\frac{I_0}{2\pi}\ln(x+iy) = -\frac{I_0}{2\pi}\left[\ln\sqrt{x^2+y^2} + i\varphi\right] \tag{469}$$

and then

$$\phi_{m0} = \mathrm{Im}(W) = -\frac{I_0}{2\pi}\arctan(y/x) = -\frac{I_0}{2\pi}\varphi \tag{470}$$

Therefore

$$\phi_{m0}^{+} - \phi_{m0}^{-} = \phi_{m0}(\varphi = 2\pi) - \phi_{m0}(\varphi = 0) = -I_0 \tag{471}$$

The treatment of the braid wires with the scalar potential ϕ_m alone would introduce branch surfaces from each wire. The detailed treatment by means of this approach may not be convenient. We will later consider the detailed solution of the boundary value problem using a combination of vector and scalar potentials and a magnetic line charge multipole description of the braid wires, which seems to constitute an efficient approach to the solution of the wire braid.

4.4 Integral Forms For Perfectly Conducting Braid Transfer And Coaxial

Inductances Per Unit Length Using Magnetic Vector Potential Approach

The simplified reciprocity approach for the magnetic coupling problem using the magnetic vector potential is now formulated for perfect conductors. In the end we are interested in only using the axial component of the vector potential, assuming the other components exist only near the braid wires. The resulting formulas are again convenient for application to nonuniform coaxial geometries which we will illustrate on the exterior ground plane and interior eccentric coaxial problems.

We use the Coulomb gauge

$$\nabla \cdot \underline{A} = 0 \tag{472}$$

and for nonmagnetic regions take

$$\underline{B} = \mu_0 \underline{H} \tag{473}$$

with

$$\nabla \times \underline{H} = \underline{J} \tag{474}$$

so that

$$\nabla \times \nabla \times \underline{A} = -\nabla^2 \underline{A} = \mu_0 \underline{J} \tag{475}$$

In general all three components of the vector potential will be present. The largest components will be associated with current flow along the braid wires. At a large distance from the braid (on the interior side) we expect that

$$\underline{A} \sim A_z \underline{e}_z \tag{476}$$

where here z is the direction of net current flow along the cable (the resultant of the two braid lattice directions). If we open circuit the cable interior, then the asymptotic axial vector potential will approach a constant level at a large distance from the braid in an analogous manner to the electric case.

Let us introduce an auxiliary potential (this has only an axial component since the current is purely axially directed in the auxiliary problem, having a cylindrical shield surface S_0 and a center conductor surface S_c)

$$\underline{A}_0 = A_{0z} \underline{e}_z \tag{477}$$

representing a coaxial mode between the center conductor and the auxiliary shield. Then let us take the difference potential

$$\underline{A}_d = \underline{A} - \underline{A}_0 \tag{478}$$

We then write

$$\nabla \cdot [\underline{A}_0 \times (\nabla \times \underline{A}_d) - \underline{A}_d \times (\nabla \times \underline{A}_0)] = (\nabla \times \underline{A}_d) \cdot (\nabla \times \underline{A}_0) - \underline{A}_0 \cdot (\nabla \times \nabla \times \underline{A}_d)$$

$$-(\nabla\times\underline{A}_0)\cdot(\nabla\times\underline{A}_d)+\underline{A}_d\cdot(\nabla\times\nabla\times\underline{A}_0) \tag{479}$$

which obviously vanishes in the source free region, so that (the volume V_0 is the auxiliary volume and the closed surface S_0 is the closed auxiliary volume surface)

$$0=\int_{V_0}\nabla\cdot[\underline{A}_0\times(\nabla\times\underline{A}_d)-\underline{A}_d\times(\nabla\times\underline{A}_0)]\,dV$$

$$=\oint_{S_0}[\underline{A}_0\times(\nabla\times\underline{A}_d)-\underline{A}_d\times(\nabla\times\underline{A}_0)]\cdot\underline{n}dS \tag{480}$$

Now noting that

$$[\underline{A}_0\times(\nabla\times\underline{A}_d)-\underline{A}_d\times(\nabla\times\underline{A}_0)]\cdot\underline{n}=\underline{n}\cdot(\underline{A}_0\times\underline{B}_d-\underline{A}_d\times\underline{B}_0)$$

$$=\underline{B}_d\cdot(\underline{n}\times\underline{A}_0)-\underline{B}_0\cdot(\underline{n}\times\underline{A}_d)=\underline{A}_0\cdot(\underline{B}_d\times\underline{n})-\underline{A}_d\cdot(\underline{B}_0\times\underline{n}) \tag{481}$$

or

$$\oint_{S_0}\underline{A}_0\cdot(\underline{n}\times\underline{B}_d)\,dS=\oint_{S_0}\underline{A}_d\cdot(\underline{n}\times\underline{B}_0)\,dS \tag{482}$$

where the unit vector $\underline{n}$ points out of the region. Since the potential in the auxiliary problem is z directed we obtain

$$\oint_{S_0}A_{0z}\underline{e}_z\cdot(\underline{n}\times\underline{B}_d)\,dS=\oint_{S_0}\underline{A}_d\cdot(\underline{n}\times\underline{B}_0)\,dS \tag{483}$$

Now we set (since the auxiliary surface and center conductor surfaces are perfect electric conductors in the auxiliary problem)

$$A_{0z}=0 \text{ on } S_0$$

$$=L_0I_0 \text{ on } S_c \tag{484}$$

and the φ directed flux over a short axial period ℓ in the auxiliary problem (between center conductor and auxiliary shield) is taken to be

$$\Phi_0=\ell L_0I_0 \tag{485}$$

(where L_0 is the inductance per unit length in the auxiliary problem) to obtain

$$L_0I_0\int_{S_c}\underline{e}_z\cdot(\underline{n}\times\underline{B}_d)\,dS=\int_{S_c}\underline{A}_d\cdot(\underline{n}\times\underline{B}_0)\,dS+\int_{S_0}\underline{A}_d\cdot(\underline{n}\times\underline{B}_0)\,dS \tag{486}$$

or with difference current

$$I_d=I-I_0 \tag{487}$$

we can write this as

$$-\ell L_0 I_0 \mu_0 I_d = \int_{S_c} A_{dz} \underline{e}_z \cdot (\underline{n} \times \underline{B}_0)\, dS + \int_{S_0} \underline{A}_d \cdot (\underline{n} \times \underline{B}_0)\, dS \tag{488}$$

On the center conductor we take the vector potential to be asymptotic to the z component where we denote the φ directed flux over a short axial period ℓ to be

$$\oint_C \underline{A} \cdot \underline{d\ell} = \int_S \underline{B} \cdot \underline{n} dS = \Phi \tag{489}$$

where the contour C is shown in Figures 19, 20, and 21 and is taken to have a $-z$ directed component along a braid wire and to close along the center conductor in the z direction. The difference flux is defined as

$$\Phi_d = \Phi - \Phi_0 \tag{490}$$

Note that the φ directed flux Φ is the flux over a short period ℓ between the center conductor and the braid wires. Now for perfectly conducting braid wires, the vanishing of the normal component of the magnetic field on the surface of a braid wire, means that the closed contour integral on the surface of a wire over any contour on the wire C'_w vanishes

$$\oint_{C'_w} \underline{A} \cdot \underline{d\ell} = 0 \tag{491}$$

Hence, the line integral of the vector potential on the surface of a braid wire depends only on the two end points. Taking the end points to be separated by the short period ℓ, we can write this as

$$\int_{C_w} \underline{A} \cdot \underline{d\ell} = A_w \ell \tag{492}$$

where the contour C_w follows a particular braid wire as shown by the blue "leg" following the braid wire of the closed contour in Figures 19, 20, and 21. If the contour C_w is taken along another wire of the same carrier, and we have adjusted the currents on the wires of the carrier to have zero net magnetic flux between them (in the perfectly conducting case) over the braid period, then we will find the same constant value A_w. Furthermore, because of symmetry between carriers we expect to find the same constant for any wire on another carrier. For the choice of zero scalar potential on the braid wires we expect that the tangential components of the low frequency vector potential to vanish on the braid wires and hence to have $A_w = 0$. With this choice, as well as noting that the end contour contributions cancel due to periodicity, we can associate the integral of the vector potential along a contour of the auxiliary shield with the magnetic flux between the shield and the braid wires. Noting with this choice that the flux per unit length on the interior is

$$\Phi = \int_{C_c} \underline{A} \cdot \underline{d\ell} \sim \int_{C_c} A_z \underline{e}_z \cdot \underline{d\ell} \tag{493}$$

where the contour C_c is along the center conductor and is z directed, using the vanishing of the auxiliary potential on the shield S_0 we can write (488) as

$$-\ell L_0 I_0 \mu_0 I_d = -\Phi_d \mu_0 I_0 + \int_{S_0} \underline{A} \cdot (\underline{n} \times \underline{B}_0)\, dS \tag{494}$$

If we fix the current so that $I = I_0$ and $I_d = 0$ then the difference flux is given by

$$\Phi_d = \frac{1}{\mu_0 I_0} \int_{S_0} \underline{A} \cdot (\underline{n} \times \underline{B}_0)\, dS \tag{495}$$

and from Faraday's law the difference voltage is

$$\oint_C \underline{E} \cdot \underline{d\ell} = V(\ell + z) - V(z) = -\frac{\partial}{\partial t} \int_S \underline{B} \cdot \underline{n} dS = i\omega\Phi \tag{496}$$

and for the auxiliary problem

$$V_0(\ell + z) - V_0(z) = -\frac{\partial}{\partial t}\Phi_0 = i\omega\Phi_0 \tag{497}$$

The difference voltage

$$V_d = V - V_0 \tag{498}$$

is then found from

$$V_d(\ell + z) - V_d(z) = i\omega\Phi_d \tag{499}$$

or from (494)

$$V_d = \ell v_d = i\omega\Phi_d = \frac{i\omega}{\mu_0 I_0} \int_{S_0} \underline{A} \cdot (\underline{n} \times \underline{B}_0)\, dS \tag{500}$$

Noting that $\underline{n} \times \underline{B}_0$ is $-z$ directed we can write this as

$$V_d = \ell v_d = i\omega\Phi_d = \frac{i\omega}{\mu_0 I_0} \int_{S_0} A_z \underline{e}_z \cdot (\underline{n} \times \underline{B}_0)\, dS \tag{501}$$

If we take the surface S_0 to be sufficiently far from the braid wires the preceding asymptotic form (where the vector potential is z directed) indicates that this expression picks out the dominant component of the vector potential.

The vector potential can be split into two parts

$$\underline{A} = \underline{A}_1 + \underline{A}_{sh} \tag{502}$$

where the first term results from interior sources creating an interior current on the transmission line and the second term results from flux penetration through the braid from an exterior current. Then splitting the sources into the exterior for the transfer impedance, or the interior for the correction to the impedance per unit length, we find

$$v_d = Z_T I_{sh} = -i\omega L_T I_{sh} = i\omega \int_{S_0} A_{shz} \underline{e}_z \cdot \left(\frac{1}{\mu_0 I_0} \underline{n} \times \underline{B}_0 \right) dS/\ell \tag{503}$$

$$v_d = -\Delta Z I_1 = i\omega \Delta L I_1 = i\omega \int_{S_0} A_{1z} \underline{e}_z \cdot \left(\frac{1}{\mu_0 I_0} \underline{n} \times \underline{B}_0 \right) dS/\ell \tag{504}$$

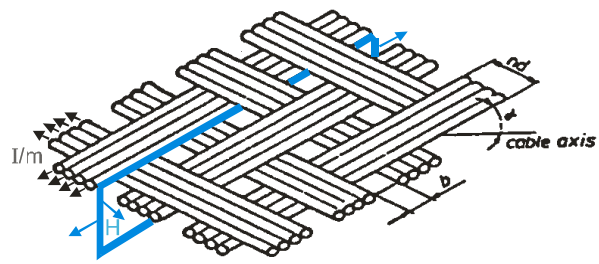

Figure 19. Magnetic flux contour following a single braid wire in the planar braid approximation (original braid figure from [7]).

This gives a transfer inductance

$$L_T = -\frac{1}{I_0 I_{sh}} \int_{S_0} A_{shz} \underline{e}_z \cdot (\underline{n} \times \underline{H}_0)\, dS/\ell \tag{505}$$

where $\underline{n}$ on S_0 in this case points outward. The preceding energy argument gave a transfer inductance

$$L_T = -\frac{1}{\ell I_1 I_{sh}} \int_V \mu_0 \underline{H}_1 \cdot \underline{H}_{sh} dV = \frac{1}{\ell I_1 I_{sh}} \oint_S \underline{A}_{sh} \cdot (\underline{n} \times \underline{H}_1)\, dS$$

$$= \frac{1}{\ell I_1 I_{sh}} \int_{S_c} \underline{A}_{sh} \cdot (\underline{n} \times \underline{H}_1)\, dS + \frac{1}{\ell I_1 I_{sh}} \int_{S_w} \underline{A}_{sh} \cdot (\underline{n} \times \underline{H}_1)\, dS = \frac{1}{\ell I_1 I_{sh}} \int_{S_c} \underline{A}_{sh} \cdot (\underline{n} \times \underline{H}_1)\, dS \tag{506}$$

which is essentially the same because $\underline{n}$ in this preceding case on S_c points inward; we can deform S_c outward to S_0 because the field from the outer shield source decays to zero in the interior region (with the center conductor open circuited) and the potential A_{shz} is essentially constant. Note that the total self inductance is

$$L_1 = L_0 + \Delta L \tag{507}$$

The minus sign in (504) is introduced because this is a passive element but the definition of the plus sign in Figure 11 is for a source.

The results (503) and (504) are analogous to the results (79) and (80) in the electric case. We will see in the next subsection that constants can be defined for the magnetic vector potential which capture the magnetic flux per unit length between the braid wires and the auxiliary shield in the same way as we found in the electric case for the electric scalar potential.

4.4.1 Local Planar Approximation

Near the auxiliary shield we can write (here we take $\underline{B}_0$ to be a locally constant value directed tangentially to the auxiliary shield)

$$\underline{B}_0 \approx \underline{B}_0 (x_0, y_0) \tag{508}$$

We define a two-dimensional position vector

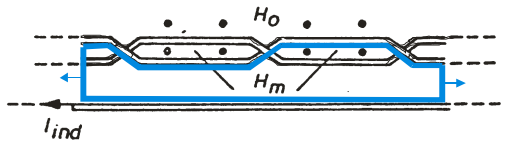

Figure 20. Cross section of flux contour following a single braid wire (original figure from [7]).

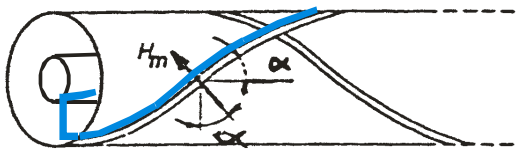

Figure 21. Magnetic flux contour in the coaxial geometry (original figure from [7]).

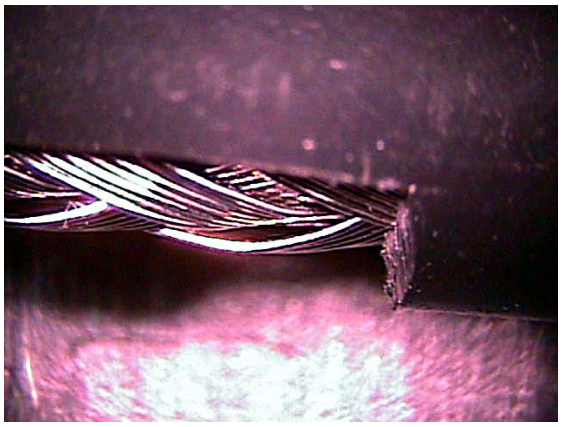

Figure 22. Photograph of high optical coverage braid (figure for Belden 8240 cable from [1]).

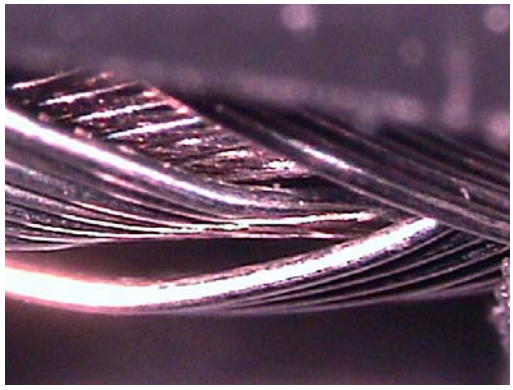

Figure 23. Closeup of high optical coverage braid (figure for Belden 8240 cable from [1]).

$$\underline{\rho} = x\underline{e}_x + y\underline{e}_y \tag{509}$$

and a vector to the auxiliary shield

$$\underline{\rho}_0 = x_0\underline{e}_x + y_0\underline{e}_y \tag{510}$$

with the smallest distance $\left|\underline{\rho} - \underline{\rho}_0\right|$ and $\underline{n} \times \left(\underline{\rho} - \underline{\rho}_0\right) = 0$. Then we can write the auxiliary potential locally as

$$\underline{A}_0 \sim -\left(\underline{\rho} - \underline{\rho}_0\right) \times \underline{B}_0\left(x_0, y_0\right) \tag{511}$$

where we note that $\nabla \times \underline{A}_0 = \underline{B}_0$ and $\nabla \cdot \underline{A}_0 = 0$, and this potential vanishes on the auxiliary shield at position x_0, y_0. The potential from the braid we take locally as

$$\underline{A} \sim -\left(\underline{\rho} - \underline{\rho}_0\right) \times \underline{B}_1\left(x_0, y_0\right) + \left(A_c + A_b\right)\underline{e}_z$$

$$\sim -\left(\underline{\rho} - \underline{\rho}_0 - \underline{n}A_b/B_1\right) \times \underline{B}_1\left(x_0, y_0\right) - \left(A_c/B_{sh}\right)\underline{n} \times \underline{B}_{sh}\left(x_0, y_0\right) \tag{512}$$

where the constant A_b/B_1 is the effective position of the braid in the normal direction (the normal in this term points out of the interior) due to the interior field $\underline{B}_1$ excited by the current I_1, and where $-A_c$ is the φ directed magnetic flux per unit length leaking through the braid from the exterior field $\underline{B}_{sh}$ excited by the current I_{sh} (note that this field is on the exterior side of the braid but to be consistent with the first term and the interior nature of the asymptotic form of the potential we take the unit normal $\underline{n}$ to point out of the interior region). Taking the axial components and approximating $\underline{B}_1 \sim \left(I_1/I_0\right)\underline{B}_0$ we find

$$A_{0z} \sim -\left(\underline{n} \cdot \left(\underline{\rho} - \underline{\rho}_0\right)\right)\underline{e}_z \cdot \left[\underline{n} \times \underline{B}_0\left(x_0, y_0\right)\right] \tag{513}$$

$$A_{shz} \sim -\left(A_c/B_0\right)\underline{e}_z \cdot \left[\underline{n} \times \underline{B}_{sh}\left(x_0, y_0\right)\right] \tag{514}$$

$$A_{1z} \sim -\left(\underline{n} \cdot \left(\underline{\rho} - \underline{\rho}_0\right) - A_b/B_0\right)\underline{e}_z \cdot \left[\underline{n} \times \underline{B}_0\left(x_0, y_0\right)\right]\left(I_1/I_0\right) \tag{515}$$

where we have denoted the generic normalization of the flux per unit length constants A_b and A_c by the generic planar field excitation B_0 to represent the constants derived from solution of a planar braid approximation driven by the field $\underline{B}_0$. Plugging these into the integral expressions for the field will require us to set $\underline{\rho} \to \underline{\rho}_0$ and hence $\underline{n} \cdot \left(\underline{\rho} - \underline{\rho}_0\right) = 0$ on the auxiliary shield. However, it will be convenient to redefine these expansions to use the origin at the braid center point in thickness x_m, y_m rather than the arbitrary auxiliary shield position; we are, in effect, expanding the solid shield to exist at the original braid mean position rather than at the arbitrary auxiliary position S_0 to better approximate the dominant fields near the braid. Thus

$$\underline{\rho}_m = x_m\underline{e}_x + y_m\underline{e}_y \tag{516}$$

$$\underline{B}_0 \approx \underline{B}_0\left(x_m, y_m\right) \tag{517}$$

$$\underline{A}_0 \sim -\left(\underline{\rho} - \underline{\rho}_0\right) \times \underline{B}_0\left(x_m, y_m\right) \tag{518}$$

$$\underline{A} \sim -\left(\underline{\rho} - \underline{\rho}_m\right) \times \underline{B}_1\left(x_m, y_m\right) + \left(-A_c + A_b\right) \underline{e}_z$$

$$\sim -\left(\underline{\rho} - \underline{\rho}_m - \underline{n} A_b / B_1\right) \times \underline{B}_1\left(x_m, y_m\right) - \left(A_c / B_{sh}\right) \underline{n} \times \underline{B}_{sh}\left(x_m, y_m\right) \tag{519}$$

and again the axial components, letting $\underline{B}_1 \sim \left(I_1 / I_0\right) \underline{B}_0$

$$A_{0z} \sim -\left(\underline{n} \cdot \left(\underline{\rho} - \underline{\rho}_0\right)\right) \underline{e}_z \cdot \left[\underline{n} \times \underline{B}_0\left(x_m, y_m\right)\right] \tag{520}$$

$$A_{shz} \sim -\left(A_c / B_0\right) \underline{e}_z \cdot \left[\underline{n} \times \underline{B}_{sh}\left(x_m, y_m\right)\right] \tag{521}$$

$$A_{1z} \sim -\left(\underline{n} \cdot \left(\underline{\rho} - \underline{\rho}_m\right) - A_b / B_0\right) \underline{e}_z \cdot \left[\underline{n} \times \underline{B}_0\left(x_m, y_m\right)\right]\left(I_1 / I_0\right) \tag{522}$$

The main effect of this selection is to redefine the constant position A_b/B_0 (the value of this constant then changes to reflect the new reference position) to be relative to the braid center line (a secondary effect is to define the fields $\underline{B}_0$ and $\underline{B}_{sh}$ to be those that would exist on the braid center line if the braid were replace by a solid conductor at this position). Plugging into the preceding expressions (note that the field $\underline{B}_{sh}$ is on the exterior side of the braid but the unit normal throughout the expression points out of the interior region of the coax)

$$Z_T = -i\omega L_T \sim -i\omega\mu_0 \int_{S_0} \left(A_c / B_0\right) \underline{e}_z \cdot \left(\frac{1}{\mu_0 I_{sh}} \underline{n} \times \underline{B}_{sh}\right) \underline{e}_z \cdot \left(\frac{1}{\mu_0 I_0} \underline{n} \times \underline{B}_0\right) dS/\ell$$

$$\sim -i\omega\mu_0 \left(A_c / B_0\right) \int_{S_0} \frac{1}{\mu_0^2 I_0 I_{sh}} \underline{B}_{sh} \cdot \underline{B}_0 dS/\ell = -i\omega\mu_0 \left(A_c / B_0\right) \int_{S_0} \frac{1}{I_0 I_{sh}} \underline{H}_{sh} \cdot \underline{H}_0 dS/\ell \tag{523}$$

$$\Delta Z = -i\omega \Delta L$$

$$\sim i\omega\mu_0 \int_{S_0} \left[\underline{n} \cdot \left(\underline{\rho}_0 - \underline{\rho}_m\right) - A_b / B_0\right] \left[\underline{e}_z \cdot \left(\frac{1}{\mu_0 I_0} \underline{n} \times \underline{B}_0\right)\right]^2 dS/\ell$$

$$\sim -i\omega\mu_0 \int_{S_0} \left(d_0 + A_b / B_0\right) \frac{1}{\mu_0^2 I_0^2} \underline{B}_0 \cdot \underline{B}_0 dS/\ell = -i\omega\mu_0 \int_{S_0} \left(d_0 + A_b / B_0\right) \frac{1}{I_0^2} \underline{H}_0 \cdot \underline{H}_0 dS/\ell \tag{524}$$

where

$$d_0 = \underline{n} \cdot \left(\underline{\rho}_m - \underline{\rho}_0\right) \tag{525}$$

is the distance from the auxiliary shield to the braid center line. Hence we find

$$L_T = \frac{1}{I_0 I_{sh}} \int_{S_0} \left(A_c / B_0\right) \underline{e}_z \cdot \left[\underline{n} \times \underline{B}_{sh}\left(x_m, y_m\right)\right] \underline{e}_z \cdot \left(\underline{n} \times \underline{H}_0\right) dS/\ell \tag{526}$$

$$\Delta L = \frac{1}{I_0^2} \int_{S_0} (d_0 + A_b/B_0) \, \underline{H}_0 \cdot \underline{H}_0 dS/\ell \tag{527}$$

These results (523) and (524) are analogous to the electrical results (79) and (80). We will now apply them to the same nonuniform coaxial problems as we did in the electric problem. We first illustrate their application on the simple uniform cylindrical coaxial geometry.

4.4.2 Simple Uniform Coaxial Geometry

In the case of the simple uniform coaxial geometry with radii a, b and b_0, where

$$\frac{1}{I_0} H_0 = \frac{1}{I_0} H_\varphi (b_0, \varphi) \to \frac{1}{2\pi b_0} \tag{528}$$

$$\frac{1}{I_0} H_\varphi (b, \varphi) \to \frac{1}{2\pi b} \tag{529}$$

$$\frac{1}{I_{sh}} H_{sh} = \frac{1}{I_{sh}} H_\varphi (b, \varphi) \to \frac{1}{2\pi b} \tag{530}$$

where b is the braid center line (these approximate fields are those that would exist with the braid replaced by a solid shield) and we denote the inductance function of the coax per unit length by

$$L(b) = \frac{\mu_0}{2\pi} \ln (b/a) \tag{531}$$

the inductance per unit length of the auxiliary shield is

$$L_0 = L(b_0) \tag{532}$$

the spacing between the braid center line and the auxiliary shield is

$$d_0 = b - b_0 \tag{533}$$

Then the preceding expressions give

$$L_T \sim \mu_0 (A_c/B_0) \int_{S_0} \frac{1}{I_0 I_{sh}} \underline{H}_{sh} \cdot \underline{H}_0 dS/\ell = \mu_0 (A_c/B_0) \frac{1}{2\pi b} \tag{534}$$

$$\Delta L \sim \mu_0 (d_0 + A_b/B_0) \int_{S_0} \frac{1}{I_0^2} \underline{H}_0 \cdot \underline{H}_0 dS/\ell$$

$$\sim \mu_0 (b - b_0 + A_b/B_0) \frac{1}{2\pi b} \tag{535}$$

The transfer inductance per unit length is

$$L_T \sim \mu_0 (A_c/B_0) \frac{1}{2\pi b} \tag{536}$$

The auxiliary problem self inductance per unit length can be written in terms of the preceding inductance

function (531)

$$L_0 \sim L(b) + (b_0 - b)\frac{\partial L}{\partial b}(b) \tag{537}$$

The self inductance per unit length correction can be written as

$$\Delta L \sim \mu_0 (b - b_0 + A_b/B_0)\frac{1}{2\pi b} = (b - b_0 + A_b/B_0)\frac{\partial L}{\partial b}(b) \tag{538}$$

Then the total self inductance per unit length is then

$$L_1 \sim L_0 + \Delta L \sim L(b) + (A_b/B_0)\frac{\partial L}{\partial b}(b) \tag{539}$$

or

$$L_1 \sim L(b + A_b/B_0) = \frac{\mu_0}{2\pi}\ln\left(\frac{b + A_b/B_0}{a}\right) \tag{540}$$

The constant A_b/B_0 associated with the braid shifts the effective center line of the braid to this new location.

4.4.3 Eccentric Coax

Consider the situation when the interior is an eccentric coax [14]. A complex potential resulting from the superposition of two line currents $\pm I_0$ is

$$W = U + iV = -\frac{\mu_0 I_0}{2\pi}\ln\left(\frac{z - iy_c}{z + iy_c}\right) = \frac{\mu_0 I_0}{2\pi} i2 \operatorname{arccot}(z/y_c) \tag{541}$$

where $z = x + iy$ and $y_c > 0$. The axial component of the perfectly conducting vector potential is taken as

$$A_{0z} = \operatorname{Re}(W) = U = -\frac{\mu_0 I_0}{2\pi}\ln\left|\frac{z - iy_c}{z + iy_c}\right| \tag{542}$$

Inverting the transformation gives

$$z = x + iy = y_c \cot\left(\frac{U + iV}{2i\frac{\mu_0 I_0}{2\pi}}\right) \tag{543}$$

or

$$x = \frac{y_c \sin\left(V/\left(\frac{\mu_0 I_0}{2\pi}\right)\right)}{\cosh\left(U/\left(\frac{\mu_0 I_0}{2\pi}\right)\right) - \cos\left(V/\left(\frac{\mu_0 I_0}{2\pi}\right)\right)} \tag{544}$$

$$y = \frac{y_c \sinh\left(U/\left(\frac{\mu_0 I_0}{2\pi}\right)\right)}{\cosh\left(U/\left(\frac{\mu_0 I_0}{2\pi}\right)\right) - \cos\left(V/\left(\frac{\mu_0 I_0}{2\pi}\right)\right)} \tag{545}$$

$$\cosh\left(U/\left(\frac{\mu_0 I_0}{2\pi}\right)\right) - \cos\left(V/\left(\frac{\mu_0 I_0}{2\pi}\right)\right) = (y_c/x)\sin\left(V/\left(\frac{\mu_0 I_0}{2\pi}\right)\right) \tag{546}$$

$$\cosh\left(U/\left(\frac{\mu_0 I_0}{2\pi}\right)\right) - (y_c/y)\sinh\left(U/\left(\frac{\mu_0 I_0}{2\pi}\right)\right) = \cos\left(V/\left(\frac{\mu_0 I_0}{2\pi}\right)\right) \tag{547}$$

Eliminating V gives

$$x\sinh\left(U/\left(\frac{\mu_0 I_0}{2\pi}\right)\right) = y\sin\left(V/\left(\frac{\mu_0 I_0}{2\pi}\right)\right) \tag{548}$$

or

$$x^2\sinh^2\left(U/\left(\frac{\mu_0 I_0}{2\pi}\right)\right) = y^2 - \left[y\cosh\left(U/\left(\frac{\mu_0 I_0}{2\pi}\right)\right) - y_c\sinh\left(U/\left(\frac{\mu_0 I_0}{2\pi}\right)\right)\right]^2 \tag{549}$$

or

$$x^2 = y^2\mathrm{csch}^2\left(U/\left(\frac{\mu_0 I_0}{2\pi}\right)\right) - \left[y\coth\left(U/\left(\frac{\mu_0 I_0}{2\pi}\right)\right) - y_c\right]^2 \tag{550}$$

or

$$x^2 = -y^2 + 2yy_c\coth\left(U/\left(\frac{\mu_0 I_0}{2\pi}\right)\right) - y_c^2 \tag{551}$$

or

$$x^2 + y^2 - 2yy_c\coth\left(U/\left(\frac{\mu_0 I_0}{2\pi}\right)\right) + y_c^2\coth^2\left(U/\left(\frac{\mu_0 I_0}{2\pi}\right)\right) = y_c^2\left[\coth^2\left(U/\left(\frac{\mu_0 I_0}{2\pi}\right)\right) - 1\right] \tag{552}$$

or

$$x^2 + \left[y - y_c\coth\left(U/\left(\frac{\mu_0 I_0}{2\pi}\right)\right)\right]^2 = y_c^2\mathrm{csch}^2\left(U/\left(\frac{\mu_0 I_0}{2\pi}\right)\right) \tag{553}$$

Thus the equipotential contours are circles centered at $x = 0$ and

$$y = y_c\coth\left(U/\left(\frac{\mu_0 I_0}{2\pi}\right)\right) \tag{554}$$

with radius

$$r = y_c\mathrm{csch}\left(U/\left(\frac{\mu_0 I_0}{2\pi}\right)\right) \tag{555}$$

If we pick two potentials $U_2 > U_1$, then the associated radii satisfy $r_2 < r_1$ (the smaller is contained within the bigger), and the associated centers satisfy $y_2 < y_1$ (the center of the smaller is below the center of the larger). Now if we take

$$y_1 = y_c\coth\left(U_1/\left(\frac{\mu_0 I_0}{2\pi}\right)\right) \tag{556}$$

$$a = y_c \mathrm{csch}\left(U_2/\left(\frac{\mu_0 I_0}{2\pi}\right)\right) \tag{557}$$

$$b = y_c \mathrm{csch}\left(U_1/\left(\frac{\mu_0 I_0}{2\pi}\right)\right) \tag{558}$$

$$d = y_c \coth\left(U_1/\left(\frac{\mu_0 I_0}{2\pi}\right)\right) - y_c \coth\left(U_2/\left(\frac{\mu_0 I_0}{2\pi}\right)\right) = \frac{y_c \sinh\left(\frac{U_2 - U_1}{\left(\frac{\mu_0 I_0}{2\pi}\right)}\right)}{\sinh\left(\frac{U_2}{\left(\frac{\mu_0 I_0}{2\pi}\right)}\right)\sinh\left(\frac{U_1}{\left(\frac{\mu_0 I_0}{2\pi}\right)}\right)} \tag{559}$$

$$\cosh\left(\frac{U_2 - U_1}{\left(\frac{\mu_0 I_0}{2\pi}\right)}\right) = \frac{a^2 + b^2 - d^2}{2ab} \tag{560}$$

Using

$$\oint_C \underline{A}_0 \cdot \underline{d\ell} = \int_S \underline{B}_0 \cdot \underline{n} dS = \Phi_0 \tag{561}$$

we have

$$\Phi_0 = U_2 - U_1 \tag{562}$$

with inductance per unit length

$$L_0 = \Phi_0 / I_0 \tag{563}$$

where

$$\mu_0 I_0 = \Phi_0 2\pi / \mathrm{Arccosh}\left(\frac{a^2 + b^2 - d^2}{2ab}\right) \tag{564}$$

$$2y_c d = \sqrt{\left(a^2 + b^2 - d^2\right)^2 - 4a^2b^2} = \sqrt{\left[(b-a)^2 - d^2\right]\left[(b+a)^2 - d^2\right]} \tag{565}$$

$$y_1 = y_c \sqrt{1 + \mathrm{csch}^2\left(U_1/\left(\frac{\mu_0 I_0}{2\pi}\right)\right)} = \sqrt{y_c^2 + b^2} \tag{566}$$

or

$$2dy_1 = \sqrt{\left(a^2 + b^2 - d^2\right)^2 - 4a^2b^2 + 4d^2b^2} = \sqrt{\left(-a^2 + b^2 + d^2\right)^2} = -a^2 + b^2 + d^2 \tag{567}$$

$$\sqrt{\left(y_c/b\right)^2 + 1} = \cosh\left(U_1/\left(\frac{\mu_0 I_0}{2\pi}\right)\right) \tag{568}$$

$$\frac{\mu_0 I_0}{2\pi} \mathrm{Arccosh} \sqrt{\left(y_c/b\right)^2 + 1} = U_1 \tag{569}$$

We want to shift from y_1 to h as the location of center of the b radius cylinder. We also want to shift the potential on the outer cylinder from U_1 to 0

$$A_{0z} = -\frac{\mu_0 I_0}{2\pi} \ln \sqrt{\frac{x^2 + (y + y_1 - h - y_c)^2}{x^2 + (y + y_1 - h + y_c)^2}} - U_1 \tag{570}$$

$$\mu_0 I_0 = \Phi_0 2\pi / \text{Arccosh}\left(\frac{a^2 + b^2 - d^2}{2ab}\right) \tag{571}$$

$$2y_c d = \sqrt{\left[(b-a)^2 - d^2\right]\left[(b+a)^2 - d^2\right]} \tag{572}$$

$$y_1 = \sqrt{y_c^2 + b^2} \tag{573}$$

$$\frac{\mu_0 I_0}{2\pi} \text{Arccosh}\,(y_1/b) = U_1 \tag{574}$$

The tangential magnetic field at the outer shield boundary is

$$\frac{1}{I_0} H_0 = \frac{1}{I_0} H_\varphi(\rho, \varphi) = -\frac{1}{\mu_0 I_0} \frac{\partial A_{0z}}{\partial \rho} = \frac{1}{2\pi} \frac{\partial}{\partial \rho} \ln \sqrt{\frac{\rho^2 \cos^2\varphi + (\rho \sin\varphi + y_1 - y_c)^2}{\rho^2 \cos^2\varphi + (\rho \sin\varphi + y_1 + y_c)^2}}$$

$$= \frac{1}{2\pi} \left[\frac{\rho + (y_1 - y_c)\sin\varphi}{\rho^2 \cos^2\varphi + (\rho \sin\varphi + y_1 - y_c)^2} - \frac{\rho + (y_1 + y_c)\sin\varphi}{\rho^2 \cos^2\varphi + (\rho \sin\varphi + y_1 + y_c)^2}\right] \tag{575}$$

where ρ is a radial coordinate centered at the center of the outer cylinder.

This can be simplified for $\rho = b$ as

$$\frac{1}{I_0} H_0 = \frac{1}{I_0} H_\varphi(b, \varphi) = \frac{1}{2\pi} \left[\frac{b + (y_1 - y_c)\sin\varphi}{b^2 \cos^2\varphi + (b \sin\varphi + y_1 - y_c)^2} - \frac{b + (y_1 + y_c)\sin\varphi}{b^2 \cos^2\varphi + (b \sin\varphi + y_1 + y_c)^2}\right]$$

$$= \frac{2y_c b}{\pi} \left[\frac{y_1 + b\sin\varphi}{\left\{b^2 + 2b(y_1 - y_c)\sin\varphi + (y_1 - y_c)^2\right\}\left\{b^2 + 2b(y_1 + y_c)\sin\varphi + (y_1 + y_c)^2\right\}}\right]$$

$$= \frac{1}{2\pi b} \frac{y_c}{y_1 + b\sin\varphi} \tag{576}$$

where

$$2y_c d = \sqrt{\left[(b-a)^2 - d^2\right]\left[(b+a)^2 - d^2\right]} \tag{577}$$

$$y_1 = \sqrt{y_c^2 + b^2} = \left(-a^2 + b^2 + d^2\right) / (2d) \tag{578}$$

$$(y_1 - y_c)(y_1 + y_c) = \left(y_1^2 - y_c^2\right) = b^2 \tag{579}$$

Then using this field along with the uniform circumferential exterior field

$$\frac{1}{I_{sh}} H_{sh} = \frac{1}{I_{sh}} H_\varphi(b, \varphi) \to \frac{1}{2\pi b} \tag{580}$$

we obtain (noting again that the field $\underline{H}_{sh}$ is the field exterior to the coax if the braid were replaced by a solid shield at the mean braid radius)

$$Z_T = -i\omega L_T \sim -i\omega\mu_0 (A_c/B_0) \int_{S_0} \frac{1}{I_0 I_{sh}} \underline{H}_{sh} \cdot \underline{H}_0 dS/\ell$$

$$\sim -i\omega\mu_0 \left(\frac{A_c/B_0}{2\pi b}\right) \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{y_c}{y_1 + b\sin\varphi}\right) d\varphi \tag{581}$$

$$\Delta Z = -i\omega\Delta L \sim -i\omega\mu_0 (d_0 + A_b/B_0) \int_{S_0} \frac{1}{I_0^2} \underline{H}_0 \cdot \underline{H}_0 dS/\ell$$

$$\sim -i\omega\mu_0 \left(\frac{d_0 + A_b/B_0}{2\pi b}\right) \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{y_c}{y_1 + b\sin\varphi}\right)^2 d\varphi \tag{582}$$

$$d_0 = b - b_0 \tag{583}$$

Noting from (130) and (199) that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{y_c}{y_1 + b\sin\varphi} d\varphi = 1 \tag{584}$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{y_c}{y_1 + b\sin\varphi}\right)^2 d\varphi = y_1/y_c \tag{585}$$

$$\sqrt{y_1^2 - b^2} = y_c \tag{586}$$

we find

$$Z_T = -i\omega L_T \sim -i\omega\mu_0 \left(\frac{A_c/B_0}{2\pi b}\right) \tag{587}$$

$$\Delta Z = -i\omega\Delta L \sim -i\omega\mu_0 \left(\frac{d_0 + A_b/B_0}{2\pi b}\right) (y_1/y_c) \tag{588}$$

The total self inductance is

$$L_1 = L_0 + \Delta L \tag{589}$$

$$\Delta L \sim \mu_0 \left(\frac{d_0 + A_b/B_0}{2\pi b}\right) (y_1/y_c) \tag{590}$$

where the inductance per unit length of the eccentric coaxial problem is

$$L(b) = \Phi_0/I_0 = \frac{\mu_0}{2\pi}\text{Arccosh}\left(\frac{a^2+b^2-d^2}{2ab}\right) = \frac{\mu_0}{2\pi}\text{Arcsinh}\left(\frac{y_c d}{ab}\right) \tag{591}$$

$$L_0 = L(b_0) \tag{592}$$

$$\frac{\partial L}{\partial b} = \frac{\mu_0}{2\pi b}\frac{b^2-a^2+d^2}{\sqrt{\left[(b+a)^2-d^2\right]\left[(b-a)^2-d^2\right]}} = \frac{\mu_0}{2\pi b}(y_1/y_c) \tag{593}$$

We can write the auxiliary inductance per unit length as

$$L_0 \sim L(b) + (b_0 - b)\frac{\partial L}{\partial b}(b) \tag{594}$$

and the correction to the inductance per unit length as

$$\Delta L \sim (d_0 + A_b/B_0)\frac{\mu_0}{2\pi b}(y_1/y_c) = (b - b_0 + A_b/B_0)\frac{\partial L}{\partial b}(b) \tag{595}$$

Then the total self inductance per unit length is

$$L_1 \sim L_0 + \Delta L \sim L(b) + (A_b/B_0)\frac{\partial L}{\partial b}(b) \tag{596}$$

or

$$L_1 \sim L(b + A_b/B_0) = \frac{\mu_0}{2\pi}\text{Arccosh}\left[\frac{a^2+(b+A_b/B_0)^2-d^2}{2a(b+A_b/B_0)}\right] \tag{597}$$

4.4.4 Ground Plane Case

When there is an exterior ground plane we write the exterior potential as

$$A_z^{sh} = \frac{\mu_0 I_{sh}}{2\pi}\ln\sqrt{\frac{x^2+(y+h_e)^2}{x^2+(y-h_e)^2}} - \frac{\mu_0 I_{sh}}{2\pi}\ln\sqrt{\frac{h+h_e}{h-h_e}} \tag{598}$$

where I_{sh} is the exterior shield current and the effective line source is placed at height

$$h_e = \sqrt{h^2 - b^2} \tag{599}$$

Letting

$$x = \rho\cos\varphi \tag{600}$$

$$y = h + \rho\sin\varphi \tag{601}$$

$$A_z^{sh} = \frac{\mu_0 I_{sh}}{2\pi} \ln \sqrt{\frac{\rho^2 - b^2 + 2h\left(h + h_e\right) + 2\left(h + h_e\right)\rho \sin\varphi}{\rho^2 - b^2 + 2h\left(h - h_e\right) + 2\left(h - h_e\right)\rho \sin\varphi}} - \frac{\mu_0 I_{sh}}{2\pi} \ln \sqrt{\frac{h + h_e}{h - h_e}} \tag{602}$$

The tangential magnetic field is

$$H_\varphi = -\frac{1}{\mu_0} \frac{\partial A_z^{sh}}{\partial \rho}$$

$$= \frac{I_{sh}}{2\pi} \left[\frac{\rho + \left(h - h_e\right) \sin\varphi}{\rho^2 - b^2 + 2h\left(h - h_e\right) + 2\left(h - h_e\right)\rho \sin\varphi} - \frac{\rho + \left(h + h_e\right) \sin\varphi}{\rho^2 - b^2 + 2h\left(h + h_e\right) + 2\left(h + h_e\right)\rho \sin\varphi} \right] \tag{603}$$

and taking $\rho = b$ the tangential field is

$$H_\varphi\left(b, \varphi\right) = \frac{I_{sh} b}{4\pi} \left(\frac{1}{h - h_e} - \frac{1}{h + h_e} \right) \frac{1}{h + b \sin\varphi}$$

$$= \frac{I_{sh}}{2\pi b} \frac{h_e}{h + b \sin\varphi} \tag{604}$$

and the potential is

$$A_z^{sh}\left(b\right) = \frac{\mu_0 I_{sh}}{2\pi} \ln \sqrt{\frac{h + h_e}{h - h_e}} - \frac{\mu_0 I_{sh}}{2\pi} \ln \sqrt{\frac{h + h_e}{h - h_e}} = 0 \tag{605}$$

with exterior inductance per unit length

$$L_{sh} = \frac{\mu_0}{2\pi} \ln \sqrt{\frac{h + h_e}{h - h_e}} = \frac{\mu_0}{2\pi} \mathrm{Arccosh}\left(h/b\right) \tag{606}$$

and we use the inductance per unit length of the exterior of the braid transmission line with the mean radius b as an approximation.

Using this exterior field

$$\frac{1}{I_{sh}} H_{sh} = \frac{1}{I_{sh}} H_\varphi\left(b, \varphi\right) \rightarrow \frac{1}{2\pi b} \frac{h_e}{h + b \sin\varphi} \tag{607}$$

along with the field of the eccentric coax we find

$$Z_T = -i\omega L_T \sim -i\omega\mu_0 \left(A_c/B_0\right) \int_{S_0} \frac{1}{I_0 I_{sh}} \underline{H}_{sh} \cdot \underline{H}_0 dS/\ell$$

$$\sim -i\omega\mu_0 \left(\frac{A_c/B_0}{2\pi b} \right) \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{y_c}{y_1 + b \sin\varphi} \right) \left(\frac{h_e}{h + b \sin\varphi} \right) d\varphi \tag{608}$$

or

$$L_T = \mu_0 \left(\frac{A_c/B_0}{2\pi b} \right) \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{y_c}{y_1 + b \sin\varphi} \right) \left(\frac{h_e}{h + b \sin\varphi} \right) d\varphi \tag{609}$$

Noting from the preceding electric field case (198) that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{h_e}{h+b\sin\varphi}\frac{y_c}{y_1+b\sin\varphi}d\varphi=\frac{y_c-h_e}{y_1-h} \tag{610}$$

and hence

$$L_T=\mu_0\left(\frac{A_c/B_0}{2\pi b}\right)\frac{y_c-h_e}{y_1-h} \tag{611}$$

If we rotate the azimuth of the eccentric coax with respect to the ground plane by $\pi/2$ then the transfer inductance becomes

$$L_T=\mu_0\left(\frac{A_c/B_0}{2\pi b}\right)\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{y_c}{y_1+b\cos\varphi}\right)\left(\frac{h_e}{h+b\sin\varphi}\right)d\varphi \tag{612}$$

Using the identity from the electric case (218)

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{h_e}{h+b\sin\varphi}\frac{y_c}{y_1+b\cos\varphi}d\varphi=\frac{y_1y_c+hh_e}{y_1^2+h_e^2}=\frac{\sqrt{y_c^2+b^2}y_c+h\sqrt{h^2-b^2}}{y_c^2+h^2} \tag{613}$$

gives

$$L_T=\mu_0\left(\frac{A_c/B_0}{2\pi b}\right)\frac{y_1y_c+hh_e}{y_1^2+h_e^2} \tag{614}$$

This source enters the interior transmission line equation as

$$\frac{dV}{dz}=Z_TI_{sh}-Z_1I \tag{615}$$

4.4.5 Evaluation Of Energy Formulas Using Approximate Planar Form Of Potentials

It is instructive to show how the preceding perturbation results can be directly derived from approximations of the energy formulas for inductance rather than using the reciprocity route. This is done by applying the preceding energy formulas stood off from the surface of the braid using an approximate planar evaluation of the potential near the braid to evaluate the self and transfer inductances.

Self Inductance Approximate Evaluation The self inductance is given by

$$L_{11}I_1^2=\ell L_1I_1^2=\int_V\mu_0H_1^2dV \tag{616}$$

Let us decompose the volume into two regions, the auxiliary part V_0 and the part ΔV where $V=V_0+\Delta V$

$$L_{11}I_1^2=\ell L_1I_1^2=\int_{V_0}\mu_0H_1^2dV+\int_{\Delta V}\mu_0H_1^2dV$$

$$\approx\int_{V_0}\mu_0H_0^2dV+\int_{\Delta V}\mu_0H_1^2dV$$

where we have approximated the magnetic field in the volume V_0 (which has a surface spaced off from the braid wires) by the auxiliary field H_0. Using the representations

$$\mu_0 \underline{H}_0 = \nabla \times \underline{A}_0$$

$$\mu_0 \underline{H}_1 = \nabla \times \underline{A}_1 \tag{617}$$

we have

$$\ell L_1 I_1^2 = \int_{V_0} (\nabla \times \underline{A}_0) \cdot \underline{H}_0 dV + \int_{\Delta V} (\nabla \times \underline{A}_1) \cdot \underline{H}_1 dV \tag{618}$$

Now using the identities

$$\nabla \cdot (\underline{A}_0 \times \underline{H}_0) = (\nabla \times \underline{A}_0) \cdot \underline{H}_0 - \underline{A}_0 \cdot (\nabla \times \underline{H}_0)$$

$$\nabla \cdot (\underline{A}_1 \times \underline{H}_1) = (\nabla \times \underline{A}_1) \cdot \underline{H}_1 - \underline{A}_1 \cdot (\nabla \times \underline{H}_1) \tag{619}$$

Then by means of Ampere's law

$$\nabla \times \underline{H}_0 = \underline{J}_0$$

$$\nabla \times \underline{H}_1 = \underline{J}_1 \tag{620}$$

we have

$$(\nabla \times \underline{A}_0) \cdot \underline{H}_0 = \underline{A}_0 \cdot \underline{J}_0 + \nabla \cdot (\underline{A}_0 \times \underline{H}_0) \tag{621}$$

$$(\nabla \times \underline{A}_1) \cdot \underline{H}_1 = \underline{A}_1 \cdot \underline{J}_1 + \nabla \cdot (\underline{A}_1 \times \underline{H}_1) \tag{622}$$

Using the divergence theorem

$$\begin{aligned} \ell L_1 I_1^2 \approx & \int_{V_0} \underline{A}_0 \cdot \underline{J}_0 dV + \int_{S_c} (\underline{A}_0 \times \underline{H}_0) \cdot \underline{n} dS + \int_{S_0} (\underline{A}_0 \times \underline{H}_0) \cdot \underline{n} dS \\ & + \int_{\Delta V} \underline{A}_1 \cdot \underline{J}_1 dV + \oint_{\Delta S} (\underline{A}_1 \times \underline{H}_1) \cdot \underline{n} dS \end{aligned} \tag{623}$$

or

$$\begin{aligned} \ell L_1 I_1^2 \approx & \int_{V_0} \underline{A}_0 \cdot \underline{J}_0 dV - \int_{S_c} \underline{A}_0 \cdot (\underline{n} \times \underline{H}_0)\, dS - \int_{S_0} \underline{A}_0 \cdot (\underline{n} \times \underline{H}_0)\, dS \\ & + \int_{\Delta V} \underline{A}_1 \cdot \underline{J}_1 dV + \oint_{\Delta S} \underline{A}_1 \cdot (\underline{n} \times \underline{H}_1)\, dS \end{aligned} \tag{624}$$

We now drop the volume current term assuming this is applied outside the perfect conductors

$$L_1 I_1^2 \approx - \int_{S_c} \underline{A}_0 \cdot (\underline{n} \times \underline{H}_0)\, dS/\ell - \int_{S_0} \underline{A}_0 \cdot (\underline{n} \times \underline{H}_0)\, dS/\ell$$

$$-\oint_{\Delta S} \underline{A}_1 \cdot (\underline{n} \times \underline{H}_1)\, dS/\ell$$

$$\approx -\int_{S_c} \underline{A}_0 \cdot (\underline{n} \times \underline{H}_0)\, dS/\ell - \int_{S_0} \underline{A}_0 \cdot (\underline{n} \times \underline{H}_0)\, dS/\ell$$

$$-\int_{S_0} \underline{A}_1 \cdot (\underline{n} \times \underline{H}_0)\, dS/\ell - \int_{S_w} \underline{A}_1 \cdot (\underline{n} \times \underline{H}_1)\, dS/\ell - \int_{S_{chassis}} \underline{A}_1 \cdot (\underline{n} \times \underline{H}_1)\, dS/\ell$$

$$\approx \int_{S_c} \underline{A}_0 \cdot \underline{K}_0 dS/\ell + \int_{S_0} \underline{A}_0 \cdot \underline{K}_0 dS/\ell$$

$$-\int_{S_0} \underline{A}_1 \cdot \underline{K}_0 dS/\ell + \int_{S_w} \underline{A}_1 \cdot \underline{K}_1 dS/\ell$$

where we take the auxiliary potential to be z directed $\underline{A}_0 = A_{0z}\underline{e}_z$, S_c is the center conductor surface and S_0 (with a standard integral symbol) represents the outer surface stood off from the braid wires, the surface S_w is the braid wire surfaces, and we have approximated the magnetic field $\underline{H}_1$ by the auxiliary magnetic field $\underline{H}_0$ on the auxiliary surface S_0, and we have dropped the chassis integral because this conductor is taken to be open circuited and we assume the leakage through the braid is small out at this conductor. We take the auxiliary potential to vanish on S_0 and to have a constant value on the perfect center conductor

$$L_1 I_1^2 \approx A_{0z}\left(S_c\right) I_1$$

$$-\int_{S_0} \underline{A}_1 \cdot \underline{K}_0 dS/\ell + \int_{S_w} \underline{A}_1 \cdot \underline{K}_1 dS/\ell$$

The first term is the auxiliary problem magnetic flux per unit length $L_0 I_1$ times the current I_1. The third term would vanish if the vector potential is set consistently with the dynamic electric field of the problem; in general it may not vanish if a static representation is used (as discussed previously). In the second term the local approximations of the vector potential at a distance from the braid wires can be used

$$\underline{A} \sim -\left(\underline{\rho} - \underline{\rho}_0\right) \times \underline{B}_1\left(x_0, y_0\right) + A_b \underline{e}_z$$

$$\sim -\left(\underline{\rho} - \underline{\rho}_0 - \underline{n} A_b / B_1\right) \times \underline{B}_1\left(x_0, y_0\right) \tag{625}$$

where the constant A_b/B_1 is the effective position of the braid in the normal direction due to the interior field $\underline{B}_1$. Taking the axial component and letting $\underline{B}_1 \sim \left(I_1/I_0\right)\underline{B}_0$ we find

$$A_{1z} \sim -\left(\underline{n} \cdot \left(\underline{\rho} - \underline{\rho}_0\right) - A_b/B_0\right) \underline{e}_z \cdot \left[\underline{n} \times \underline{B}_0\left(x_0, y_0\right)\right]\left(I_1/I_0\right) \tag{626}$$

Plugging this into the integral expressions for the field will require us to set $\underline{\rho} \to \underline{\rho}_0$ and hence $\underline{n} \cdot \left(\underline{\rho} - \underline{\rho}_0\right) = 0$ on the auxiliary shield. However, it is convenient to redefine these expansions to use the origin at the braid center point in thickness x_m, y_m rather than the arbitrary auxiliary shield position. Thus

$$\underline{\rho}_m = x_m \underline{e}_x + y_m \underline{e}_y \tag{627}$$

$$\underline{B}_0 \approx \underline{B}_0\left(x_m, y_m\right) \tag{628}$$

$$\underline{A} \sim -\left(\underline{\rho} - \underline{\rho}_m\right) \times \underline{B}_1\left(x_m, y_m\right) + A_b \underline{e}_z$$

$$\sim -\left(\underline{\rho} - \underline{\rho}_m - \underline{n} A_b / B_1\right) \times \underline{B}_1\left(x_m, y_m\right) \tag{629}$$

and again the axial component, letting $\underline{B}_1 \sim \left(I_1/I_0\right) \underline{B}_0$

$$A_{1z} \sim -\left(\underline{n} \cdot \left(\underline{\rho} - \underline{\rho}_m\right) - A_b/B_0\right) \underline{e}_z \cdot \left[\underline{n} \times \underline{B}_0\left(x_m, y_m\right)\right]\left(I_1/I_0\right) \tag{630}$$

The main effect of this selection is to redefine the constant position A_b/B_0 (it changes value) to be relative to the braid center line (a secondary effect is to define the fields $\underline{B}_0$ to be that which would exist on the braid center line if the braid were replace by a solid conductor at this position). Plugging into the preceding expression

$$\left(L_1 - L_0\right) I_1^2 = \Delta L I_1^2 \approx -\int_{S_0} \underline{A}_1 \cdot \underline{K}_0 dS/\ell + \int_{S_w} \underline{A}_1 \cdot \underline{K}_1 dS/\ell$$

$$\approx -\frac{1}{\mu_0} \int_{S_0} \underline{A}_1 \cdot \left(\underline{n} \times \underline{B}_0\right) dS/\ell + \int_{S_w} \underline{A}_1 \cdot \underline{K}_1 dS/\ell$$

$$\approx -\mu_0 \int_{S_0} \left[\underline{n} \cdot \left(\underline{\rho}_0 - \underline{\rho}_m\right) - A_b/B_0\right] \frac{1}{\mu_0^2 I_0^2} \left[\underline{e}_z \cdot \left(\frac{1}{\mu_0 I_0} \underline{n} \times \underline{B}_0\right)\right]^2 dS/\ell$$

$$+ \int_{S_w} \underline{A}_1 \cdot \underline{K}_1 dS/\ell$$

$$\sim \mu_0 \int_{S_0} \left(d_0 + A_b/B_0\right) \frac{1}{\mu_0^2 I_0^2} \underline{B}_0 \cdot \underline{B}_0 dS/\ell = \mu_0 \int_{S_0} \left(d_0 + A_b/B_0\right) \frac{1}{I_0^2} \underline{H}_0 \cdot \underline{H}_0 dS/\ell$$

$$+ \int_{S_w} \underline{A}_1 \cdot \underline{K}_1 dS/\ell \tag{631}$$

where

$$d_0 = \underline{n} \cdot \left(\underline{\rho}_m - \underline{\rho}_0\right) \tag{632}$$

is the distance from the auxiliary shield to the braid center line.

Transfer Inductance Approximate Evaluation The mutual inductance (and transfer inductance per unit length L_T) is given by

$$M I_1 I_2 = L_T \ell I_1 I_2 = \int_V \mu_0 \underline{H}_1 \cdot \underline{H}_2 dV \tag{633}$$

The original volume consists of both regions about the braid wires (inward toward the center conductor as well as outward toward the chassis). Decomposing the volume into the auxiliary volume V_0 from the center conductor outward toward the braid wires plus $\Delta V = V - V_0$

$$MI_1I_2 = L_T\ell I_1 I_2 = \int_{V_0} \mu_0 \underline{H}_1 \cdot \underline{H}_2 dV + \int_{\Delta V} \mu_0 \underline{H}_1 \cdot \underline{H}_2 dV \tag{634}$$

Approximating the field $\underline{H}_1$ in region V_0 by $\underline{H}_0$

$$M = L_T\ell \approx \frac{1}{I_0 I_2} \int_{V_0} \mu_0 \underline{H}_0 \cdot \underline{H}_2 dV + \frac{1}{I_1 I_2} \int_{\Delta V} \mu_0 \underline{H}_1 \cdot \underline{H}_2 dV \tag{635}$$

In the first V_0 integral we use

$$\nabla \cdot (\underline{A}_0 \times \underline{H}_2) = \underline{H}_2 \cdot (\nabla \times \underline{A}_0) - \underline{A}_0 \cdot (\nabla \times \underline{H}_2) = \underline{H}_2 \cdot (\nabla \times \underline{A}_0) - \underline{A}_0 \cdot \underline{J}_2 \tag{636}$$

$$\nabla \times \underline{H}_2 = \underline{J}_2 \tag{637}$$

to find

$$\int_{V_0} \mu_0 \underline{H}_0 \cdot \underline{H}_2 dV = \oint_{S_0} (\underline{A}_0 \times \underline{H}_2) \cdot \underline{n} dS = -\oint_{S_0} \underline{A}_0 \cdot (\underline{n} \times \underline{H}_2)\, dS$$

$$= -\int_{S_0} \underline{A}_0 \cdot (\underline{n} \times \underline{H}_2)\, dS - \int_{S_c} \underline{A}_0 \cdot (\underline{n} \times \underline{H}_2)\, dS$$

$$\sim -\int_{S_0} A_{0z} \underline{e}_z \cdot (\underline{n} \times \underline{H}_2)\, dS - \int_{S_c} A_{0z} \underline{e}_z \cdot (\underline{n} \times \underline{H}_2)\, dS \tag{638}$$

where we have dropped the current term since the integration is taken outside the conductors and the final expression results because the auxiliary field is z directed. We can take the auxiliary potential A_{0z} to vanish on the auxiliary surface S_0 and the first term is zero. The auxiliary potential is constant on the center conductor, but because we again take the center conductor to be open circuited in the 2 problem the net current is zero and the second term vanishes.

In the second ΔV integral we use

$$\nabla \cdot (\underline{A}_2 \times \underline{H}_1) = \underline{H}_1 \cdot (\nabla \times \underline{A}_2) - \underline{A}_2 \cdot (\nabla \times \underline{H}_1) = \underline{H}_1 \cdot (\nabla \times \underline{A}_2) - \underline{A}_2 \cdot \underline{J}_1 \tag{639}$$

$$\nabla \times \underline{H}_1 = \underline{J}_1 \tag{640}$$

to find

$$\int_{\Delta V} \mu_0 \underline{H}_1 \cdot \underline{H}_2 dV = \oint_{\Delta S} (\underline{A}_2 \times \underline{H}_1) \cdot \underline{n} dS = -\oint_{\Delta S} \underline{A}_2 \cdot (\underline{n} \times \underline{H}_1)\, dS$$

$$= -\int_{S_0} \underline{A}_2 \cdot (\underline{n} \times \underline{H}_1)\, dS - \int_{S_w} \underline{A}_2 \cdot (\underline{n} \times \underline{H}_1)\, dS - \int_{S_{chassis}} \underline{A}_2 \cdot (\underline{n} \times \underline{H}_1)\, dS$$

$$\sim -\int_{S_0} \underline{A}_2 \cdot (\underline{n} \times \underline{H}_1)\, dS - \int_{S_w} \underline{A}_2 \cdot (\underline{n} \times \underline{H}_1)\, dS - \int_{S_{chassis}} A_{2z} \underline{e}_z \cdot (\underline{n} \times \underline{H}_1)\, dS \tag{641}$$

where we have dropped the current term by taking the volume to be outside the conductors, and the final result follows because the potential approaches a scalar z-directed quantity far from the local braid region on the chassis. Because the vector potential A_{2z} is a constant on the perfect chassis, and because we take the chassis to be open circuited in the 1 problem, the final integral vanishes

$$L_T \ell I_1 I_2 = -\int_{S_0} \underline{A}_2 \cdot (\underline{n} \times \underline{H}_1)\, dS - \int_{S_w} \underline{A}_2 \cdot (\underline{n} \times \underline{H}_1)\, dS$$

Replacing the magnetic field $\underline{H}_1 \approx \underline{H}_0$ on S_0 gives (note here that $\underline{n}$ points inward on S_0 toward the center conductor since it arose from ΔV and thus the current density on the auxiliary shield is $\underline{K}_0 = \underline{n} \times \underline{H}_0$, however on S_w the unit vector $\underline{n}$ points into the wires and the current density on these wires is $\underline{K}_1 = -\underline{n} \times \underline{H}_1$)

$$\begin{aligned} L_T &\approx -\frac{1}{I_0 I_2} \int_{S_0} \underline{A}_2 \cdot (\underline{n} \times \underline{H}_0)\, dS/\ell - \frac{1}{I_1 I_2} \int_{S_w} \underline{A}_2 \cdot (\underline{n} \times \underline{H}_1)\, dS/\ell \\ &\approx -\frac{1}{I_0 I_2} \int_{S_0} \underline{A}_2 \cdot \underline{K}_0 dS/\ell + \frac{1}{I_1 I_2} \int_{S_w} \underline{A}_2 \cdot \underline{K}_1 dS/\ell \end{aligned} \tag{642}$$

Setting $I_2 = -I_{sh}$ and $\underline{A}_2 = \underline{A}_{sh}$

$$\begin{aligned} L_T &\approx \frac{1}{I_0 I_{sh}} \int_{S_0} \underline{A}_{sh} \cdot (\underline{n} \times \underline{H}_0)\, dS/\ell + \frac{1}{I_1 I_{sh}} \int_{S_w} \underline{A}_{sh} \cdot (\underline{n} \times \underline{H}_1)\, dS/\ell \\ &\approx \frac{1}{I_0 I_{sh}} \int_{S_0} \underline{A}_{sh} \cdot \underline{K}_0 dS/\ell - \frac{1}{I_1 I_{sh}} \int_{S_w} \underline{A}_{sh} \cdot \underline{K}_1 dS/\ell \\ &\approx \frac{1}{I_0 I_{sh}} \int_{S_0} A_{shz} K_{0z} dS/\ell - \frac{1}{I_1 I_{sh}} \int_{S_w} \underline{A}_{sh} \cdot \underline{K}_1 dS/\ell \end{aligned} \tag{643}$$

where in the first term we have used the fact that the auxiliary problem has z directed current density on the surface of the perfect conductors $\underline{K}_0 = K_{0z} \underline{e}_z$. The vector potential from the exterior drive $\underline{A}_{sh}$ can be selected so that on the perfect conductors of S_w we have $\underline{n} \times \underline{A}_{sh} = 0$ because of the vanishing of the tangential electric field on the braid perfect conductors, and then the final term vanishes ($\underline{n}$ points inward in this expression)

$$L_T \approx \frac{1}{I_0 I_{sh}} \int_{S_0} \underline{A}_{sh} \cdot (\underline{n} \times \underline{H}_0)\, dS/\ell \approx \frac{1}{I_0 I_{sh}} \int_{S_0} A_{shz} K_{0z} dS/\ell \tag{644}$$

In this case we then insert the form of the potential in the vicinity of the auxiliary shield (this asymptotic potential is on the interior side of the shield and has $\underline{n}$ pointing outward from the interior region as in (514), but the field $\underline{B}_{sh}$ is on the exterior side of the shield when the shield is replaced by a solid conductor at the auxiliary shield position)

$$A_{shz} \sim -(A_c/B_0)\, \underline{e}_z \cdot [\underline{n} \times \underline{B}_{sh}(x_0, y_0)] \tag{645}$$

to find

$$L_T \approx -\frac{1}{I_0 I_{sh}} \int_{S_0} (A_c/B_0)\, \underline{e}_z \cdot [\underline{n} \times \underline{B}_{sh}(x_0, y_0)]\, K_{0z} dS/\ell$$

$$\approx \int_{S_0} (\mu_0 A_c/B_0)\, \underline{e}_z \cdot \left[\frac{1}{\mu_0 I_{sh}} \underline{n} \times \underline{B}_{sh}(x_0, y_0)\right] \underline{e}_z \cdot \left[\frac{1}{\mu_0 I_0} \underline{n} \times \underline{B}_0(x_0, y_0)\right] dS/\ell \tag{646}$$

which is the same as the previously derived form from reciprocity (526) (for better accuracy and less ambiguity the fields can now be replaced by those evaluated at the braid mean position instead of at the auxiliary shield position, $\underline{n}$ points outward in this expression)

$$L_T \approx \int_{S_0} (\mu_0 A_c/B_0)\, \underline{e}_z \cdot \left[\frac{1}{\mu_0 I_{sh}} \underline{n} \times \underline{B}_{sh}(x_m, y_m)\right] \underline{e}_z \cdot \left[\frac{1}{\mu_0 I_0} \underline{n} \times \underline{B}_0(x_m, y_m)\right] dS/\ell \tag{647}$$

4.5 Magnetic Vector Potential Multipole Representation

The exterior magnetic field must be represented with unknown basis functions to set up the detailed braid solution. One approach is to choose electric currents along the N wire segments of the braid. The vector potential from these is then

$$\underline{A} \approx \sum_{n=1}^{N} A_{z_n} \underline{e}_{z_n} \tag{648}$$

where the unit vectors $\underline{e}_{z_n}$ are along the various N wire segments making up the braid. Multipole moments can then be introduced to match the boundary condition around the circumference of the wires. If we include only the axial multipole currents, the multipole representation can be written as

$$A_{z_n} = \sum_{m=0}^{M} A_{nm} A_{z_n}^{(m)} \tag{649}$$

where $A_{z_n}^{(m)}$ represents the mth line multipole moment of segment n along vector direction $\underline{e}_{z_n}$. This is illustrated in Figure 24. We can write ($\underline{m}^{(0)} = I_n$)

$$A_{z_n}^{(0)} = \frac{\mu_0 I_n}{4\pi} \ln \left[\frac{(z - z_n' + \Delta_n/2) + \sqrt{\rho^2 + (z - z_n' + \Delta_n/2)^2}}{(z - z_n' - \Delta_n/2) + \sqrt{\rho^2 + (z - z_n' - \Delta_n/2)^2}}\right] \tag{650}$$

$$A_{z_n}^{(m)} = \underline{m}^{(0)} \underline{m}^{(1)} ... \underline{m}^{(m)} \cdot \nabla_t^m \left(A_{z_n}^{(0)}/I_n\right) \tag{651}$$

Collecting terms and using (333) this can be written as

$$A_{z_n} = -\frac{\mu_0}{4\pi} \sum_{m=0}^{M} \left[m_e^{(m)} \cos(m\varphi) + m_o^{(m)} \sin(m\varphi)\right] L_m\left(\rho, \frac{\partial}{\partial \rho}\right) \ln \left[\frac{(s - s_n/2) + \sqrt{\rho^2 + (s - s_n/2)^2}}{(s + s_n/2) + \sqrt{\rho^2 + (s + s_n/2)^2}}\right] \tag{652}$$

which has $2M$ unknowns.

When the wires are perfectly conducting the normal component of the magnetic field vanishes on the surface

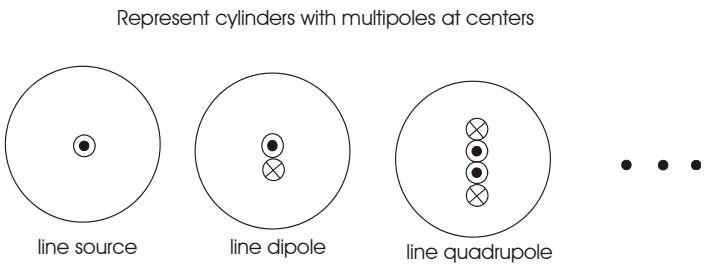

Figure 24. Illustration of line current multipole moments.

$$H_n = 0 \tag{653}$$

or using

$$\nabla \times \underline{A} = \underline{B} = \mu_0 \underline{H} \tag{654}$$

we can enforce

$$\underline{e}_{n'} \cdot \nabla \times \underline{A} = 0 \tag{655}$$

at $2M$ discrete angles

$$\varphi_{m'} = \pi m'/M \ , \ m' = 0, ..., 2M-1 \tag{656}$$

Noting that

$$\underline{e}_\rho \cdot (\nabla \times \underline{A}) = \frac{1}{\rho}\frac{\partial}{\partial \varphi} A_z - \frac{\partial}{\partial z} A_\varphi \tag{657}$$

$$A_\varphi = \underline{e}_\varphi \cdot \underline{A} = -A_x \sin\varphi + A_y \cos\varphi \tag{658}$$

we find that (the first expression follows because $\oint_C \underline{A} \cdot \underline{d\ell} = \Phi$ but there is no magnetic flux within the perfectly conducting wire, the second expression follows because A_z is single valued)

$$\int_0^{2\pi} A_\varphi a d\varphi = 0 \tag{659}$$

$$\int_0^{2\pi} \frac{\partial}{\partial \varphi} A_z d\varphi = A_z(2\pi) - A_z(0) = 0 \tag{660}$$

on wire n', where A_{nm} are amplitudes adjusted to match the wire boundary conditions. Hence in the perfectly conducting case there is no $m = 0$ term present in this condition. The $m = 0$ term of each A_{z_n} is tied to the net current being carried by the nth wire segment. However, the preceding azimuthal derivative eliminates this term from the $H_\rho(\rho = a)$ condition. Hence the enforcement of the condition can be done at $2M - 1$ points (for example, leaving out the $m' = 0$ position). This makes physical sense because the net wire current cannot be determined by the fact that the surface of the wire is a perfect electric conductor. The net wire currents must be determined by the combination of a zero flux condition between wires over a period (so that there is no accumulation of potential difference over many periods) in addition to the imposed injected total current of the braid.

A problem with the preceding approach is that there will in general be transverse components of the electric current density on the wire surface. Hence, so far this setup represents an approximation and extra basis functions must be included for a more rigorous representation of the field. In fact we might have restricted the treatment to forcing A_z to be constant around the azimuth of the wire (the first term of the normal magnetic field expression (657)) instead of nulling the normal magnetic field from both terms. Rather than proceeding to develop such transverse basis functions we switch to a mixed potential representation in the next subsection.

4.6 Scalar Potential And Magnetic Vector Potential Construction

To set up the magnetic braid problem in a manner similar to the electric problem, we use a combination of the magnetic vector potential and magnetic scalar potential. The magnetic vector potential is used to represent the net current carried by each braid wire, while the magnetic scalar potential matches the boundary conditions on the wire surface. Because the magnetic scalar potential represents the difference problem where the braid wire carries no net current, the branch surfaces from the braid wires are avoided, while retaining the simplicity of the scalar description. We take the vector potential to be generated by current filaments at the centers of the wires and denote this potential as $\underline{A}_f$. This formulation includes both axial and transverse components of the current on the wire surfaces. The magnetic induction and magnetic field are then given by

$$\underline{B} = \mu_0 \underline{H} = -\mu_0 \nabla \phi_m + \nabla \times \underline{A}_f \tag{661}$$

In the perfectly conducting case the magnetic scalar potential is used to restore the boundary condition of zero normal magnetic field on the metallic surfaces

$$\underline{n} \cdot \underline{H} = 0 \tag{662}$$

or

$$\mu_0 \underline{n} \cdot \nabla \phi_m = \underline{n} \cdot \nabla \times \underline{A}_f \tag{663}$$

on each of the braid wire segments. The magnetic flux Φ through a surface S from this representation is the sum

$$\Phi = \oint_C \underline{A}_f \cdot \underline{d\ell} - \mu_0 \int_S \nabla \phi_m \cdot \underline{n} dS \tag{664}$$

The φ directed magnetic flux of interest is that between a single wire of a single strip of the braid and the center conductor of the coax.

We assume that there is electrical contact between the wires of a carrier strip at the drive point but no contact between strips and little additional contact between braid wires so that the wire currents remain fixed along their length. In the cylindrical coax the current I/N_s is thus injected into each strip, and by symmetry this is the same on all the N_s strips. The current injected into each wire segment I_n (this will not change over the period of segments) of the carrier strip is then determined by enforcing no net magnetic flux between wires in the carrier over a braid period ℓ.

4.6.1 Flux Constants

From the preceding vector potential formulation, and in particular the local planar approximation, we see that our objective in that section was to evaluate the magnetic flux per unit length constants A_c and A_b. These magnetic flux constants entered the preceding vector potential formulation to find the transfer impedance and self-impedance corrections. To find these constants using the present mixed potential approach we can drive a planar periodic representation of the braid with a uniform field that is tangential to the braid surface as illustrated in Figure 25

$$\underline{B}_0 = \mu_0 \underline{H}_0 \tag{665}$$

When driven from the exterior side of the braid we evaluate

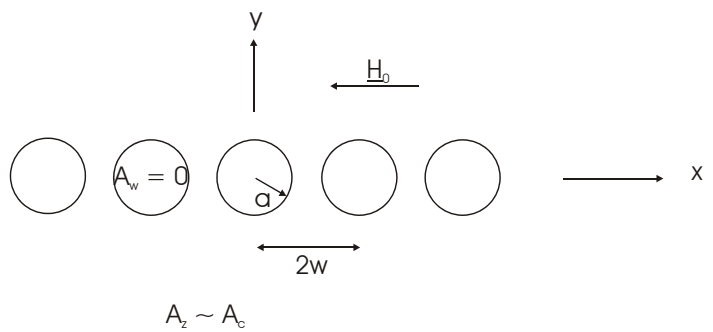

Figure 25. Illustration of planar braid driven by uniform magnetic field to determine magnetic flux per unit length constants.

$$-\ell A_c = \oint_C \underline{A}_f \cdot \underline{d\ell} - \mu_0 \int_S \nabla\phi_m \cdot \underline{n} dS \tag{666}$$

where the surface S corresponds to that of Figure 19 and the contour surrounds S. In addition, when the planar braid is driven by the field from the interior side, then using the same contour

$$\ell\left(d_0 B_0 + A_b\right) = \oint_C \underline{A}_f \cdot \underline{d\ell} - \mu_0 \int_S \nabla\phi_m \cdot \underline{n} dS \tag{667}$$

where d_0 is the normal distance from the center plane of the braid to the return leg of the contour (the edge of the surface S).

4.6.2 Inductance Per Unit Length

We now show that we can evaluate the self impedance from the energy argument directly using the mixed potentials on the surfaces of the wires. The inductance can be found from the stored magnetic energy by means of

$$\frac{1}{2}LI^2 = \int_V \frac{1}{2}\mu_0 H^2 dV \tag{668}$$

Taking

$$\mu_0 \underline{H} = \nabla \times \underline{A}_f - \mu_0 \nabla\phi_m \tag{669}$$

we find

$$\frac{1}{2}LI^2 = \frac{1}{2\mu_0}\int_V \left[\left(\nabla \times \underline{A}_f\right) \cdot \left(\nabla \times \underline{A}_f\right) - 2\mu_0 \nabla\phi_m \cdot \nabla \times \underline{A}_f + \mu_0^2 \nabla\phi_m \cdot \nabla\phi_m\right] dV \tag{670}$$

Now using the identities

$$\nabla \cdot \left(\underline{A}_f \times \nabla \times \underline{A}_f\right) = \left(\nabla \times \underline{A}_f\right) \cdot \left(\nabla \times \underline{A}_f\right) - \underline{A}_f \cdot \left(\nabla \times \nabla \times \underline{A}_f\right) \tag{671}$$

$$\nabla \cdot \left(\phi_m \nabla\phi_m\right) = \nabla\phi_m \cdot \nabla\phi_m + \phi_m \nabla^2 \phi_m \tag{672}$$

$$\nabla \cdot \left(\phi_m \nabla \times \underline{A}_f\right) = 0 + \nabla\phi_m \cdot \nabla \times \underline{A}_f \tag{673}$$

Then by means of

$$\nabla \times \nabla \times \underline{A}_f = 0 \tag{674}$$

$$\nabla^2 \phi_m = 0 \tag{675}$$

and using the divergence theorem

$$LI^2 = \frac{1}{\mu_0}\oint_S \left(\underline{A}_f \times \nabla \times \underline{A}_f - 2\mu_0 \phi_m \nabla \times \underline{A}_f + \mu_0^2 \phi_m \nabla\phi_m\right) \cdot \underline{n} dS \tag{676}$$

or

$$LI^2 = \frac{1}{\mu_0} \oint_S \left(\underline{A}_f \times \underline{B}_f - \mu_0 \phi_m \underline{B}_f - \mu_0 \phi_m \underline{B} \right) \cdot \underline{n} dS \tag{677}$$

where we combined one of the middle terms with the final term to obtain the total magnetic induction (we could also have twice the final term with the middle terms, leaving minus the final term)

$$LI^2 = \frac{1}{\mu_0} \oint_S \left(\underline{A}_f \times \underline{B}_f - 2\mu_0 \phi_m \underline{B} - \mu_0^2 \phi_m \nabla \phi_m \right) \cdot \underline{n} dS \tag{678}$$

Noting that on the perfect conductors that the final term vanishes and it also vanishes from periodicity on the end surfaces (with the sign of $\underline{n}$ reversing) we have

$$LI^2 = \frac{1}{\mu_0} \oint_S \left(\underline{A}_f \times \underline{B}_f - \mu_0 \phi_m \underline{B}_f \right) \cdot \underline{n} dS \tag{679}$$

or

$$LI^2 = \frac{1}{\mu_0} \oint_S \left(\underline{A}_f \times \underline{B}_f - \mu_0^2 \phi_m \nabla \phi_m \right) \cdot \underline{n} dS \tag{680}$$

and then

$$LI^2 = -\frac{1}{\mu_0} \oint_S \left[\underline{A}_f \cdot \left(\underline{n} \times \underline{B}_f \right) + \mu_0 \phi_m \underline{n} \cdot \underline{B}_f \right] dS \tag{681}$$

or

$$LI^2 = -\frac{1}{\mu_0} \oint_S \left[\underline{A}_f \cdot \left(\underline{n} \times \underline{B}_f \right) - \mu_0^2 \phi_m \underline{n} \cdot \nabla \phi_m \right] dS \tag{682}$$

If we take the volume V to consist of the region surrounding the perfect center conductor, excluding the braid wires, and including the outer space to the return chassis, then the surface integral can be broken into the various parts

$$LI^2 = I_1 A_{fz}(S_c) - \frac{1}{\mu_0} \int_{S_w} \left[\underline{A}_f \cdot \left(\underline{n} \times \underline{B}_f \right) + \mu_0 \phi_m \underline{n} \cdot \underline{B}_f \right] dS$$

$$-\frac{1}{\mu_0} \int_{S_{chassis}} \left[\underline{A}_f \cdot \left(\underline{n} \times \underline{B}_f \right) + \mu_0 \phi_m \underline{n} \cdot \underline{B}_f \right] dS \tag{683}$$

or

$$LI^2 = I_1 A_{fz}(S_c) - \frac{1}{\mu_0} \int_{S_w} \left[\underline{A}_f \cdot \left(\underline{n} \times \underline{B}_f \right) - \mu_0^2 \phi_m \underline{n} \cdot \nabla \phi_m \right] dS$$

$$-\frac{1}{\mu_0} \int_{S_{chassis}} \left[\underline{A}_f \cdot \left(\underline{n} \times \underline{B}_f \right) - \mu_0^2 \phi_m \underline{n} \cdot \nabla \phi_m \right] dS \tag{684}$$

where we again drop the end surfaces when this is applied over a short axial period ℓ; because the currents and fields are periodic over a short axial distance ℓ (including the filament current) we assume that the

filament vector potential $\underline{A}_f$ is constructed to be periodic also; because the magnetic flux is periodic over a short axial period ℓ we assume that the magnetic scalar part ϕ_m is also. In the preceding expression we have taken $\underline{n} \times \underline{B}_f$ to be predominantly z directed on the perfect center conductor, A_{fz} to be a constant around the center conductor over a short axial distance, and $\underline{n} \cdot \underline{B}_f$ to be small on the center conductor.

Self Inductance Per Unit Length For the moment let us take the outer loop to be open circuited, and assume the magnetic field near the chassis is negligible, so that we can drop the final integral (the current I in this case is identified as the current on the center conductor I_1)

$$\ell L_1 I_1^2 = \ell I_1 A_{fz}(S_c) - \frac{1}{\mu_0} \int_{S_w} \left[\underline{A}_f \cdot \left(\underline{n} \times \underline{B}_f \right) + \mu_0 \phi_m \underline{n} \cdot \underline{B}_f \right] dS \tag{685}$$

or

$$\ell L_1 I_1^2 = \ell I_1 A_{fz}(S_c) - \frac{1}{\mu_0} \int_{S_w} \left[\underline{A}_f \cdot \left(\underline{n} \times \underline{B}_f \right) - \mu_0^2 \phi_m \underline{n} \cdot \nabla \phi_m \right] dS \tag{686}$$

On the surface of the braid wires we define the surface current density $\underline{K}_f$ (note here that the unit normal $\underline{n}$ points into the conductor) by

$$\underline{K}_f = -\underline{n} \times \underline{H}_f \tag{687}$$

and then

$$L_1 I_1^2 = I_1 A_{fz}(S_c) + \int_{S_w} \left(\underline{A}_f \cdot \underline{K}_f - \phi_m \underline{n} \cdot \underline{B}_f \right) dS/\ell$$

$$= I_1 A_{fz}(S_c) + \int_{S_w} \left[\underline{A}_f \cdot \underline{K}_f - \mu_0 \phi_m \left(\underline{n} \cdot \underline{H}_f \right) \right] dS/\ell \tag{688}$$

or

$$L_1 I_1^2 = I_1 A_{fz}(S_c) + \int_{S_w} \left[\underline{A}_f \cdot \underline{K}_f + \mu_0 \phi_m \underline{n} \cdot \nabla \phi_m \right] dS/\ell \tag{689}$$

Now in the cylindrical case by symmetry we assume that the contribution from each strip carrier is the same (only true on cylindrical conductor geometry, not on asymmetric geometries)

$$L_1 I_1^2 = I_1 A_{fz}(S_c) + N_s \int_{S_s} \left(\underline{A}_f \cdot \underline{K}_f - \phi_m \underline{n} \cdot \underline{B}_f \right) dS/\ell \tag{690}$$

or

$$L_1 I_1^2 = I_1 A_{fz}(S_c) + N_s \int_{S_s} \left[\underline{A}_f \cdot \underline{K}_f + \mu_0 \phi_m \underline{n} \cdot \nabla \phi_m \right] dS/\ell \tag{691}$$

which can be written as

$$L_1 I_1 = A_{fz}(S_c) + \int_{S_s} \left[\underline{A}_f \cdot \left(\underline{K}_f \frac{N_s}{I_1} \right) - \mu_0 \phi_m \left(\frac{N_s}{I_1} \underline{n} \cdot \underline{H}_f \right) \right] dS/\ell \tag{692}$$

or

$$L_1 I_1 = A_{fz}(S_c) + \int_{S_s} \left[\underline{A}_f \cdot \left(\underline{K}_f \frac{N_s}{I_1} \right) + \mu_0 \phi_m \frac{N_s}{I_1} \underline{n} \cdot \nabla \phi_m \right] dS/\ell \tag{693}$$

These results define the inductance per unit length L_1 in terms of the potential solution of the coax. The surface integral is over the wire conductors in the braid carrier strips for an axial braid period.

4.6.3 Transfer Inductance Per Unit Length

The magnetic flux current relations in a two port can be written as

$$\Phi_1 = L_{11} I_1 + M_{12} I_2 \tag{694}$$

$$\Phi_2 = M_{21} I_1 + L_{22} I_2 \tag{695}$$

From Faraday's law

$$\nabla \times \underline{E} = -\frac{\partial}{\partial t} \underline{B} \tag{696}$$

or in integral form

$$\oint_C \underline{E} \cdot \underline{d\ell} = -\int_S \frac{\partial}{\partial t} \underline{B} dS \tag{697}$$

we can write

$$\frac{\partial V}{\partial z} = -\frac{\partial \Phi}{\partial t} \tag{698}$$

The voltage current relations in a two port inductive circuit can thus be written as

$$V_1(z+\ell) - V_1(z) = -\frac{\partial \Phi_1}{\partial t} = -L_{11} \frac{dI_1}{dt} - M_{12} \frac{dI_2}{dt} \tag{699}$$

$$V_2(z+\ell) - V_2(z) = -\frac{\partial \Phi_2}{\partial t} = -M_{21} \frac{dI_1}{dt} - L_{22} \frac{dI_2}{dt} \tag{700}$$

In a reciprocal media the cross terms are equal $M_{12} = M_{21} = M$. The power removed from a periodic section is minus the derivative of the magnetic energy

$$[V_1(z+\ell) - V_1(z)] I_1 + [V_2(z+\ell) - V_2(z)] I_2 = -\frac{d}{dt} \left(\frac{1}{2} L_{11} I_1^2 + M I_1 I_2 + \frac{1}{2} L_{22} I_2^2 \right) \tag{701}$$

Taking two sources we write the total magnetic field as

$$\underline{H} = \underline{H}_1 + \underline{H}_2 \tag{702}$$

The 1 problem is defined to have a current I_1 on the center conductor and z directed with return on the braid and the chassis open circuited (no current); the 2 problem is defined to have a current I_2 on the chassis and z directed with return on the braid and the center conductor open circuited (no current). Equating the energies from the circuit and field gives

$$\frac{1}{2}L_{11}I_1^2 + MI_1I_2 + \frac{1}{2}L_{22}I_2^2 = \int_V \frac{1}{2}\mu_0 H^2 dV$$

$$= \int_V \frac{1}{2}\mu_0 H_1^2 dV + \int_V \mu_0 \underline{H}_1 \cdot \underline{H}_2 dV + \int_V \frac{1}{2}\mu_0 H_2^2 dV \tag{703}$$

The self inductances $L_{11} = \ell L_1$ and $L_{22} = \ell L_{sh}$ (where L_{sh} is the inductance per unit length in the outer transmission line) are the first and final terms

$$L_{11}I_1^2 = \ell L_1 I_1^2 = \int_V \frac{1}{2}\mu_0 H_1^2 dV \tag{704}$$

$$L_{22}I_2^2 = \ell L_{sh} I_2^2 = \int_V \frac{1}{2}\mu_0 H_2^2 dV \tag{705}$$

The mutual inductance M is the cross term

$$M = \frac{1}{I_1 I_2}\int_V \mu_0 \underline{H}_1 \cdot \underline{H}_2 dV \tag{706}$$

We intend to take conductor 1 as the center conductor and conductor 2 as the chassis. The sign of the mutual inductance in the preceding expressions depends on whether the magnetic flux generated in circuit 1 by the current I_2 has the same sign as the flux generated by current 1 in circuit 1. Now setting the interior to be open circuited and $I_2 = I_{chassis} = -I_{sh}$ (I_{sh} is the z directed braid current) with $\Phi_2 = -\Phi_{sh}$, and defining the transfer inductance by means of $M = L_T \ell$, we find

$$\Phi_1 = L_{11}I_1 - MI_{sh} = L_1 \ell I_1 - L_T \ell I_{sh} \tag{707}$$

$$\Phi_{sh} = -MI_1 + L_{22}I_{sh} = -L_T \ell I_1 + \ell L_{sh} I_{sh} \tag{708}$$

The voltage along the inner transmission line from Faraday's law applied over a period is

$$V_1(\ell + z) - V_1(z) = v_s \ell = -\frac{\partial}{\partial t}\Phi_1 = i\omega\Phi_1 \tag{709}$$

where the voltage source per unit length (setting $I_1 = 0$ when the center conductor is open circuited) is

$$v_s = -i\omega L_T I_{sh} = Z_T I_{sh} \tag{710}$$

and the transfer inductance per unit length, replacing I_2 by $-I_{sh}$ and $\underline{H}_2$ by $\underline{H}_{sh}$ is

$$L_T = -\frac{1}{\ell I_1 I_{sh}}\int_V \mu_0 \underline{H}_1 \cdot \underline{H}_{sh} dV \tag{711}$$

Now if we suppose the braid has a large aperture we see that the penetrant magnetic field $\underline{H}_{sh}$, for positive shield current I_{sh}, produces a magnetic flux around the center conductor of the inner line with the opposite sign as that produced by the center conductor current I_1 and thus the minus sign in (707) introduced in the mutual impedance, or in the transfer impedance to make it positive for low optical coverage cables (the negative porpoising contribution will result in negative values of L_T for high optical coverage cables). The positive reference of the source voltage is on the positive z side of the elements since the inner transmission line equation is written as

$$\frac{dV_1}{dz} + Z_1 I_1 = v_s \tag{712}$$

Taking

$$\underline{B} = \mu_0 \underline{H} = \nabla \times \underline{A}_f - \mu_0 \nabla \phi_m \tag{713}$$

gives

$$L_T = \frac{1}{\ell I_1 I_2} \int_V \mu_0 \underline{H}_1 \cdot \underline{H}_2 dV = \frac{1/\mu_0}{\ell I_1 I_2} \int_V \left(\nabla \times \underline{A}_{f1} - \mu_0 \nabla \phi_{m1}\right) \cdot \left(\nabla \times \underline{A}_{f2} - \mu_0 \nabla \phi_{m2}\right) dV \tag{714}$$

Now we have the identities (and their permutation in indices)

$$\nabla \cdot \left(\underline{A}_{f2} \times \nabla \times \underline{A}_{f1}\right) = \left(\nabla \times \underline{A}_{f1}\right) \cdot \left(\nabla \times \underline{A}_{f2}\right) - \underline{A}_{f2} \cdot \left(\nabla \times \nabla \times \underline{A}_{f1}\right)$$

$$= \left(\nabla \times \underline{A}_{f1}\right) \cdot \left(\nabla \times \underline{A}_{f2}\right) - \mu_0 \underline{A}_{f2} \cdot \underline{J}_{f1} \tag{715}$$

$$\nabla \cdot \left(\phi_{m2} \nabla \phi_{m1}\right) = \nabla \phi_{m1} \cdot \nabla \phi_{m2} + \phi_{m2} \nabla^2 \phi_{m1}$$

$$= \nabla \phi_{m1} \cdot \nabla \phi_{m2} \tag{716}$$

$$\nabla \cdot \left(\phi_{m2} \nabla \times \underline{A}_{f1}\right) = 0 + \nabla \phi_{m2} \cdot \nabla \times \underline{A}_{f1} \tag{717}$$

where we have used Ampere's law

$$\nabla \times \underline{H} = \underline{J} \tag{718}$$

or

$$\nabla \times \nabla \times \underline{A} = \mu_0 \underline{J} \tag{719}$$

and Gauss's law

$$\nabla \cdot \underline{B} = 0 \tag{720}$$

$$L_T = \frac{1/\mu_0}{\ell I_1 I_2} \int_V \left(\nabla \times \underline{A}_{f1}\right) \cdot \left(\nabla \times \underline{A}_{f2}\right) dV$$

$$-\frac{1}{\ell I_1 I_2} \int_V \left(\nabla \times \underline{A}_{f1}\right) \cdot \nabla \phi_{m2} dV$$

$$-\frac{1}{\ell I_1 I_2} \int_V \nabla \phi_{m1} \cdot \left(\nabla \times \underline{A}_{f2}\right) dV$$

$$+\frac{\mu_0}{\ell I_1 I_2}\int_V \nabla\phi_{m1}\cdot\nabla\phi_{m2}dV \tag{721}$$

or

$$L_T = \frac{1}{I_1 I_2}\oint_S \left(\underline{A}_{f2}\times\underline{H}_{f1}\right)\cdot\underline{n}dS/\ell + \frac{1}{I_1 I_2}\int_V \underline{A}_{f2}\cdot\underline{J}_{f1}dV/\ell$$

$$-\frac{1}{I_1 I_2}\oint_S \phi_{m2}\left(\nabla\times\underline{A}_{f1}\right)\cdot\underline{n}dS/\ell$$

$$-\frac{1}{I_1 I_2}\oint_S \phi_{m1}\left(\nabla\times\underline{A}_{f2}-\mu_0\nabla\phi_{m2}\right)\cdot\underline{n}dS/\ell \tag{722}$$

The final term vanishes on all perfectly conducting surfaces since there is no normal magnetic field present $\underline{H}_2\cdot\underline{n}=0$. Hence we find

$$L_T = -\frac{1}{I_1 I_2}\oint_S \left[\underline{A}_{f2}\cdot\left(\underline{n}\times\underline{H}_{f1}\right)+\mu_0\phi_{m2}\underline{H}_{f1}\cdot\underline{n}\right]dS/\ell + \frac{1}{I_1 I_2}\int_V \underline{A}_{f2}\cdot\underline{J}_{f1}dV/\ell \tag{723}$$

If we apply this to a volume outside the conductors then the volume integral vanishes

$$L_T = -\frac{1}{I_1 I_2}\oint_S \left[\underline{A}_{f2}\cdot\left(\underline{n}\times\underline{H}_{f1}\right)+\mu_0\phi_{m2}\underline{H}_{f1}\cdot\underline{n}\right]dS/\ell$$

$$= -\frac{1}{I_1 I_2}\oint_S \left[\underline{A}_{f1}\cdot\left(\underline{n}\times\underline{H}_{f2}\right)+\mu_0\phi_{m1}\underline{H}_{f2}\cdot\underline{n}\right]dS/\ell \tag{724}$$

where we have also introduced the final reciprocal expression in the second equality. Now using the first expression and changing from $I_2=-I_{sh}$, $\underline{H}_{f2}=\underline{H}_{fsh}$, and $\phi_{m2}=\phi_{msh}$ gives

$$L_T = \frac{1}{I_1 I_{sh}}\oint_S \left[\underline{A}_{fsh}\cdot\left(\underline{n}\times\underline{H}_{f1}\right)+\mu_0\phi_{msh}\underline{H}_{f1}\cdot\underline{n}\right]dS/\ell \tag{725}$$

Noting that there is no net current on the chassis in the 1 problem, it is likely that we can eliminate the chassis from the surface integral by taking the filament contribution in the chassis to have a magnetic field aligned with the surface.

We can alternatively follow the steps

$$L_T = \frac{1/\mu_0}{\ell I_1 I_2}\int_V \left(\nabla\times\underline{A}_{f1}-\mu_0\nabla\phi_{m1}\right)\cdot\left(\nabla\times\underline{A}_{f2}-\mu_0\nabla\phi_{m2}\right)dV$$

$$= \frac{1/\mu_0}{\ell I_1 I_2}\int_V \nabla\cdot\left[\underline{A}_{f2}\times\left(\nabla\times\underline{A}_{f1}-\mu_0\nabla\phi_{m1}\right)-\mu_0\phi_{m2}\left(\nabla\times\underline{A}_{f1}-\mu_0\nabla\phi_{m1}\right)\right]dV$$

$$+\frac{1/\mu_0}{\ell I_1 I_2}\int_V \left[\underline{A}_{f2}\cdot\left(\nabla\times\nabla\times\underline{A}_{f1}\right)\right]dV$$

$$= \frac{1/\mu_0}{\ell I_1 I_2} \oint_S \underline{n} \cdot \left[\underline{A}_{f2} \times \left(\nabla \times \underline{A}_{f1} - \mu_0 \nabla \phi_{m1} \right) - \mu_0 \phi_{m2} \left(\nabla \times \underline{A}_{f1} - \mu_0 \nabla \phi_{m1} \right) \right] dS$$

$$+ \frac{1}{\ell I_1 I_2} \int_V \underline{A}_{f2} \cdot \underline{J}_{f1} dV \tag{726}$$

Now because the total magnetic field satisfies $\underline{n} \cdot \underline{H}_1 = 0$ on the perfect electric conductor surfaces we can drop the second term in brackets

$$L_T = \frac{1}{\ell I_1 I_2} \oint_S \underline{n} \cdot \left(\underline{A}_{f2} \times \underline{H}_1 \right) dS + \frac{1}{\ell I_1 I_2} \int_V \underline{A}_{f2} \cdot \underline{J}_{f1} dV$$

$$= -\frac{1}{I_1 I_2} \oint_S \underline{A}_{f2} \cdot \left(\underline{n} \times \underline{H}_1 \right) dS/\ell + \frac{1}{I_1 I_2} \int_V \underline{A}_{f2} \cdot \underline{J}_{f1} dV/\ell \tag{727}$$

If we choose the volume V to be outside of the conductors we can drop the final term

$$L_T = -\frac{1}{I_1 I_2} \oint_S \underline{A}_{f2} \cdot \left(\underline{n} \times \underline{H}_1 \right) dS/\ell \tag{728}$$

In the perfect electric conductor case we can write this as

$$L_T = \frac{1}{I_1 I_2} \oint_S \underline{A}_{f2} \cdot \underline{K}_1 dS/\ell = \frac{1}{I_1 I_2} \oint_S \underline{A}_{f1} \cdot \underline{K}_2 dS/\ell \tag{729}$$

where we have included the result with permuted indices as the final equality. Now using the first expression and changing from $I_2 = -I_{sh}$, $\underline{A}_{f2} = \underline{A}_{fsh}$, gives

$$L_T = -\frac{1}{I_1 I_{sh}} \oint_S \underline{A}_{fsh} \cdot \underline{K}_1 dS/\ell \tag{730}$$

If we take the filament in such a place that the potential on the chassis is asymptotic to $\underline{A}_{fsh} \sim A_{fshz} \underline{e}_z$ with A_{fshz} asymptotically constant on the chassis, then with no net current on the chassis in the 1 problem the surface integral will vanish and we find

$$L_T = -\frac{1}{I_1 I_{sh}} \int_{S_c} \underline{A}_{fsh} \cdot \underline{K}_1 dS/\ell - \frac{1}{I_1 I_{sh}} \int_{S_w} \underline{A}_{fsh} \cdot \underline{K}_1 dS/\ell \tag{731}$$

Notice that this form of the transfer inductance only requires us to have knowledge of the filament potential as well as the current density on the conductors.

The transfer inductance in this perfectly conducting case should also be connected to the magnetic flux between the center conductor and braid wires $\Phi_1 = \Phi_T$ with $I_1 = 0$

$$-\ell L_T I_{sh} = \Phi_T = \int_S \underline{B}_{sh} \cdot \underline{n} dS \tag{732}$$

when driven by an electric shield current $I_{chassis} = -I_{sh}$.

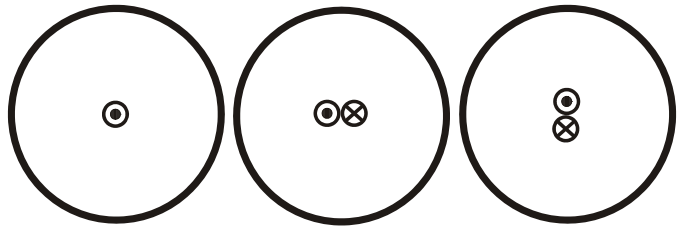

Figure 26. Line charge multipole moments. We do not use the monopole moment but instead use a current filament.

4.6.4 Braid Wire Current Distribution

The above argument of zero net flux between braid wires, for determining the braid wire current distribution can be replaced by an energy argument based on (681) or (682). We determine the braid wire current distribution by minimizing the preceding magnetic energy

$$0 = \frac{\partial}{\partial I_j}\left(LI^2\right) = -\frac{1}{\mu_0}\frac{\partial}{\partial I_j}\oint_S \left[\underline{A}_f \cdot \left(\underline{n} \times \underline{B}_f\right) + \mu_0 \phi_m \underline{n} \cdot \underline{B}_f\right] dS \ , \ j = 2, ..., N_s N_w \tag{733}$$

or

$$0 = \frac{\partial}{\partial I_j}\left(LI^2\right) = -\frac{1}{\mu_0}\frac{\partial}{\partial I_j}\oint_S \left[\underline{A}_f \cdot \left(\underline{n} \times \underline{B}_f\right) - \mu_0^2 \phi_m \underline{n} \cdot \nabla \phi_m\right] dS \ , \ j = 2, ..., N_s N_w \tag{734}$$

with the total braid wire current given

$$\sum_{j=1}^{N_w N_s} I_j = I \tag{735}$$

and N_w is the number of wires in a carrier strip and N_s is the number of carrier strips. This would have to be applied to determine the braid wire current distribution both for the problem where the inner coax is driven $I = I_c$ with $I_{sh} = 0$ and for the problem where the outer coax is driven $I = I_{sh}$ with $I_c = 0$. This approach results in a system of equations where the coefficients only involve integration on the conductor surfaces S. This is useful in this mixed potential approach to avoid having to integrate the magnetic flux off the conductor surfaces.

4.6.5 Magnetic Multipole Representation

With the preceding mixed potential representation we need the magnetic scalar potential representation for a sequence of magnetic line charge multipoles as illustrated in Figure 26.

The magnetic scalar potential will be represented by a magnetic multipole summation similar to the electric problem and by an electric current filament (and the field given by the magnetic vector potential). The solution for a varying magnetic line charge $q_m(z)$ is

$$\phi_m = \frac{1}{4\pi\mu_0}\int q_m(z')\frac{dz'}{|\underline{r} - \underline{r}'|} \tag{736}$$

If the charge is discretized as pulses of strength q_n, length Δ_n, centered at z'_n, we can write

$$\phi_m = \frac{q_{mn}}{4\pi\mu_0}\int_{z'_n-\Delta_n/2}^{z'_n+\Delta_n/2}\frac{dz'}{\sqrt{\rho^2+(z-z')^2}} = \frac{q_{mn}}{4\pi\mu_0}\int_{(z-z'_n-\Delta_n/2)/\rho}^{(z-z'_n+\Delta_n/2)/\rho}\frac{du}{\sqrt{1+u^2}}$$

$$= \frac{q_{mn}}{4\pi\mu_0}\left[\text{Arcsinh}\left(\frac{z-z'_n+\Delta_n/2}{\rho}\right) - \text{Arcsinh}\left(\frac{z-z'_n-\Delta_n/2}{\rho}\right)\right]$$

$$= \frac{q_{mn}}{4\pi\mu_0}\ln\left[\frac{(z-z'_n+\Delta_n/2)+\sqrt{\rho^2+(z-z'_n+\Delta_n/2)^2}}{(z-z'_n-\Delta_n/2)+\sqrt{\rho^2+(z-z'_n-\Delta_n/2)^2}}\right] \tag{737}$$

or

$$\phi_{mn}^{(0)} = \frac{p_m^{(0)}}{4\pi\mu_0}\ln\left[\frac{(z-z'_n+\Delta_n/2)+\sqrt{\rho^2+(z-z'_n+\Delta_n/2)^2}}{(z-z'_n-\Delta_n/2)+\sqrt{\rho^2+(z-z'_n-\Delta_n/2)^2}}\right] \tag{738}$$

The magnetic multipole moments are defined in a similar way to the electric problem. Note that we expect $q_{mn} = \underline{p}_m^{(0)}$ to vanish in the case of perfectly conducting braid wires, but we retain it since it may be required in the finitely conducting case, where magnetic flux could enter from a neighboring segment.

$$\phi_{mtot} = \sum_{n=1}^{N}\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\phi_{mn} = \sum_{n=1}^{N}\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\sum_{m=0}^{M}\phi_{mn}^{(m)} = \sum_{n=1}^{N}\frac{1}{4\pi\mu_0}\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}$$

$$\sum_{m=0}^{M}\underline{p}_m^{(0)}...\underline{p}_m^{(m)}\cdot\nabla_t^m\ln\left[\frac{(s-s_n/2-ju_{sn}-kv_{sn})+\sqrt{\left|\underline{\rho}-j\underline{u}_{\rho n}-k\underline{v}_{\rho n}\right|^2+(s-s_n/2-ju_{sn}-kv_{sn})^2}}{(s+s_n/2-ju_{sn}-kv_{sn})+\sqrt{\left|\underline{\rho}-j\underline{u}_{\rho n}-k\underline{v}_{\rho n}\right|^2+(s+s_n/2-ju_{sn}-kv_{sn})^2}}\right]$$

$$= \sum_{n=1}^{N}\frac{1}{4\pi\mu_0}\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}$$

$$\sum_{m=0}^{M}\underline{p}_m^{(0)}...\underline{p}_m^{(m)}\cdot\nabla_t^m\ln\left[\frac{(s-s_j/2-ju_{sj}-kv_{sj})+\left|\underline{r}-\underline{r}_j^+-j\underline{u}-k\underline{v}\right|}{(s+s_j/2-ju_{sj}-kv_{sj})+\left|\underline{r}-\underline{r}_j^--j\underline{u}-k\underline{v}\right|}\right] \tag{739}$$

In general we could alternatively collect all the $\cos(m\varphi)$ and $\sin(m\varphi)$ terms and write

$$\phi_{mn} = -\frac{1}{4\pi\mu_0}\sum_{m=0}^{M}\left[p_{me}^{(m)}\cos(m\varphi)+p_{mo}^{(m)}\sin(m\varphi)\right]L_m\left(\rho,\frac{\partial}{\partial\rho}\right)\ln\left[\frac{(s-s_n/2)+\sqrt{\rho^2+(s-s_n/2)^2}}{(s+s_n/2)+\sqrt{\rho^2+(s+s_n/2)^2}}\right] \tag{740}$$

where the operator L_m includes all the appropriate differential forms in ρ (in general it also includes arbitrary constants which we will take to be selected on the basis of the two-dimensional Laplacian relations as discussed in the preceding electric section). There are $2M$ multipole coefficients for each of the N wire segments which must be determined from matching around the wire segments (note that $p_{mo}^{(0)} = 0 = p_{mo}^{(M)}$ are not unknowns).

The electric current filament contribution is

$$\underline{A}_f = \frac{\mu_0}{4\pi} \sum_{n=1}^{N} I_n \underline{e}_{s_n} \int \frac{ds'}{|\underline{r} - \underline{r}'|} =$$

$$\frac{\mu_0}{4\pi} \sum_{n=1}^{N} \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} I_n \underline{e}_{s_n} \ln \left[\frac{(s - s_n/2 - ju_{sn} - kv_{sn}) + \sqrt{\left|\underline{\rho} - j\underline{u}_{\rho n} - k\underline{v}_{\rho n}\right|^2 + (s - s_n/2 - ju_{sn} - kv_{sn})^2}}{(s + s_n/2 - ju_{sn} - kv_{sn}) + \sqrt{\left|\underline{\rho} - j\underline{u}_{\rho n} - k\underline{v}_{\rho n}\right|^2 + (s + s_n/2 - ju_{sn} - kv_{sn})^2}} \right] \tag{741}$$

The boundary condition to be enforced on the perfectly conducting braid wires is

$$\underline{n} \cdot \underline{H} = \underline{n} \cdot \left(\frac{1}{\mu_0} \nabla \times \underline{A}_f - \nabla \phi_m \right) = 0 \tag{742}$$

or

$$\mu_0 \frac{\partial \phi_m}{\partial n} = \underline{n} \cdot \nabla \times \underline{A}_f \tag{743}$$

We sample this equation at $2M$ azimuthal points around each segment (we know that there is no net magnetic flux emanating from a segment in the perfect electric conductor case and therefore we expect the $m = 0$ coefficient to vanish) which generates $N \times (2M)$ equations for the $N \times (2M)$ unknowns ($2M$ multipole unknowns on each of the N segments) associated with the scalar potential ϕ_m. We also impose no net magnetic flux between wires in a carrier strip over a braid period. Furthermore, we want each carrier strip in the uniform coax to carrier net current I/N_s and for insulated wires the wire currents remain the same over the course of a period. These final conditions generate a further N conditions imposed on the coefficients I_n in the filament vector potential $\underline{A}_f$.

In the circular uniform current case, suppose there are N_w wires per carrier strip and N_s carrier strips. In this case the current on each carrier strip is the same and equal to $I_s = I/N_s$. The wire currents must add to this $I_s = \sum_{n=1}^{N_w} I_{wn}$. There are $N_w - 1$ zero flux conditions between the wires on a carrier strip. These conditions are sufficient to fix all the net currents on the wire segments.

5 FINITE CONDUCTIVITY OF BRAID WIRES

We now consider modifications to the preceding magnetic problem when the braid wires have large but finite conductivity. We begin with a complex power approach but also include reciprocity approaches to the formulation. Finally we introduce the detailed boundary value problems for the wire braid based on line multipole representations.

5.1 Poynting Theorem (Self Impedance) And Faraday's Law (Transfer Impedance) Approach

Faraday's law is

$$\nabla \times \underline{E} = i\omega\mu \underline{H} \tag{744}$$

Inside the conductor let us ignore the displacement current term in Maxwell's equation

$$\nabla \times \underline{H} = \underline{J} = \sigma \underline{E} \tag{745}$$

Then the time harmonic Poynting vector

$$\underline{S} = \frac{1}{2}\underline{E} \times \underline{H}^* \tag{746}$$

has divergence (again we are ignoring the displacement current term)

$$\nabla \cdot \underline{S} = \frac{1}{2}\underline{H}^* \cdot \nabla \times \underline{E} - \frac{1}{2}\underline{E} \cdot \nabla \times \underline{H}^* = i\omega \frac{1}{2}\mu \underline{H} \cdot \underline{H}^* - \frac{1}{2}\underline{E} \cdot \underline{J}^* \tag{747}$$

Integration over the braid wire volume and use of the divergence theorem then yields

$$\oint_{S_w} \underline{S} \cdot \underline{n} dS = i\omega \int_{V_w} \frac{1}{2}\mu \underline{H} \cdot \underline{H}^* dV - \int_{V_w} \frac{1}{2}\sigma \underline{E} \cdot \underline{E}^* dV \tag{748}$$

where the unit normal points out of the braid wire conductor. Minus this quantity corresponds to the peak stored magnetic energy (times $-i\omega$) associated with the internal inductance of the braid and the average losses associated with the braid conduction. We can set this equal to the internal impedance per unit length times a braid period

$$\frac{1}{2} Z_1^{int} \ell \left| I_1 \right|^2 = -\oint_{S_w} \underline{S} \cdot \underline{n} dS = -i\omega \int_{V_w} \frac{1}{2}\mu \underline{H} \cdot \underline{H}^* dV + \int_{V_w} \frac{1}{2}\sigma \underline{E} \cdot \underline{E}^* dV \tag{749}$$

The external contribution to the impedance per unit length is found from (ignoring the electric energy term in this impedance per unit length section of the report)

$$\frac{1}{2} Z_1^{ext} \ell \left| I_1 \right|^2 = -i\omega \int_{V_e} \frac{1}{2}\mu_0 \underline{H} \cdot \underline{H}^* dV \tag{750}$$

where the volume V_e is external to the conductors. If the external magnetic field is represented by the vector potential

$$\mu_0 \underline{H} = \nabla \times \underline{A}$$

and external to the conductors (ignoring displacement currents since these were included in the electric problem)

$$\nabla \times \underline{H} = 0 \tag{751}$$

we can write

$$\frac{1}{2} Z_1^{ext} \ell \left| I_1 \right|^2 = -i\omega \frac{1}{2} \int_{V_e} (\nabla \times \underline{A}) \cdot \underline{H}^* dV$$

$$= -i\omega \frac{1}{2} \int_{V_e} \left[\nabla \cdot (\underline{A} \times \underline{H}^*) + \underline{A} \cdot (\nabla \times \underline{H}^*) \right] dV$$

$$= i\omega \frac{1}{2} \oint_{S_e} \underline{A} \cdot (\underline{n} \times \underline{H}^*) \, dS \tag{752}$$

where here $\underline{n}$ points out of the volume, the surface integral S_e is on the surface of the wire braid of the shield as well as on the center conductor (we are ignoring the outer return for the self impedance of the coax), and the volume V_e is external to the braid wires. Alternatively, if we have the external mixed potential representation

$$\underline{B} = \mu_0 \underline{H} = \nabla \times \underline{A}_f - \mu_0 \nabla \phi_m \tag{753}$$

we can write

$$\frac{1}{2} Z_1^{ext} \ell \left| I_1 \right|^2 = -i\omega \int_{V_e} \frac{1}{2\mu_0}$$

$$\left[\left(\nabla \times \underline{A}_f \right) \cdot \left(\nabla \times \underline{A}_f^* \right) - \mu_0 \nabla \phi_m \cdot \left(\nabla \times \underline{A}_f^* \right) - \mu_0 \nabla \phi_m^* \cdot \left(\nabla \times \underline{A}_f \right) + \mu_0^2 \nabla \phi_m \cdot \nabla \phi_m^* \right] dV \tag{754}$$

Again using the identities

$$\nabla \cdot \left(\underline{A}_f \times \nabla \times \underline{A}_f^* \right) = \left(\nabla \times \underline{A}_f \right) \cdot \left(\nabla \times \underline{A}_f^* \right) - \underline{A}_f \cdot \left(\nabla \times \nabla \times \underline{A}_f^* \right) \tag{755}$$

$$\nabla \cdot \left(\phi_m \nabla \phi_m^* \right) = \nabla \phi_m \cdot \nabla \phi_m^* + \phi_m \nabla^2 \phi_m^* \tag{756}$$

$$\nabla \cdot \left(\phi_m \nabla \times \underline{A}_f^* \right) = 0 + \nabla \phi_m \cdot \nabla \times \underline{A}_f^* \tag{757}$$

with

$$\nabla \times \nabla \times \underline{A}_f = 0 \tag{758}$$

$$\nabla^2 \phi_m = 0 \tag{759}$$

gives

$$\frac{1}{2} Z_1^{ext} \ell \left| I_1 \right|^2 = -i\omega \int_{V_e} \frac{1}{2\mu_0}$$

$$\nabla \cdot \left[\left(\underline{A}_f \times \nabla \times \underline{A}_f^* \right) - \mu_0 \left(\phi_m \nabla \times \underline{A}_f^* \right) - \mu_0 \left(\phi_m^* \nabla \times \underline{A}_f \right) + \mu_0^2 \left(\phi_m \nabla \phi_m^* \right) \right] dV \tag{760}$$

or

$$\frac{1}{2} Z_1^{ext} \ell \left| I_1 \right|^2 = -i\omega \oint_{S_e} \frac{1}{2\mu_0}$$

$$\left[\left(\underline{A}_f \times \nabla \times \underline{A}_f^* \right) - \mu_0 \left(\phi_m \nabla \times \underline{A}_f^* \right) - \mu_0 \left(\phi_m^* \nabla \times \underline{A}_f \right) + \mu_0^2 \left(\phi_m \nabla \phi_m^* \right) \right] \cdot \underline{n} dS \tag{761}$$

or

$$\frac{1}{2}Z_1^{ext}\ell\left|I_1\right|^2 = -i\omega\frac{1}{2}\oint_{S_e}$$

$$\left[\left(\underline{A}_f \times \underline{H}_f^*\right) - \mu_0\left(\phi_m\underline{H}_f^*\right) - \mu_0\left(\phi_m^*\underline{H}_f\right) + \mu_0\left(\phi_m\nabla\phi_m^*\right)\right]\cdot\underline{n}dS \tag{762}$$

These results require integration on the conductor surfaces but not out in the free space volume.

This approach partitions the self impedance per unit length into a part inside the wires and a part outside the wires, where

$$Z_1 = Z_1^{ext} + Z_1^{int} \tag{763}$$

If we apply a z directed current to the return $I_2 = I_{chassis} = -I_{sh}$ with the center conductor open circuited, then the application of Faraday's law (C is the contour in the z direction along the center conductor with return along a braid wire including the periodic legs spaced by a periodic distance ℓ)

$$\oint_C \underline{E}\cdot\underline{d\ell} = V\left(\ell + z\right) - V\left(z\right) - \int_{C_w}\underline{E}\cdot\underline{d\ell} = -\frac{\partial}{\partial t}\int_S \underline{B}\cdot\underline{n}dS = -\frac{\partial}{\partial t}\oint_C \underline{A}\cdot\underline{d\ell} = i\omega\Phi_T \tag{764}$$

not only includes the magnetic flux induced between the center conductor and the braid wires Φ_T but also the integration of the surface electric field along the braid wires (we still take the center conductor to be perfect and define the integration contour to be along a braid wire in the positive z direction). These now create a voltage

$$v_s\ell = i\omega\Phi_T + \int_{C_w}\underline{E}\cdot\underline{d\ell} \tag{765}$$

from which we define the transfer impedance Z_T as

$$v_s = Z_T I_{sh} \tag{766}$$

The contour C here is a magnetic flux contour along the center conductor with a return along a braid wire of a particular carrier strip over a braid period. Because we will adjust the individual wire currents in a carrier strip to make the difference voltage between wires vanish over a period this will give the same result for different choices of wires.

5.2 Impedance Matrix & Complex Power Approach

The previous approach used to formulate the capacitance and inductance matrices can also be used in this case. Applying Faraday's law

$$\oint_C \underline{E}\cdot\underline{d\ell} = i\omega\Phi \tag{767}$$

to a section of line with the magnetic fluxes connected to the electric currents by

$$-i\omega\Phi_1 = Z_{11}I_1 + Z_{12}I_2 \tag{768}$$

$$-i\omega\Phi_2 = Z_{21}I_2 + Z_{22}I_2 \tag{769}$$

gives

$$V_1(z+\ell) - V_1(z) = -Z_{11}I_1 - Z_{12}I_2 \tag{770}$$

$$V_2(z+\ell) - V_2(z) = -Z_{21}I_1 - Z_{22}I_2 \tag{771}$$

In a reciprocal media the cross terms are equal $Z_{12} = Z_{21}$. The complex power removed from a periodic section is

$$\frac{1}{2}\left[V_1(z+\ell) - V_1(z)\right]I_1^* + \frac{1}{2}\left[V_2(z+\ell) - V_2(z)\right]I_2^* = -\frac{1}{2}Z_{11}I_1I_1^* - \frac{1}{2}Z_{12}I_2I_1^* - \frac{1}{2}Z_{21}I_1I_2^* - \frac{1}{2}Z_{22}I_2I_2^* \tag{772}$$

Faraday's law is

$$\nabla \times \underline{E} = i\omega\mu\underline{H} \tag{773}$$

Inside the conductor we ignore the displacement current term in Maxwell's equation

$$\nabla \times \underline{H} = \underline{J} = \sigma\underline{E} \tag{774}$$

and outside we also ignore displacement currents (since these were included in the electric problem)

$$\nabla \times \underline{H} = 0 \tag{775}$$

since these were included in the electric problem. Then the time harmonic Poynting vector

$$\underline{S} = \frac{1}{2}\underline{E} \times \underline{H}^* \tag{776}$$

has divergence

$$\nabla \cdot \underline{S} = \frac{1}{2}\underline{H}^* \cdot \nabla \times \underline{E} - \frac{1}{2}\underline{E} \cdot \nabla \times \underline{H}^* = i\omega\frac{1}{2}\mu\underline{H} \cdot \underline{H}^* - \frac{1}{2}\underline{E} \cdot \underline{J}^* \tag{777}$$

where we have ignored the external electric energy term in this series impedance treatment. Integration over the transmission line volume and use of the divergence theorem then yields

$$\oint_S \underline{S} \cdot \underline{n}dS = i\omega\int_V \frac{1}{2}\mu\underline{H} \cdot \underline{H}^* dV - \int_{V_w} \frac{1}{2}\sigma\underline{E} \cdot \underline{E}^* dV$$

$$= i\omega\int_{V_e} \frac{1}{2}\mu_0\underline{H} \cdot \underline{H}^* dV + i\omega\int_{V_w} \frac{1}{2}\mu\underline{H} \cdot \underline{H}^* dV - \int_{V_w} \frac{1}{2}\sigma\underline{E} \cdot \underline{E}^* dV$$

$$= i\omega\int_{V_e} \frac{1}{2}\mu_0\underline{H} \cdot \underline{H}^* dV + \oint_{S_w} \underline{S} \cdot \underline{n}dS \tag{778}$$

where the unit normal points out of each region (including the V_w region in the final surface integral). This quantity corresponds to the power leaving a region of the line which is minus the average stored magnetic

energy (times $-i\omega$) and minus the average losses associated with the braid conduction. Setting the Pointing vector integral (778) equal to (772) and multiplying by two gives

$$\left[V_1\left(z+\ell\right)-V_1\left(z\right)\right]I_1^*+\left[V_2\left(z+\ell\right)-V_2\left(z\right)\right]I_2^*=-Z_{11}I_1I_1^*-Z_{12}I_2I_1^*-Z_{21}I_1I_2^*-Z_{22}I_2I_2^*$$

$$=2\oint_S \underline{S}\cdot\underline{n}dS=i\omega\int_V \mu\underline{H}\cdot\underline{H}^*dV-\int_{V_w}\sigma\underline{E}\cdot\underline{E}^*dV$$

$$=i\omega\int_{V_e}\mu_0\underline{H}\cdot\underline{H}^*dV+2\oint_{S_w}\underline{S}\cdot\underline{n}dS \tag{779}$$

Note that we have left out the center conductor and chassis from the surface integrals since they are assumed to be perfect conductors with a vanishing normal Poynting vector component. We may want to leave the Poynting vector surface integral on the braid wires instead of using a volume integral inside the braid wires (this alternative would allow us to include the interior as an added surface integral as in the final expression where V_e is the volume external to the braid wires).

Taking two sources we write the total fields as

$$\underline{H}=\underline{H}_1+\underline{H}_2 \tag{780}$$

$$\underline{E}=\underline{E}_1+\underline{E}_2 \tag{781}$$

gives

$$Z_{11}I_1I_1^*+Z_{12}I_2I_1^*+Z_{21}I_1I_2^*+Z_{22}I_2I_2^*$$

$$=-i\omega\int_V \mu\left(\underline{H}_1+\underline{H}_2\right)\cdot\left(\underline{H}_1+\underline{H}_2\right)^*dV+\int_{V_w}\sigma\left(\underline{E}_1+\underline{E}_2\right)\cdot\left(\underline{E}_1+\underline{E}_2\right)^*dV \tag{782}$$

or

$$Z_{11}=R_{11}-iX_{11}=-i\omega\frac{1}{\left|I_1\right|^2}\int_V \mu\underline{H}_1\cdot\underline{H}_1^*dV+\frac{1}{\left|I_1\right|^2}\int_{V_w}\sigma\underline{E}_1\cdot\underline{E}_1^*dV \tag{783}$$

$$Z_{22}=R_{22}-iX_{22}=-i\omega\frac{1}{\left|I_2\right|^2}\int_V \mu\underline{H}_2\cdot\underline{H}_2^*dV+\frac{1}{\left|I_2\right|^2}\int_{V_w}\sigma\underline{E}_2\cdot\underline{E}_2^*dV \tag{784}$$

$$Z_{12}\left(I_1I_2^*+I_2I_1^*\right)=Z_{21}\left(I_2I_1^*+I_1I_2^*\right)=Z_{12}2\,\mathrm{Re}\left(I_1I_2^*\right)=Z_{21}2\,\mathrm{Re}\left(I_1I_2^*\right)$$

$$=-i\omega\int_V \mu\left(\underline{H}_1\cdot\underline{H}_2^*+\underline{H}_2\cdot\underline{H}_1^*\right)dV+\int_{V_w}\sigma\left(\underline{E}_1\cdot\underline{E}_2^*+\underline{E}_2\cdot\underline{E}_1^*\right)dV$$

$$=-i\omega\int_V \mu 2\,\mathrm{Re}\left(\underline{H}_1\cdot\underline{H}_2^*\right)dV+\int_{V_w}\sigma 2\,\mathrm{Re}\left(\underline{E}_1\cdot\underline{E}_2^*\right)dV \tag{785}$$

or

$$Z_{12} = R_{12} - iX_{12} = Z_{21} = R_{21} - iX_{21}$$

$$= -i\omega \frac{1}{\mathrm{Re}\left(I_1 I_2^*\right)} \int_V \mu \,\mathrm{Re}\left(\underline{H}_1 \cdot \underline{H}_2^*\right) dV + \frac{1}{\mathrm{Re}\left(I_1 I_2^*\right)} \int_{V_w} \sigma \,\mathrm{Re}\left(\underline{E}_1 \cdot \underline{E}_2^*\right) dV \tag{786}$$

Alternatively, using the surface integral form on the braid wires, we can write (the unit normal $\underline{n}$ on the following braid wire surface integrals, which are denoted as unclosed, is taken to point inward)

$$Z_{11} I_1 I_1^* + Z_{12} I_2 I_1^* + Z_{21} I_1 I_2^* + Z_{22} I_2 I_2^*$$

$$= -i\omega \int_{V_e} \mu_0 \left(\underline{H}_1 + \underline{H}_2\right) \cdot \left(\underline{H}_1 + \underline{H}_2\right)^* dV + \int_{S_w} \left[\underline{n} \times \left(\underline{E}_1 + \underline{E}_2\right)\right] \cdot \left(\underline{H}_1 + \underline{H}_2\right)^* dS \tag{787}$$

$$Z_{11} I_1 I_1^* = -i\omega \int_{V_e} \mu_0 \underline{H}_1 \cdot \underline{H}_1^* dV + \int_{S_w} \left(\underline{n} \times \underline{E}_1\right) \cdot \underline{H}_1^* dS \tag{788}$$

$$Z_{22} I_2 I_2^* = -i\omega \int_{V_e} \mu_0 \underline{H}_2 \cdot \underline{H}_2^* dV + \int_{S_w} \left(\underline{n} \times \underline{E}_2\right) \cdot \underline{H}_2^* dS \tag{789}$$

$$Z_{12} I_2 I_1^* + Z_{21} I_1 I_2^*$$

$$= -i\omega \int_{V_e} \mu_0 \left(\underline{H}_1 \cdot \underline{H}_2^* + \underline{H}_2 \cdot \underline{H}_1^*\right) dV + \int_{S_w} \left[\left(\underline{n} \times \underline{E}_1\right) \cdot \underline{H}_2^* + \left(\underline{n} \times \underline{E}_2\right) \cdot \underline{H}_1^*\right] dS \tag{790}$$

or

$$Z_{11} = -iX_{11}^{ext} + Z_{11}^{int} = -i\omega \frac{1}{\left|I_1\right|^2} \int_{V_e} \mu_0 \underline{H}_1 \cdot \underline{H}_1^* dV + \frac{1}{\left|I_1\right|^2} \int_{S_w} \left(\underline{n} \times \underline{E}_1\right) \cdot \underline{H}_1^* dS \tag{791}$$

$$Z_{22} = -iX_2^{ext} + Z_{22}^{int} = -i\omega \frac{1}{\left|I_2\right|^2} \int_{V_e} \mu_0 \underline{H}_2 \cdot \underline{H}_2^* dV + \frac{1}{\left|I_2\right|^2} \int_{S_w} \left(\underline{n} \times \underline{E}_2\right) \cdot \underline{H}_2^* dS \tag{792}$$

where the volume V_e is external to the conductors and because we expect $Z_{12} = Z_{21}$ for reciprocal media

$$Z_{12} \left(I_2 I_1^* + I_1 I_2^*\right) = Z_{21} \left(I_2 I_1^* + I_1 I_2^*\right)$$

$$= -i\omega \int_{V_e} \mu_0 \left(\underline{H}_1 \cdot \underline{H}_2^* + \underline{H}_2 \cdot \underline{H}_1^*\right) dV + \int_{S_w} \left[\left(\underline{n} \times \underline{E}_1\right) \cdot \underline{H}_2^* + \left(\underline{n} \times \underline{E}_2\right) \cdot \underline{H}_1^*\right] dS \tag{793}$$

or

$$Z_{12} = -i\omega \frac{1}{\mathrm{Re}\left(I_1^* I_2\right)} \mathrm{Re} \int_{V_e} \mu_0 \left(\underline{H}_2 \cdot \underline{H}_1^*\right) dV + \frac{1}{\mathrm{Re}\left(I_1^* I_2\right)} \int_{S_w} \left[\left(\underline{n} \times \underline{E}_1\right) \cdot \underline{H}_2^* + \left(\underline{n} \times \underline{E}_2\right) \cdot \underline{H}_1^*\right] dS$$

$$= Z_{21} = -i\omega \frac{1}{\mathrm{Re}\,(I_1 I_2^*)} \mathrm{Re} \int_{V_e} \mu_0 \left(\underline{H}_1 \cdot \underline{H}_2^*\right) dV + \frac{1}{\mathrm{Re}\,(I_1 I_2^*)} \int_{S_w} \left[(\underline{n} \times \underline{E}_1) \cdot \underline{H}_2^* + (\underline{n} \times \underline{E}_2) \cdot \underline{H}_1^*\right] dS \quad (794)$$

Note that for a symmetric braid (mirror symmetry from inside to outside), the calculation of each term in the integral for the planar approximation will be identical, and hence there is no extra work associated with the addition of the two terms.

5.2.1 Vector Potential Representation

If the external magnetic field is represented by the vector potential $\mu_0 \underline{H}_1 = \nabla \times \underline{A}_1$, satisfying

$$\nabla \times \underline{H}_1 = 0 \quad (795)$$

external to the conductors, we can write

$$Z_{11}^{ext} = Z_1^{ext} \ell = -i\omega \frac{1}{|I_1|^2} \int_{V_e} (\nabla \times \underline{A}_1) \cdot \underline{H}_1^* dV$$

$$= -i\omega \frac{1}{|I_1|^2} \int_{V_e} \left[\nabla \cdot (\underline{A}_1 \times \underline{H}_1^*) + \underline{A}_1 \cdot (\nabla \times \underline{H}_1^*)\right] dV$$

$$= -i\omega \frac{1}{|I_1|^2} \oint_{S_e} \underline{n} \cdot (\underline{A}_1 \times \underline{H}_1^*)\, dS = i\omega \frac{1}{|I_1|^2} \oint_{S_e} \underline{A}_1 \cdot (\underline{n} \times \underline{H}_1^*)\, dS \quad (796)$$

where the surface integral is on the surface of the wire braid of the shield as well as on the center conductor (we are ignoring the outer chassis return for the self impedance of the coax). Combining this with the internal term gives (the unit vector $\underline{n}$ points into the braid wires in the second term and out of S_e in the first term)

$$Z_{11} = Z_1 \ell = -i\omega \frac{1}{|I_1|^2} \oint_{S_e} (\underline{n} \times \underline{A}_1) \cdot \underline{H}_1^* dS + \frac{1}{|I_1|^2} \int_{S_w} (\underline{n} \times \underline{E}_1) \cdot \underline{H}_1^* dS \quad (797)$$

Now if we construct the vector potential such that on the braid wire surfaces

$$i\omega\, (\underline{n} \times \underline{A}_1) = \underline{n} \times \underline{E}_1 \quad (798)$$

then the surface integral terms on S_w cancel out. Our philosophy in the electric section was to treat only the current as varying along the line length and the voltage as fixed over a periodic section; our philosophy in the magnetic section was to treat only the voltage as varying along the line length and the current as fixed over a periodic section; this assumption will be continued in the lossy treatment of the line impedance here. Thus we are left with (the unit vector $\underline{n}$ points into the center conductor)

$$Z_{11} = Z_1 \ell = -i\omega \frac{1}{|I_1|^2} \int_{S_c} (\underline{n} \times \underline{A}_1) \cdot \underline{H}_1^* dS \quad (799)$$

Also using the vector potential to represent the second field $\mu_0 \underline{H}_2 = \nabla \times \underline{A}_2$, again satisfying

$$\nabla \times \underline{H}_2 = 0 \quad (800)$$

external to the conductors, we can write

$$\int_{V_e} \mu_0 \underline{H}_1 \cdot \underline{H}_2^* dV = \int_{V_e} (\nabla \times \underline{A}_1) \cdot \underline{H}_2^* dV$$

$$= \int_{V_e} \left[\nabla \cdot (\underline{A}_1 \times \underline{H}_2^*) + \underline{A}_1 \cdot (\nabla \times \underline{H}_2^*)\right] dV$$

$$= \oint_{S_e} \underline{n} \cdot (\underline{A}_1 \times \underline{H}_2^*)\, dS = -\oint_{S_e} \underline{A}_1 \cdot (\underline{n} \times \underline{H}_2^*)\, dS$$

$$= \oint_{S_e} (\underline{n} \times \underline{A}_1) \cdot \underline{H}_2^* dS \tag{801}$$

Then

$$Z_{12} I_2 I_1^* + Z_{21} I_1 I_2^*$$

$$= -i\omega \oint_{S_e} \left[(\underline{n} \times \underline{A}_1) \cdot \underline{H}_2^* + (\underline{n} \times \underline{A}_2) \cdot \underline{H}_1^*\right] dS$$

$$+ \int_{S_w} \left[(\underline{n} \times \underline{E}_1) \cdot \underline{H}_2^* + (\underline{n} \times \underline{E}_2) \cdot \underline{H}_1^*\right] dS \tag{802}$$

where the unit vector $\underline{n}$ points into the braid wires in the final integral. Again if on the braid wire surface S_w we have

$$i\omega\, (\underline{n} \times \underline{A}_1) = \underline{n} \times \underline{E}_1 \tag{803}$$

$$i\omega\, (\underline{n} \times \underline{A}_2) = \underline{n} \times \underline{E}_2 \tag{804}$$

then the braid wire surface integrals cancel and we find (where $\underline{n}$ points into the conductors)

$$Z_{12} I_2 I_1^* + Z_{21} I_1 I_2^*$$

$$= -i\omega \left(\int_{S_c} + \int_{S_{chassis}}\right) \left[(\underline{n} \times \underline{A}_1) \cdot \underline{H}_2^* + (\underline{n} \times \underline{A}_2) \cdot \underline{H}_1^*\right] dS \tag{805}$$

5.2.2 Mixed Potential Representation

Alternatively, if we have the external mixed potential representation

$$\underline{B} = \mu_0 \underline{H} = \nabla \times \underline{A}_f - \mu_0 \nabla \phi_m \tag{806}$$

we can write

$$Z_{11}I_1I_1^* = Z_1\ell\left|I_1\right|^2 = -i\omega\int_{V_e}\mu_0\underline{H}_1\cdot\underline{H}_1^*dV + \int_{S_w}\left(\underline{n}\times\underline{E}_1\right)\cdot\underline{H}_1^*dS$$

$$= -i\frac{\omega}{\mu_0}\int_{V_e}\left(\nabla\times\underline{A}_{f1}-\mu_0\nabla\phi_{m1}\right)\cdot\left(\nabla\times\underline{A}_{f1}-\mu_0\nabla\phi_{m1}\right)^*dV$$

$$+\int_{S_w}\left(\underline{n}\times\underline{E}_1\right)\cdot\underline{H}_1^*dS$$

$$= -i\frac{\omega}{\mu_0}\int_{V_e}\left[\left(\nabla\times\underline{A}_{f1}\right)\cdot\left(\nabla\times\underline{A}_{f1}^*\right)-\mu_0\nabla\phi_{m1}\cdot\left(\nabla\times\underline{A}_{f1}^*\right)-\mu_0\nabla\phi_{m1}^*\cdot\left(\nabla\times\underline{A}_{f1}\right)+\mu_0^2\nabla\phi_{m1}\cdot\nabla\phi_{m1}^*\right]dV$$

$$+\int_{S_w}\left(\underline{n}\times\underline{E}_1\right)\cdot\underline{H}_1^*dS \tag{807}$$

Again using the identities

$$\nabla\cdot\left(\underline{A}_f\times\nabla\times\underline{A}_f^*\right) = \left(\nabla\times\underline{A}_f\right)\cdot\left(\nabla\times\underline{A}_f^*\right)-\underline{A}_f\cdot\left(\nabla\times\nabla\times\underline{A}_f^*\right) \tag{808}$$

$$\nabla\cdot\left(\phi_m\nabla\phi_m^*\right) = \nabla\phi_m\cdot\nabla\phi_m^*+\phi_m\nabla^2\phi_m^* \tag{809}$$

$$\nabla\cdot\left(\phi_m\nabla\times\underline{A}_f^*\right) = 0+\nabla\phi_m\cdot\nabla\times\underline{A}_f^* \tag{810}$$

with

$$\nabla\times\nabla\times\underline{A}_f = 0 \tag{811}$$

$$\nabla^2\phi_m = 0 \tag{812}$$

gives

$$Z_1\ell\left|I_1\right|^2 = -i\frac{\omega}{\mu_0}\int_{V_e}$$

$$\nabla\cdot\left[\left(\underline{A}_{f1}\times\nabla\times\underline{A}_{f1}^*\right)-\mu_0\left(\phi_{m1}\nabla\times\underline{A}_{f1}^*\right)-\mu_0\left(\phi_{m1}^*\nabla\times\underline{A}_{f1}\right)+\mu_0^2\left(\phi_{m1}\nabla\phi_{m1}^*\right)\right]dV$$

$$+\int_{S_w}\left(\underline{n}\times\underline{E}_1\right)\cdot\underline{H}_1^*dS \tag{813}$$

or

$$Z_1\ell\left|I_1\right|^2 = -i\frac{\omega}{\mu_0}\oint_{S_e}$$

$$\left[\left(\underline{A}_{f1}\times\nabla\times\underline{A}_{f1}^*\right)-\mu_0\left(\phi_{m1}\nabla\times\underline{A}_{f1}^*\right)-\mu_0\left(\phi_{m1}^*\nabla\times\underline{A}_{f1}\right)+\mu_0^2\left(\phi_{m1}\nabla\phi_{m1}^*\right)\right]\cdot\underline{n}dS$$

$$+\int_{S_w}(\underline{n}\times\underline{E}_1)\cdot\underline{H}_1^* dS \tag{814}$$

or

$$Z_1\ell\left|I_1\right|^2=-i\omega\oint_{S_e}$$

$$\left[\left(\underline{A}_{f1}\times\underline{H}_{f1}^*\right)-\mu_0\left(\phi_{m1}\underline{H}_{f1}^*\right)-\mu_0\left(\phi_{m1}^*\underline{H}_{f1}\right)+\mu_0\left(\phi_{m1}\nabla\phi_{m1}^*\right)\right]\cdot\underline{n}dS$$

$$+\int_{S_w}(\underline{n}\times\underline{E}_1)\cdot\underline{H}_1^* dS \tag{815}$$

These results require integration on the conductor surfaces but not out in the free space volume which is an advantage since we must solve the problem on these surfaces.

For the transfer impedance we begin with

$$Z_{21}\left(I_1I_2^*+I_1^*I_2\right)=Z_{12}\left(I_1I_2^*+I_1^*I_2\right)=-i\omega\int_{V_e}\mu_0\underline{H}_1\cdot\underline{H}_2^* dV-i\omega\int_{V_e}\mu_0\underline{H}_1^*\cdot\underline{H}_2 dV$$

$$+\int_{S_w}\left[(\underline{n}\times\underline{E}_1)\cdot\underline{H}_2^*+(\underline{n}\times\underline{E}_2)\cdot\underline{H}_1^*\right]dS \tag{816}$$

Using the identities (and their conjugates)

$$\nabla\cdot\left(\underline{A}_{f1}\times\nabla\times\underline{A}_{f2}^*\right)=\left(\nabla\times\underline{A}_{f1}\right)\cdot\left(\nabla\times\underline{A}_{f2}^*\right)-\underline{A}_{f1}\cdot\left(\nabla\times\nabla\times\underline{A}_{f2}^*\right) \tag{817}$$

$$\nabla\cdot\left(\phi_{m1}\nabla\phi_{m2}^*\right)=\nabla\phi_{m1}\cdot\nabla\phi_{m2}^*+\phi_{m1}\nabla^2\phi_{m2}^* \tag{818}$$

$$\nabla\cdot\left(\phi_{m1}\nabla\times\underline{A}_{f2}^*\right)=0+\nabla\phi_{m1}\cdot\nabla\times\underline{A}_{f2}^* \tag{819}$$

$$\nabla\cdot\left(\phi_{m2}^*\nabla\times\underline{A}_{f1}\right)=0+\nabla\phi_{m2}^*\cdot\nabla\times\underline{A}_{f1} \tag{820}$$

with

$$\nabla\times\nabla\times\underline{A}_{f2}=0 \tag{821}$$

$$\nabla^2\phi_{m2}=0 \tag{822}$$

gives

$$Z_{21}\,\mathrm{Re}\left(I_1I_2^*\right)=Z_{12}\,\mathrm{Re}\left(I_1^*I_2\right)$$

$$= -i\omega \operatorname{Re} \int_{V_e} \frac{1}{\mu_0} \left(\nabla \times \underline{A}_{f1} - \mu_0 \nabla \phi_{m1}\right) \cdot \left(\nabla \times \underline{A}^*_{f2} - \mu_0 \nabla \phi^*_{m2}\right) dV$$

$$+ \frac{1}{2} \int_{S_w} \left[(\underline{n} \times \underline{E}_1) \cdot \underline{H}^*_2 + (\underline{n} \times \underline{E}_2) \cdot \underline{H}^*_1\right] dS$$

$$= -i\omega \operatorname{Re} \int_{V_e} \frac{1}{\mu_0} \left(\nabla \times \underline{A}_{f1}\right) \cdot \left(\nabla \times \underline{A}^*_{f2}\right) dV + i\omega \operatorname{Re} \int_{V_e} \nabla \times \underline{A}_{f1} \cdot \nabla \phi^*_{m2} dV$$

$$+ i\omega \operatorname{Re} \int_{V_e} \nabla \phi_{m1} \cdot \nabla \times \underline{A}^*_{f2} dV - i\omega \operatorname{Re} \int_{V_e} \mu_0 \nabla \phi_{m1} \cdot \nabla \phi^*_{m2} dV$$

$$+ \frac{1}{2} \int_{S_w} \left[(\underline{n} \times \underline{E}_1) \cdot \underline{H}^*_2 + (\underline{n} \times \underline{E}_2) \cdot \underline{H}^*_1\right] dS$$

$$= -i\omega \operatorname{Re} \int_{V_e} \frac{1}{\mu_0} \nabla \cdot \left(\underline{A}_{f1} \times \nabla \times \underline{A}^*_{f2}\right) dV + i\omega \operatorname{Re} \int_{V_e} \nabla \cdot \left(\phi^*_{m2} \nabla \times \underline{A}_{f1}\right) dV$$

$$+ i\omega \operatorname{Re} \int_{V_e} \nabla \cdot \left(\phi_{m1} \nabla \times \underline{A}^*_{f2}\right) dV - i\omega \operatorname{Re} \int_{V_e} \mu_0 \nabla \cdot \left(\phi_{m1} \nabla \phi^*_{m2}\right) dV$$

$$+ \frac{1}{2} \int_{S_w} \left[(\underline{n} \times \underline{E}_1) \cdot \underline{H}^*_2 + (\underline{n} \times \underline{E}_2) \cdot \underline{H}^*_1\right] dS$$

$$= -i\omega \operatorname{Re} \oint_{S_e} \frac{1}{\mu_0} \underline{n} \cdot \left(\underline{A}_{f1} \times \nabla \times \underline{A}^*_{f2}\right) dS + i\omega \operatorname{Re} \oint_{S_e} \underline{n} \cdot \left(\phi^*_{m2} \nabla \times \underline{A}_{f1}\right) dS$$

$$+ i\omega \operatorname{Re} \oint_{S_e} \underline{n} \cdot \left(\phi_{m1} \nabla \times \underline{A}^*_{f2}\right) dS - i\omega \operatorname{Re} \oint_{S_e} \mu_0 \underline{n} \cdot \left(\phi_{m1} \nabla \phi^*_{m2}\right) dV$$

$$+ \frac{1}{2} \int_{S_w} \left[(\underline{n} \times \underline{E}_1) \cdot \underline{H}^*_2 + (\underline{n} \times \underline{E}_2) \cdot \underline{H}^*_1\right] dS$$

$$= -i\omega \operatorname{Re} \oint_{S_e} \underline{n} \cdot \left(\underline{A}_{f1} \times \underline{H}^*_{f2}\right) dS + i\omega\mu_0 \operatorname{Re} \oint_{S_e} \phi^*_{m2} \underline{n} \cdot \underline{H}_{f1} dS$$

$$+ i\omega\mu_0 \operatorname{Re} \oint_{S_e} \phi_{m1} \underline{n} \cdot \underline{H}^*_{f2} dS - i\omega\mu_0 \operatorname{Re} \oint_{S_e} \phi_{m1} \underline{n} \cdot \nabla \phi^*_{m2} dS$$

$$+ \frac{1}{2} \int_{S_w} \left[(\underline{n} \times \underline{E}_1) \cdot \underline{H}^*_2 + (\underline{n} \times \underline{E}_2) \cdot \underline{H}^*_1\right] dS$$

$$= -i\omega \operatorname{Re} \oint_{S_e} \left(\underline{n} \times \underline{A}_{f1}\right) \cdot \underline{H}^*_{f2} dS + i\omega\mu_0 \operatorname{Re} \oint_{S_e} \phi^*_{m2} \underline{n} \cdot \underline{H}_{f1} dS$$

$$+i\omega\mu_0 \operatorname{Re} \oint_{S_e} \phi_{m1} \underline{n} \cdot \underline{H}_{f2}^* dS - i\omega\mu_0 \operatorname{Re} \oint_{S_e} \phi_{m1} \underline{n} \cdot \nabla \phi_{m2}^* dS$$

$$+\frac{1}{2} \int_{S_w} \left[(\underline{n} \times \underline{E}_1) \cdot \underline{H}_2^* + (\underline{n} \times \underline{E}_2) \cdot \underline{H}_1^* \right] dS$$

$$= -i\omega \operatorname{Re} \oint_{S_e} \left(\underline{n} \times \underline{A}_{f1} - \mu_0 \phi_{m1} \underline{n} \right) \cdot \underline{H}_{f2}^* dS + i\omega\mu_0 \operatorname{Re} \oint_{S_e} \phi_{m2}^* \underline{n} \cdot \underline{H}_{f1} dS - i\omega\mu_0 \operatorname{Re} \oint_{S_e} \phi_{m1} \underline{n} \cdot \nabla \phi_{m2}^* dS$$

$$+\frac{1}{2} \int_{S_w} \left[(\underline{n} \times \underline{E}_1) \cdot \underline{H}_2^* + (\underline{n} \times \underline{E}_2) \cdot \underline{H}_1^* \right] dS$$

$$= -i\omega \operatorname{Re} \oint_{S_e} \left(\underline{n} \times \underline{A}_{f1} \right) \cdot \underline{H}_{f2}^* dS + i\omega \operatorname{Re} \oint_{S_e} \underline{n} \cdot \left(\mu_0 \phi_{m2}^* \underline{H}_{f1} \right) dS + i\omega \operatorname{Re} \oint_{S_e} \phi_{m1} \underline{n} \cdot \left(\nabla \times \underline{A}_{f2}^* - \mu_0 \nabla \phi_{m2}^* \right) dS$$

$$+\frac{1}{2} \int_{S_w} (\underline{n} \times \underline{E}_1) \cdot \underline{H}_2^* dS + \frac{1}{2} \int_{S_w} (\underline{n} \times \underline{E}_2) \cdot \underline{H}_1^* dS$$

$$= -i\omega \operatorname{Re} \oint_{S_e} \left(\underline{n} \times \underline{A}_{f1} \right) \cdot \underline{H}_{f2}^* dS + i\omega\mu_0 \operatorname{Re} \oint_{S_e} \phi_{m2}^* \underline{n} \cdot \underline{H}_{f1} dS + i\omega\mu_0 \operatorname{Re} \oint_{S_e} \phi_{m1} \underline{n} \cdot \underline{H}_2^* dS$$

$$+\frac{1}{2} \int_{S_w} \left[(\underline{n} \times \underline{E}_1) \cdot \underline{H}_2^* + (\underline{n} \times \underline{E}_2) \cdot \underline{H}_1^* \right] dS$$

$$= -i\omega \operatorname{Re} \oint_{S_e} \left(\underline{n} \times \underline{A}_{f1} - \mu_0 \phi_{m1} \underline{n} \right) \cdot \underline{H}_2^* dS - i\omega \operatorname{Re} \oint_{S_e} \left(\underline{n} \times \underline{A}_{f1} \right) \cdot \nabla \phi_{m2}^* dS + i\omega\mu_0 \operatorname{Re} \oint_{S_e} \phi_{m2}^* \underline{n} \cdot \underline{H}_{f1} dS$$

$$+\frac{1}{2} \int_{S_w} \left[(\underline{n} \times \underline{E}_1) \cdot \underline{H}_2^* + (\underline{n} \times \underline{E}_2) \cdot \underline{H}_1^* \right] dS \tag{823}$$

5.2.3 Perturbation Approximation With Vector Potential

We now want to approximate the integrals in the expressions for the impedance when the impedance length scale $Z_s / (\omega\mu_0)$ is small compared to the global transverse geometry of the cable. The surface impedance of the wires Z_s is $Z_s = (1-i) / (\sigma\delta)$ when the skin depth $\delta = \sqrt{2/(\omega\mu\sigma)}$ is small compared to the wire radius a. However, for large skin depths we take the surface impedance to be approximately $Z_s \approx 1/(\sigma\Delta)$ where the effective thickness of the braid is taken approximately as $\Delta = O(2a)$. When the impedance length scale is small the global current distribution is not significantly perturbed from the perfectly conducting case (in the circular cylindrical geometry it remains uniform even when this length scale become large). In this limit we can thus make use of the current distribution from the perfectly conducting solid shield cable of the same global geometry to determine the impedance per unit length of the braid using the approximate planar form of the vector potential near the braid.

Suppose we take the external volume in the self impedance to be written as $V_e = V_0 + \Delta V$ and

$$Z_{11} = -i\omega \frac{1}{|I_1|^2} \int_{V_0} \mu_0 \underline{H}_1 \cdot \underline{H}_1^* dV - i\omega \frac{1}{|I_1|^2} \int_{\Delta V} \mu_0 \underline{H}_1 \cdot \underline{H}_1^* dV + \frac{1}{|I_1|^2} \int_{S_w} (\underline{n} \times \underline{E}_1) \cdot \underline{H}_1^* dS \tag{824}$$

In the region V_0 we approximate the field as $\underline{H}_0$

$$Z_{11} \approx -i\omega \frac{1}{|I_0|^2} \int_{V_0} \mu_0 \underline{H}_0 \cdot \underline{H}_0^* dV - i\omega \frac{1}{|I_1|^2} \int_{\Delta V} \mu_0 \underline{H}_1 \cdot \underline{H}_1^* dV + \frac{1}{|I_1|^2} \int_{S_w} (\underline{n} \times \underline{E}_1) \cdot \underline{H}_1^* dS \tag{825}$$

Using the vector potential representation we can rewrite the second volume integral as

$$-i\omega \frac{1}{|I_1|^2} \int_{\Delta V} (\nabla \times \underline{A}_1) \cdot \underline{H}_1^* dV$$

$$= -i\omega \frac{1}{|I_1|^2} \int_{\Delta V} \left[\nabla \cdot (\underline{A}_1 \times \underline{H}_1^*) + \underline{A}_1 \cdot (\nabla \times \underline{H}_1^*) \right] dV$$

$$= -i\omega \frac{1}{|I_1|^2} \left(\int_{S_0} + \int_{S_w} \right) (\underline{n} \times \underline{A}_1) \cdot \underline{H}_1^* dS \tag{826}$$

Assuming that the vector potential is set up to satisfy

$$i\omega \left(\underline{n} \times \underline{A}_1 \right) = \underline{n} \times \underline{E}_1 \tag{827}$$

we see that the S_w integrals cancel out and we are left with

$$Z_{11} \approx -i\omega \frac{1}{|I_0|^2} \int_{V_0} \mu_0 \underline{H}_0 \cdot \underline{H}_0^* dV - i\omega \frac{1}{|I_1|^2} \int_{S_0} (\underline{n} \times \underline{A}_1) \cdot \underline{H}_1^* dS \tag{828}$$

Using the same identities on the first term yields

$$Z_{11} \approx -i\omega \frac{1}{|I_0|^2} \left(\int_{S_c} + \int_{S_0} \right) (\underline{n} \times \underline{A}_0) \cdot \underline{H}_0^* dS - i\omega \frac{1}{|I_1|^2} \int_{S_0} (\underline{n} \times \underline{A}_1) \cdot \underline{H}_1^* dS \tag{829}$$

The first term represents the perfectly conducting inductance contribution of the auxiliary problem $L_0 \ell$. The second term represents the contribution from the braid wire geometry (versus the continuous shield) as well as the finite conductivity of the braid wires. We approximate the final term by taking the field as the field in the auxiliary problem $\underline{H}_1^* \to \underline{H}_0^*$ with $I_1^* \to I_0^*$

$$Z_{11} = Z_1 \ell \approx -i\omega L_0 \ell - i\omega \frac{1}{I_1 I_0^*} \int_{S_0} (\underline{n} \times \underline{A}_1) \cdot \underline{H}_0^* dS$$

$$\approx -i\omega L_0 \ell + i\omega \frac{1}{I_1 I_0^*} \int_{S_0} \underline{A}_1 \cdot (\underline{n} \times \underline{H}_0^*) \, dS \tag{830}$$

where we note that the unit normal points inward since it came from the ΔV integration. The entire contribution of the loss is contained in how the constant offset in the vector potential is changed to a complex value relative to the perfect electric conductor case. Now using the asymptotic form of the potential

$$A_{1z} \sim -\left(\underline{n} \cdot \left(\underline{\rho} - \underline{\rho}_0 \right) - A_b / B_0 \right) \underline{e}_z \cdot \left[\underline{n} \times \underline{B}_0 \left(x_0, y_0 \right) \right] \left(I_1 / I_0 \right) \tag{831}$$

or referenced to the mean braid radius

$$A_{1z} \sim -\left(\underline{n}\cdot\left(\underline{\rho}-\underline{\rho}_m\right)-A_b/B_0\right)\underline{e}_z\cdot\left[\underline{n}\times\underline{B}_0\left(x_m,y_m\right)\right]\left(I_1/I_0\right) \tag{832}$$

we find

$$Z_1 = Z_0+\Delta Z \approx -i\omega L_0 - i\omega\frac{\mu_0}{|I_0|^2}\int_{S_0}\left(d_0+A_b/B_0\right)\left[\underline{e}_z\cdot\left(\underline{n}\times\underline{H}_0\right)\right]\left[\underline{e}_z\cdot\left(\underline{n}\times\underline{H}_0^*\right)\right]dS/\ell$$

$$\approx -i\omega L_0 - i\omega\mu_0\frac{1}{|I_0|^2}\int_{S_0}\left(d_0+A_b/B_0\right)\underline{H}_0\cdot\underline{H}_0^* dS/\ell \tag{833}$$

where we have reversed the sign of the unit normal to point outward in the $\underline{n}\times\underline{H}_0^*$ factor and where

$$d_0 = \underline{n}\cdot\left(\underline{\rho}_m-\underline{\rho}_0\right) \tag{834}$$

This is the same result as found previously except that the offset constant A_b/B_0 will now be complex to account for diffusion into the braid wires.

For the transfer impedance we start with

$$Z_{12}\left(I_1I_2^*+I_2I_1^*\right) = Z_{21}\left(I_1I_2^*+I_2I_1^*\right)$$

$$= -i\omega\int_{V_e}\mu_0\underline{H}_2\cdot\underline{H}_1^* dV + \int_{S_w}\left(\underline{n}\times\underline{E}_2\right)\cdot\underline{H}_1^* dS$$

$$-i\omega\int_{V_e}\mu_0\underline{H}_1\cdot\underline{H}_2^* dV + \int_{S_w}\left(\underline{n}\times\underline{E}_1\right)\cdot\underline{H}_2^* dS \tag{835}$$

Breaking the external volume into two parts $V_e = V_0+\Delta V$ (in this case the auxiliary volume will itself consist of two parts $V_0 = V_{01}+V_{02}$ where V_{01} extends from the center conductor outward toward the braid wires and V_{02} extends from the chassis inward toward the braid wires, and ΔV is adjacent to the braid wires, excluding the braid wire volume, with a part inside the center region and a part outside in the chassis region) we write

$$Z_{12}\left(I_1I_2^*+I_2I_1^*\right) = Z_{21}\left(I_1I_2^*+I_2I_1^*\right)$$

$$= \left(-iX_{12}^{ext}+Z_{12}^{int}\right)2\,\mathrm{Re}\left(I_1I_2^*\right) = \left(-iX_{21}^{ext}+Z_{21}^{int}\right)2\,\mathrm{Re}\left(I_2I_1^*\right)$$

$$= -i\omega\int_{V_0}\mu_0\underline{H}_2\cdot\underline{H}_1^* dV - i\omega\int_{\Delta V}\mu_0\underline{H}_2\cdot\underline{H}_1^* dV$$

$$-i\omega\int_{V_0}\mu_0\underline{H}_1\cdot\underline{H}_2^* dV - i\omega\int_{\Delta V}\mu_0\underline{H}_1\cdot\underline{H}_2^* dV$$

$$+\int_{S_w}\left[\left(\underline{n}\times\underline{E}_2\right)\cdot\underline{H}_1^*+\left(\underline{n}\times\underline{E}_1\right)\cdot\underline{H}_2^*\right]dS \tag{836}$$

Due to the open circuit conditions on the chassis in the 1 problem, and the open circuit center conductor conditions in the 2 problem we now neglect the V_0 volume integrals. In the ΔV integrals we use $\mu_0 \underline{H}_1 = \nabla \times \underline{A}_1$ and $\mu_0 \underline{H}_2 = \nabla \times \underline{A}_2$ along with

$$(\nabla \times \underline{A}_1) \cdot \underline{H}_2^* = \nabla \cdot (\underline{A}_1 \times \underline{H}_2^*) - \underline{A}_1 \cdot (\nabla \times \underline{H}_2^*) = \nabla \cdot (\underline{A}_1 \times \underline{H}_2^*) - \underline{A}_1 \cdot \underline{J}_2^* \tag{837}$$

$$(\nabla \times \underline{A}_2) \cdot \underline{H}_1^* = \nabla \cdot (\underline{A}_2 \times \underline{H}_1^*) - \underline{A}_2 \cdot (\nabla \times \underline{H}_1^*) = \nabla \cdot (\underline{A}_2 \times \underline{H}_1^*) - \underline{A}_2 \cdot \underline{J}_1^* \tag{838}$$

to find

$$Z_{12} \left(I_1 I_2^* + I_2 I_1^* \right) = Z_{21} \left(I_1 I_2^* + I_2 I_1^* \right)$$

$$= \left(-iX_{12}^{ext} + Z_{12}^{int} \right) 2 \operatorname{Re} \left(I_1 I_2^* \right) = \left(-iX_{21}^{ext} + Z_{21}^{int} \right) 2 \operatorname{Re} \left(I_2 I_1^* \right)$$

$$\approx -i\omega \oint_{\Delta S} (\underline{A}_2 \times \underline{H}_1^*) \cdot \underline{n} dS - i\omega \oint_{\Delta S} (\underline{A}_1 \times \underline{H}_2^*) \cdot \underline{n} dS$$

$$+ \int_{S_w} \left[(\underline{n} \times \underline{E}_2) \cdot \underline{H}_1^* + (\underline{n} \times \underline{E}_1) \cdot \underline{H}_2^* \right] dS$$

$$\approx -i\omega \left(\int_{S_{01}} + \int_{S_w} \right) (\underline{A}_2 \times \underline{H}_1^*) \cdot \underline{n} dS - i\omega \left(\int_{S_{02}} + \int_{S_w} \right) (\underline{A}_1 \times \underline{H}_2^*) \cdot \underline{n} dS$$

$$+ \int_{S_w} \left[(\underline{n} \times \underline{E}_2) \cdot \underline{H}_1^* + (\underline{n} \times \underline{E}_1) \cdot \underline{H}_2^* \right] dS$$

$$\approx -i\omega \left(\int_{S_{01}} + \int_{S_w} \right) (\underline{n} \times \underline{A}_2) \cdot \underline{H}_1^* dS - i\omega \left(\int_{S_{02}} + \int_{S_w} \right) (\underline{n} \times \underline{A}_1) \cdot \underline{H}_2^* dS$$

$$+ \int_{S_w} \left[(\underline{n} \times \underline{E}_2) \cdot \underline{H}_1^* + (\underline{n} \times \underline{E}_1) \cdot \underline{H}_2^* \right] dS \tag{839}$$

where $\underline{n}$ points inward on S_{01}, outward on S_{02}, and into the conductors on S_w. Now assuming that the problem has been solved with

$$i\omega (\underline{n} \times \underline{A}_1) = \underline{n} \times \underline{E}_1 \tag{840}$$

$$i\omega (\underline{n} \times \underline{A}_2) = \underline{n} \times \underline{E}_2 \tag{841}$$

the integrals on S_w cancel and we have (we drop the first integral on S_{02} since $\underline{H}_1^*$ is small there and the second integral on S_{01} because $\underline{H}_2^*$ is small there)

$$Z_{12} 2 \operatorname{Re} \left(I_1 I_2^* \right) = Z_{21} 2 \operatorname{Re} \left(I_2 I_1^* \right)$$

$$\approx -i\omega \int_{S_{01}} (\underline{n} \times \underline{A}_2) \cdot \underline{H}_1^* dS - i\omega \int_{S_{02}} (\underline{n} \times \underline{A}_1) \cdot \underline{H}_2^* dS \tag{842}$$

Now we set $Z_{12} = Z_{21} = Z_T \ell$, $\underline{H}_2^* = \underline{H}_{sh}^*$, $\underline{A}_2 = \underline{A}_{sh}$ and $I_2 = -I_{sh}$ and we modify the unit normals to point out of the center conductor region 1 on S_{01} and out of the chassis region 2 on S_{02} (this is done to be consistent with the following asymptotic forms)

$$Z_T 2 \operatorname{Re}(I_1 I_{sh}^*) \approx -i\omega \int_{S_{01}} (\underline{n} \times \underline{A}_{sh}) \cdot \underline{H}_1^* dS/\ell - i\omega \int_{S_{02}} (\underline{n} \times \underline{A}_1) \cdot \underline{H}_{sh}^* dS/\ell$$

$$\approx i\omega \int_{S_{01}} (\underline{n} \times \underline{H}_0^*) \cdot \underline{A}_{sh} dS/\ell + i\omega \int_{S_{02}} (\underline{n} \times \underline{H}_{sh0}^*) \cdot \underline{A}_1 dS/\ell \tag{843}$$

where the introduction of the 0 in the magnetic fields of the final expression is done as an approximation, using the perfectly conducting solid shield replacement of the braid.

We see then that the entire effect of the loss addition is accounted for by a complex addition to the potentials $\underline{A}_{sh}$ and $\underline{A}_1$ on $S_0 = S_{01} + S_{02}$. Using our previous asymptotic form (here $\underline{n}$ points out of the center conductor region 1on S_{01})

$$A_{shz} \sim -(A_c/B_0)\, \underline{e}_z \cdot [\underline{n} \times \underline{B}_{sh0}(x_m, y_m)] \tag{844}$$

in addition to the dual (here $\underline{n}$ points out of the chassis region 2 on S_{02})

$$A_{1z} \sim -(A_c/B_0)\, \underline{e}_z \cdot [\underline{n} \times \underline{B}_0(x_m, y_m)] \tag{845}$$

we find

$$Z_T 2 \operatorname{Re}(I_0 I_{sh0}^*)$$

$$\approx -i\omega (A_c/B_0) \int_{S_{01}} [\underline{e}_z \cdot (\underline{n} \times \underline{H}_0^*)]\, \underline{e}_z \cdot [\underline{n} \times \underline{B}_{sh0}(x_m, y_m)]\, dS/\ell$$

$$-i\omega (A_c/B_0) \int_{S_{02}} [\underline{e}_z \cdot (\underline{n} \times \underline{H}_{sh0}^*)]\, \underline{e}_z \cdot [\underline{n} \times \underline{B}_0(x_m, y_m)]\, dS/\ell \tag{846}$$

where we have modified the unit normals in this expression to point outward. Taking the currents in the 0 problems to be real, and the resulting magnetic fields to be real (because these are now the perfectly conducting solid shields in the 0 problem), this becomes

$$2Z_T \approx -i\omega (A_c/B_0) \frac{1}{I_0 I_{sh0}} \int_{S_{01}} [\underline{e}_z \cdot (\underline{n} \times \underline{H}_0)]\, \underline{e}_z \cdot [\underline{n} \times \underline{B}_{sh0}(x_m, y_m)]\, dS/\ell$$

$$-i\omega (A_c/B_0) \frac{1}{I_0 I_{sh0}} \int_{S_{02}} [\underline{e}_z \cdot (\underline{n} \times \underline{H}_{sh0})]\, \underline{e}_z \cdot [\underline{n} \times \underline{B}_0(x_m, y_m)]\, dS/\ell \tag{847}$$

or because the two terms are the same as the surfaces are taken to coalesce at the mean braid position ($\underline{n}$ points outward in this expression)

$$Z_T \approx -i\omega \left(A_c/B_0\right) \frac{1}{I_0 I_{sh0}} \int_{S_0} \left[\underline{e}_z \cdot \left(\underline{n} \times \underline{H}_0\right)\right] \underline{e}_z \cdot \left[\underline{n} \times \underline{B}_{sh0}\left(x_m, y_m\right)\right] dS/\ell$$

This expression is the same as found previously for the perfect conductor case except that the constant A_c/B_0 is modified to a complex value accounting for the diffusion into the wires of the braid.

5.3 Examples Of Impedance Parameters With Finitely Conducting Braid

In the case of lossy braid wires we expect the arguments for the perfectly conducting case to be modified in the sense that the planar braid constants A_b/B_0 and A_c/B_0 will become complex. Hence in the uniform coaxial case from (534) and (535) we expect to find

$$Z_T \sim -i\omega\mu_0 \left(A_c/B_0\right) \int_{S_0} \frac{1}{I_0 I_{sh}} \underline{H}_{sh} \cdot \underline{H}_0 dS/\ell = -i\omega\mu_0 \left(A_c/B_0\right) \frac{1}{2\pi b} \tag{848}$$

$$\Delta Z \sim -i\omega\mu_0 \left(d_0 + A_b/B_0\right) \int_{S_0} \frac{1}{I_0^2} \underline{H}_0 \cdot \underline{H}_0 dS/\ell$$

$$\sim -i\omega\mu_0 \left(b - b_0 + A_b/B_0\right) \frac{1}{2\pi b} \tag{849}$$

In the eccentric coax case from (587) and (588) we expect

$$Z_T \sim -i\omega\mu_0 \left(\frac{A_c/B_0}{2\pi b}\right) \tag{850}$$

$$\Delta Z \sim -i\omega\mu_0 \left(\frac{d_0 + A_b/B_0}{2\pi b}\right) \left(y_1/y_c\right) \tag{851}$$

If an outer ground plane is also added to the eccentric coax from (611) we expect the transfer impedance to become

$$Z_T = -i\omega\mu_0 \left(\frac{A_c/B_0}{2\pi b}\right) \frac{y_c - h_e}{y_1 - h} \tag{852}$$

If we rotate the azimuth of the eccentric coax with respect to the ground plane by $\pi/2$ then from (614) we expect

$$Z_T = -i\omega\mu_0 \left(\frac{A_c/B_0}{2\pi b}\right) \frac{y_1 y_c + h h_e}{y_1^2 + h_e^2} \tag{853}$$

5.4 Reciprocity With Perfectly Conducting Solution

An alternative convenient decomposition, which allows us to use the preceding perfectly conducting self impedance per unit length for the "external" piece of the impedance (this decomposition is different than that of the preceding section in that the "internal" piece includes contributions to the losses and stored energy within the wires in addition to changes in the stored energy outside the wires due to the finite conductivity), is based on the perfectly conducting case plus corrections for finite conductivity discussed in this section on reciprocity.

Consider the two problems of the perfectly conducting braid wires, labeled with a subscript p, and the finitely conducting braid wires. Maxwell's equations for the two problems are

$$\nabla \times \underline{E}_p = i\omega\mu\underline{H}_p \tag{854}$$

$$\nabla \times \underline{H}_p = \underline{J}_p - i\omega\varepsilon\underline{E}_p \tag{855}$$

$$\nabla \times \underline{E} = i\omega\mu\underline{H} \tag{856}$$

$$\nabla \times \underline{H} = \underline{J} - i\omega\varepsilon\underline{E} \tag{857}$$

Then we can write

$$\nabla \cdot \left(\underline{E}_p \times \underline{H} - \underline{E} \times \underline{H}_p\right) = \underline{H} \cdot \left(\nabla \times \underline{E}_p\right) - \underline{E}_p \cdot (\nabla \times \underline{H}) - \underline{H}_p \cdot (\nabla \times \underline{E}) + \underline{E} \cdot \left(\nabla \times \underline{H}_p\right)$$

$$= i\omega\mu\underline{H} \cdot \underline{H}_p + \underline{E} \cdot \underline{J}_p - \underline{E}_p \cdot \underline{J} + i\omega\varepsilon\underline{E}_p \cdot \underline{E} - i\omega\mu\underline{H}_p \cdot \underline{H} - i\omega\varepsilon\underline{E} \cdot \underline{E}_p = \underline{E} \cdot \underline{J}_p - \underline{E}_p \cdot \underline{J} \tag{858}$$

or using the divergence theorem

$$\oint_S \left(\underline{E}_p \times \underline{H} - \underline{E} \times \underline{H}_p\right) \cdot \underline{n} dS = \oint_S \left[\underline{H} \cdot \left(\underline{n} \times \underline{E}_p\right) - \underline{H}_p \cdot (\underline{n} \times \underline{E})\right] dS$$

$$= \oint_S \left[\underline{E} \cdot \left(\underline{n} \times \underline{H}_p\right) - \underline{E}_p \cdot (\underline{n} \times \underline{H})\right] dS = \int_V \left(\underline{E} \cdot \underline{J}_p - \underline{E}_p \cdot \underline{J}\right) dV \tag{859}$$

Let us take the volume to include the braid wires. The surface integral is on the center conductor and the outer chassis (which are both taken to be perfect conductors for which $\underline{n} \times \underline{E}_p = 0 = \underline{n} \times \underline{E}$ and hence the surface integrals on these two surfaces vanish. The surface integrals also exist on the input and output ports. Internal to the braid wires

$$\underline{J}_p = \underline{K}_p \delta(n) \tag{860}$$

$$\underline{E}_p = 0 \tag{861}$$

and

$$\int_{S_{ports}} \left(\underline{E}_p \times \underline{H} - \underline{E} \times \underline{H}_p\right) \cdot \underline{n} dS = \int_{S_{ports}} \left[\underline{E} \cdot \left(\underline{n} \times \underline{H}_p\right) - \underline{E}_p \cdot (\underline{n} \times \underline{H})\right] dS = \int_{S_w} \underline{E} \cdot \underline{K}_p dV \tag{862}$$

We expect the dominant contribution to the integral on the left hand side of (862) to arise outside and slightly away from the detailed braid wires. The dominant fields in this integral for a forward propagating wave will be transverse to z and will have transverse distributions (this is an E-wave and ψ is the axial electric Hertz potential)

$$\underline{E}_p/V_p = \underline{f}_V = \nabla_t \psi \sim \underline{E}/V \tag{863}$$

$$\underline{H}_p/I_p = \underline{f}_I = (V_p/I_p)\,/\,(\omega\mu_0/\Gamma)\,\underline{e}_z \times \underline{E}_p/V_p \approx (Z_0/\eta)\,\underline{e}_z \times \underline{f}_V \sim \underline{H}/I \tag{864}$$

where the Hertz potential satisfies

$$\left(\nabla_t^2 + k^2 - \Gamma^2\right)\psi \approx \nabla_t^2 \psi = 0 \tag{865}$$

Inserting this into the preceding expression gives

$$\int_{S_{port}} \left[\underline{E}\cdot\left(\underline{n}\times\underline{H}_p\right) - \underline{E}_p\cdot\left(\underline{n}\times\underline{H}\right)\right] dS \sim (VI_p - V_pI)\,(Z_0/\eta)\int_{S_{port}} \nabla\psi\cdot\left(\underline{n}\times\underline{e}_z\times\nabla\psi\right) dS$$

$$\sim -(VI_p - V_pI)\,(Z_0/\eta)\int_{S_{port}} \nabla\psi\cdot\nabla\psi dS = -(VI_p - V_pI) \tag{866}$$

For the circular symmetric case with inner port radius ρ_1 and outer port radius ρ_2 we have

$$\psi = \ln\rho/\ln\left(\rho_2/\rho_1\right) \tag{867}$$

$$E_\rho/V = f_V = \underline{e}_\rho\cdot\nabla\psi = \frac{1}{\rho\ln\left(\rho_2/\rho_1\right)} \tag{868}$$

$$H_\varphi/I = f_I = (Z_0/\eta)\,f_V = \frac{1}{2\pi\rho} \tag{869}$$

$$Z_0 = \frac{\eta}{2\pi}\ln\left(\rho_2/\rho_1\right) \tag{870}$$

5.4.1 Self Impedance Per Unit Length

For the self impedance of the inner coax we open circuit the outer port by putting

$$I_{sh} = 0 \tag{871}$$

taking the outer port boundaries to be perfect magnetic conductors with

$$\underline{n}\times\underline{H} = 0 = \underline{n}\times\underline{H}_p \tag{872}$$

Then for the inner coaxial ports (with nonzero current) we have

$$\int_{S_{ports}} \left[\underline{E}\cdot\left(\underline{n}\times\underline{H}_p\right) - \underline{E}_p\cdot\left(\underline{n}\times\underline{H}\right)\right] dS =$$

$$-\left[V(z+\ell) - V(z)\right]I_p + \left[V_p(z+\ell) - V_p(z)\right]I = \int_{S_w} \underline{E}\cdot\underline{K}_p dV \tag{873}$$

Setting the currents to be the same $I_p = I$ gives

$$V(z+\ell)-V(z)=V_p(z+\ell)-V_p(z)-\frac{1}{I_p}\int_{S_w}\underline{E}\cdot\underline{K}_p dV \tag{874}$$

Setting this to the impedance per unit length (the self impedance acts as a passive term and hence the sign change on the left side)

$$-Z_1\ell I_p=V(z+\ell)-V(z) \tag{875}$$

$$-Z_{1p}\ell I_p=V_p(z+\ell)-V_p(z) \tag{876}$$

we find

$$Z_1=Z_{1p}+\frac{1}{\ell I_p^2}\int_{S_w}\underline{E}\cdot\underline{K}_p dV \tag{877}$$

or

$$Z_1=Z_{1p}+\frac{1}{\ell}\int_{S_w}(\underline{E}/I_p)\cdot\left(\underline{K}_p/I_p\right)dV \tag{878}$$

where both fields are normalized by the same current which was injected to generate them (in one case the braid is perfectly conducting and in the other it is finitely conducting). The perfectly conducting term is purely inductive

$$Z_{1p}=Z_1^{ext}=-i\omega L_1^{ext} \tag{879}$$

The additional term is evaluated by an integration on the surface of the braid wires (note here that the surface electric field $\underline{E}=\underline{E}_1$ is generated by the return current on the braid wires from a drive to the center conductor $I=I_p$)

$$Z_1^{int}=\frac{1}{\ell}\int_{S_w}(\underline{E}_1/I_p)\cdot\left(\underline{K}_p/I_p\right)dV \tag{880}$$

with

$$Z_1=Z_1^{int}+Z_1^{ext} \tag{881}$$

5.4.2 Transfer Impedance Per Unit Length

Now we take the center conductor to be open circuited $I=0$ with a current $-I_{sh}$ on the chassis return (the braid is carrying z directed current I_{sh}). However in the perfectly conducting problem we take the current I_p to be nonzero on the center conductor with zero current on the chassis return $-I_{shp}=0$. Then we can write

$$\int_{S_{ports}}\left[\underline{E}\cdot\left(\underline{n}\times\underline{H}_p\right)-\underline{E}_p\cdot\left(\underline{n}\times\underline{H}\right)\right]dS=$$

$$-\left[V(z+\ell)-V(z)\right]I_p+\left[V_{shp}(z+\ell)-V_{shp}(z)\right]I_{sh}=\int_{S_w}\underline{E}\cdot\underline{K}_p dV \tag{882}$$

We see from this expression that if we take the braid wires to be perfectly conducting in both problems

the right hand side vanishes (with $V \to V_p$ and $I_{sh} \to I_{shp}$, but remember that each problem is driven by a current on the alternate transmission line) and by dividing by I_p and I_{shp} we find that the transfer inductances are reciprocal

$$-\left[V_p\left(z+\ell\right)-V_p\left(z\right)\right]I_p+\left[V_{shp}\left(z+\ell\right)-V_{shp}\left(z\right)\right]I_{shp}=0 \tag{883}$$

Using this in the preceding expression gives

$$\left[V\left(z+\ell\right)-V\left(z\right)\right]=\left[V_p\left(z+\ell\right)-V_p\left(z\right)\right]I_{sh}/I_{shp}-\frac{1}{I_p}\int_{S_w}\underline{E}\cdot\underline{K}_p dV \tag{884}$$

Setting $I_{sh}=I_{shp}$

$$\left[V\left(z+\ell\right)-V\left(z\right)\right]=\left[V_p\left(z+\ell\right)-V_p\left(z\right)\right]-\frac{1}{I_p}\int_{S_w}\underline{E}\cdot\underline{K}_p dV \tag{885}$$

Now writing this for the transfer as (the transfer impedance results in an active source for the interior)

$$Z_T\ell I_{sh}=V\left(z+\ell\right)-V\left(z\right) \tag{886}$$

$$Z_{Tp}\ell I_{shp}=V_p\left(z+\ell\right)-V_p\left(z\right) \tag{887}$$

we find

$$Z_T=Z_{Tp}-\frac{1}{\ell I_{shp}I_p}\int_{S_w}\underline{E}\cdot\underline{K}_p dV \tag{888}$$

or

$$Z_T=Z_{Tp}-\frac{1}{\ell}\int_{S_w}\left(\underline{E}/I_{shp}\right)\cdot\left(\underline{K}_p/I_p\right)dV \tag{889}$$

where the fields are normalized by the currents injected to generate them (in the perfectly conducting case the current is injected on the center conductor and in the finitely conducting case it is injected on the shield with a return on chassis). The perfectly conducting term is a pure reactance

$$Z_{Tp}=Z_T^{ext}=-i\omega L_T^{ext} \tag{890}$$

The additional term is evaluated by an integration on the braid wires (note here that the surface electric field $\underline{E}=\underline{E}_{sh}$ is generated by the shield current on the braid wires from a drive to the braid wires $I_{sh}=I_{shp}$, whereas the current $\underline{K}_p$ is generated by a drive to the center conductor I_p, the field and current will be in opposite directions here)

$$Z_T^{int}=\frac{1}{\ell}\int_{S_w}\left(-\underline{E}_{sh}/I_{shp}\right)\cdot\left(\underline{K}_p/I_p\right)dV \tag{891}$$

where

$$Z_T=Z_T^{int}+Z_T^{ext} \tag{892}$$

The voltage source in the transmission line equation

$$\frac{dV_1}{dz} + Z_1 I_1 = v_s \tag{893}$$

is given by

$$v_s = Z_T I_{sh} \tag{894}$$

The next sections consider the detailed solution of the braid using internal field solutions in the wires in addition to matching across the braid surface boundaries.

5.5 Vector Potential Approach For Lossy Wires

As in the perfectly conducting case we first consider an approximate approach of using only the axial components of the electric current density in the braid wires. If the wires are lossy the preceding boundary condition of no normal magnetic field at the surface is replaced by matching of the tangential electric and magnetic fields on the surface of the wire segments; because we have included only the axial components of the current (along each of the wire directions) we cannot match all components of the field on the surface, but say, only the local axial component and its normal derivative (azimuthal magnetic field). The local electric field is thus matched at the wire surface

$$i\omega A_s = i\omega A_z = E_{z_{n'}} \tag{895}$$

The tangential magnetic field is also matched on each wire

$$\frac{1}{\mu_0}\frac{\partial A_s}{\partial n} = \frac{1}{\mu_0}\frac{\partial A_z}{\partial n} = \frac{1}{i\omega\mu}\frac{\partial E_{z_{n'}}}{\partial n} \tag{896}$$

A point matching scheme in the azimuth about the wire $\varphi = \varphi_{nm'} = \pi m'/M$, $m' = 0, 1, ..., 2M-1$ can be used to set up the system of equations for the unknown multipole coefficients A_{nm} and B_{nm} on the outside, and C_{nm} and D_{nm} on the inside.

5.5.1 Symmetric Mode And Wire Currents

As in the perfectly conducting case we do not expect the wire currents to be determined from the preceding matching even though we have two conditions on the surface (instead of the single for the perfectly conducting case) since we now have unknowns inside the wire. Instead these net currents must be found from the total injected current of the braid and a condition of no net voltage difference between wires over a braid period. From Faraday's law on a contour including the surfaces of two wires over a braid period

$$\oint_{C_F} \underline{E} \cdot \underline{d\ell} = i\omega \int_{S_F} \underline{B} \cdot \underline{n} dS \tag{897}$$

we can write

$$\sum_n \left(E^1_{z_n} - E^2_{z_n}\right) \Delta z_n + \Delta V = i\omega\Phi_{12} \tag{898}$$

where the two electric fields are those on the surface of the two wires summed over the segments making up a braid period, and the magnetic flux enclosed is Φ_{12}. This can be applied with $\Delta V = 0$ to all the braid wires in a carrier strip to give $N_w - 1$ conditions on the currents I_n. In the symmetrical case the total strip current is constrained by the total braid current divided by the number of carrier strips

$$\sum_{n=1}^{N_w} I_n = I/N_s \tag{899}$$

5.5.2 Approximate Solution Inside Wires

The solution inside the wire to be used in the preceding boundary equations is found as follows. Inside each wire we take the axial electric field to solve the Helmholtz equation (time dependence $e^{-i\omega t}$ is suppressed and z here is aligned with a local segment axis)

$$\left(\nabla_t^2 + \gamma^2\right) E_z = 0 \tag{900}$$

$$\gamma^2 = i\omega\mu\sigma \tag{901}$$

$$\gamma = (1+i)/\delta \tag{902}$$

$$\delta = \sqrt{2/(\omega\mu\sigma)} \tag{903}$$

with solutions

$$E_z = \sum_{m=0}^{M} \frac{J_m(\gamma\rho)}{J_m(\gamma a)} \left[C_{nm}\cos(m\varphi) + D_{nm}\sin(m\varphi)\right] \tag{904}$$

Because the net current on each segment of a particular braid wire will be taken to be the same, the filament term constants C_{n0} are identical in each segment along a particular wire. We also take the wires to be at the same scalar potential, which can be selected to vanish for this common mode case (if the currents have odd symmetry for a differential mode we cannot always take the scalar potential to vanish)

$$\phi(\rho = a) = 0 \tag{905}$$

Hence on each of the wire surfaces

$$E_{z_n} = \sum_{m=0}^{M} \left[C_{nm}\cos(m\varphi) + D_{nm}\sin(m\varphi)\right] = i\omega A_z \tag{906}$$

where E_{z_n} is inside the nth wire segment but A_z is made up of contributions from all the wires. Note that the $m = 0$ equation is enforcing the monopole moment continuity of the electric field across the interface.

Also we have the normal derivative

$$\frac{1}{i\omega\mu}\frac{\partial}{\partial n}E_{z_n} \sim \frac{1}{i\omega\mu}\frac{\partial}{\partial\rho}E_z \tag{907}$$

$$= \frac{\gamma}{i\omega\mu}\sum_{m=0}^{M}\frac{J_m'(\gamma a)}{J_m(\gamma a)}\left[C_{nm}\cos(m\varphi_n) + D_{nm}\sin(m\varphi_n)\right] = \frac{1}{\mu_0}\frac{\partial A_z}{\partial n} \sim \frac{1}{\mu_0}\frac{\partial A_s}{\partial n} \tag{908}$$

Using the orthogonality relations

$$\int_0^{2\pi} \cos(m\varphi_n)\cos(m'\varphi_n)\,d\varphi_n = \frac{2\pi}{\varepsilon_m}\delta_{mm'} \tag{909}$$

$$\int_0^{2\pi} \sin(m\varphi_n)\cos(m'\varphi_n)\,d\varphi_n = 0 \tag{910}$$

$$\int_0^{2\pi} \sin(m\varphi_n)\sin(m'\varphi_n)\,d\varphi_n = \pi\delta_{mm'} \tag{911}$$

we can solve for the unknown constants

$$C_{nm} = i\omega\frac{\varepsilon_m}{2\pi}\int_0^{2\pi} A_s\,(n\text{th wire})\cos(m\varphi_n)\,d\varphi_n \tag{912}$$

$$D_{nm} = i\omega\frac{1}{\pi}\int_0^{2\pi} A_s\,(n\text{th wire})\sin(m\varphi_n)\,d\varphi_n \tag{913}$$

In terms of discrete values

$$\varphi_{nj} = \pi j/M\ ,\ j = 0, 1, ..., 2M-1 \tag{914}$$

$$\sum_{j=0}^{2M-1} \cos(mj\pi/N)\cos(m'j\pi/N) = \frac{4M}{\varepsilon_m\varepsilon_{M-m}}\delta_{mm'} \tag{915}$$

$$\sum_{j=1}^{2M-1} \sin(mj\pi/M)\sin(m'j\pi/M) = M\delta_{mm'}\ ,\ m \neq 0, M \tag{916}$$

$$\sum_{j=1}^{2M-1} \sin(mj\pi/M)\cos(m'j\pi/M) = 0 \tag{917}$$

we find

$$C_{nm} = i\omega\frac{\varepsilon_m}{M}\sum_{j=0}^{2M-1} A_s\left(\varphi_{nj}\right)\cos\left(m\varphi_{nj}\right) \tag{918}$$

$$D_{nm} = i\omega\frac{2}{M}\sum_{j=1}^{2M-1} A_s\left(\varphi_{nj}\right)\sin\left(m\varphi_{nj}\right) \tag{919}$$

and

$$\frac{\gamma}{i\omega\mu}\frac{J_m'(\gamma a)}{J_m(\gamma a)}C_{nm} = \frac{1}{\mu_0}\frac{\varepsilon_m}{M}\sum_{j=0}^{2M-1}\frac{\partial A_s}{\partial n}\left(\varphi_{nj}\right)\cos\left(m\varphi_{nj}\right) \tag{920}$$

$$\frac{\gamma}{i\omega\mu}\frac{J_m'(\gamma a)}{J_m(\gamma a)}D_{nm} = \frac{1}{\mu_0}\frac{2}{M}\sum_{j=1}^{2M-1}\frac{\partial A_s}{\partial n}(\varphi_{nj})\sin(m\varphi_{nj}) \tag{921}$$

Substituting from the preceding gives

$$0 = \frac{\varepsilon_m}{M}\sum_{j=0}^{2M-1}\left[\frac{\partial A_s}{\partial n}(\varphi_{nj}) - \left(\frac{\mu_0}{\mu}\gamma\right)\frac{J_m'(\gamma a)}{J_m(\gamma a)}A_s(\varphi_{nj})\right]\cos(m\varphi_{nj}) \ , \ m = 0, 1, ..., M \tag{922}$$

$$0 = \frac{2}{M}\sum_{j=1}^{2M-1}\left[\frac{\partial A_s}{\partial n}(\varphi_{nj}) - \left(\frac{\mu_0}{\mu}\gamma\right)\frac{J_m'(\gamma a)}{J_m(\gamma a)}A_s(\varphi_{nj})\right]\sin(m\varphi_{nj}) \ , \ m = 1, ..., M \tag{923}$$

where $\varepsilon_m = 2$ unless $m = 0$ where it is unity, $\gamma = \sqrt{i\omega\mu\sigma} = (1+i)/\delta$ and these are enforced for $n = 1, ..., N$ wire segments ($N = N_w N_s$). They thus constitute $(2M) \times N$ equations.

Again a problem with the preceding approach is that there will in general be transverse components of the electric current density inside the wires. Hence, so far this setup represents an approximation and extra basis functions must be included for a more rigorous representation of the internal field. Instead of proceeding to develop such basis functions we again switch to a mixed potential representation in the next subsection using Hertz potentials internal to the wires and the scalar and filament vector potentials outside.

5.6 General Internal Field Representation

The internal field must also be represented in the finitely conducting case. The most convenient characterization may be to use Hertz potentials or the axial components of the two vector potentials within the braid wires. In this section z is a local coordinate along the axis of a braid wire segment. We assume that conduction currents dominate over displacement currents within the wires. The magnetic potential is taken as

$$\underline{\Pi} = \psi \underline{e}_z \tag{924}$$

which is connected to the magnetic vector potential by

$$\underline{A} = \mu\sigma\underline{\Pi} \tag{925}$$

The electric potential is taken as

$$\underline{\Pi}_e = \psi_e \underline{e}_z \tag{926}$$

which is connected to the electric vector potential by

$$\underline{A}_e = \mu\sigma\underline{\Pi}_e \tag{927}$$

The fields are given by

$$\underline{H} = \frac{1}{\mu\sigma}\left(\nabla\nabla\cdot\underline{A}_e + \gamma^2\underline{A}_e\right) + \frac{1}{\mu}\nabla\times\underline{A} = \nabla\nabla\cdot\underline{\Pi}_e + \gamma^2\underline{\Pi}_e + \sigma\nabla\times\underline{\Pi} \tag{928}$$

$$\underline{E} = \frac{1}{\mu\sigma}\left(\nabla\nabla\cdot\underline{A} + \gamma^2\underline{A}\right) + \frac{i\omega}{\sigma}\nabla\times\underline{A}_e = \nabla\nabla\cdot\underline{\Pi} + \gamma^2\underline{\Pi} + i\omega\mu\nabla\times\underline{\Pi}_e \tag{929}$$

where the scalar axial components satisfy the Helmholtz equation

$$\left(\nabla^2 + \gamma^2\right)(\psi, \psi_e) = 0 \tag{930}$$

with propagation constant

$$\gamma^2 = i\omega\mu\sigma \tag{931}$$

The solutions of the Helmholtz equation can be used to find the fields in the nth cylindrical waveguide segment for a given cylindrical mode m as

$$\psi = e^{i\alpha z}J_m(\zeta\rho)\left[A_{nm}\cos(m\varphi) + B_{nm}\sin(m\varphi)\right] \approx e^{i\alpha z}J_m(\gamma\rho)\left[A_{nm}\cos(m\varphi) + B_{nm}\sin(m\varphi)\right] \tag{932}$$

$$\psi_e = e^{i\alpha z}J_m(\zeta\rho)\left[C_{nm}\sin(m\varphi) + D_{nm}\cos(m\varphi)\right] \approx e^{i\alpha z}J_m(\gamma\rho)\left[C_{nm}\sin(m\varphi) + D_{nm}\cos(m\varphi)\right] \tag{933}$$

where

$$\zeta^2 = \gamma^2 - \alpha^2 \tag{934}$$

$$\gamma^2 = i\omega\mu\sigma \tag{935}$$

and in the final expressions we have approximated the solution for the axial propagation constant α small compared to the internal propagation constant γ or $\alpha << \gamma$. This approximation assumes that the axial variation of the current density in the wire is slow compared to the skin depth. We will simplify the analysis at present by assuming this is valid, but it could conceivably be violated near the wire crossover point in the braid. Because the braid wires carry a net current in the same z direction (or $\underline{e}_{s_n}$ directions), this common mode current density should not concentrate near the region of crossover near-contact as oppositely directed currents would tend to do, and hence we might expect this approach to be approximately valid. The potential component ψ represents the axial component of the current density, whereas the potential component ψ_e represents the transverse components of the current. The magnetic field components for a given cylindrical mode m are then

$$H_z = \left(\frac{\partial^2}{\partial z^2} + \gamma^2\right)\psi_e \approx \gamma^2 e^{i\alpha z}J_m(\gamma\rho)\left[C_{nm}\sin(m\varphi) + D_{nm}\cos(m\varphi)\right] \tag{936}$$

$$H_\varphi = -\sigma\frac{\partial\psi}{\partial\rho} + \left(\frac{\partial^2}{\rho\partial\varphi\partial z}\right)\psi_e$$

$$\approx -\sigma\gamma e^{i\alpha z}J_m'(\gamma\rho)\left[A_{nm}\cos(m\varphi) + B_{nm}\sin(m\varphi)\right] + i\alpha\frac{m}{\rho}e^{i\alpha z}J_m(\gamma\rho)\left[C_{nm}\cos(m\varphi) - D_{nm}\sin(m\varphi)\right] \tag{937}$$

$$H_\rho = \sigma \frac{1}{\rho} \frac{\partial \psi}{\partial \varphi} + \left(\frac{\partial^2}{\partial \rho \partial z} \right) \psi_e$$

$$\approx -\sigma \frac{m}{\rho} e^{i\alpha z} J_m(\gamma\rho) \left[A_{nm} \sin(m\varphi) - B_{nm} \cos(m\varphi) \right] + i\alpha\gamma e^{i\alpha z} J'_m(\gamma\rho) \left[C_{nm} \sin(m\varphi) + D_{nm} \cos(m\varphi) \right] \tag{938}$$

Let us take the z variation to be slow enough that we can approximate it as a linear function for each nth segment and a particular mth Fourier azimuthal mode (here the arrow indicates an inverse transform replacement of the Fourier coefficients and kernel by the spatial function)

$$e^{i\alpha z} A_{nm} \to F^e_{nm}(z) = F^e_{nm} p_n(z) \tag{939}$$

$$e^{i\alpha z} B_{nm} \to F^o_{nm}(z) = F^o_{nm} p_n(z) \tag{940}$$

$$e^{i\alpha z} C_{nm} \to G^e_{nm}(z) = G^e_{nm} \widetilde{p}_n(z) \tag{941}$$

$$e^{i\alpha z} D_{nm} \to G^o_{nm}(z) = G^o_{nm} \widetilde{p}_n(z) \tag{942}$$

The linear interpolation functions associated with ψ are

$$p_n(z) = \frac{z - z_{n-1}}{z_n - z_{n-1}} \ , \ z_n > z > z_{n-1}$$

$$= \frac{z_{n+1} - z}{z_{n+1} - z_n} \ , \ z_{n+1} > z > z_n$$

$$= 0 \ , \ \text{otherwise} \tag{943}$$

and the grid values, where the H field values are well defined and continuous, are taken to be z_n. The other linear interpolation functions associated with ψ_e are

$$\widetilde{p}_n(z) = \frac{z - \widetilde{z}_{n-1}}{\widetilde{z}_n - \widetilde{z}_{n-1}} \ , \ \widetilde{z}_n > z > \widetilde{z}_{n-1}$$

$$= \frac{\widetilde{z}_{n+1} - z}{\widetilde{z}_{n+1} - \widetilde{z}_n} \ , \ \widetilde{z}_{n+1} > z > \widetilde{z}_n$$

$$= 0 \ , \ \text{otherwise} \tag{944}$$

where, for example, we can take the staggered grid, where the E field values are continuous and well defined, to be

$$\widetilde{z}_n = (z_{n+1} + z_n)/2 \tag{945}$$

The potentials are then given by

$$\psi \approx J_m(\gamma\rho)\left[F_{nm}^e(z)\cos(m\varphi) + F_{nm}^o(z)\sin(m\varphi)\right] \tag{946}$$

$$\psi_e \approx J_m(\gamma\rho)\left[G_{nm}^e(z)\sin(m\varphi) + G_{nm}^o(z)\cos(m\varphi)\right] \tag{947}$$

and the magnetic field is

$$H_z \approx \gamma^2 J_m(\gamma\rho)\left[G_{nm}^e(z)\sin(m\varphi) + G_{nm}^o(z)\cos(m\varphi)\right] \tag{948}$$

$$H_\varphi \approx -\sigma\gamma J_m'(\gamma\rho)\left[F_{nm}^e(z)\cos(m\varphi) + F_{nm}^o(z)\sin(m\varphi)\right]$$

$$+\frac{m}{\rho}J_m(\gamma\rho)\left[G_{nm}^{e\prime}(z)\cos(m\varphi) - G_{nm}^{o\prime}(z)\sin(m\varphi)\right] \tag{949}$$

$$H_\rho \approx -\sigma\frac{m}{\rho}J_m(\gamma\rho)\left[F_{nm}^e(z)\sin(m\varphi) - F_{nm}^o(z)\cos(m\varphi)\right]$$

$$+\gamma J_m'(\gamma\rho)\left[G_{nm}^{e\prime}(z)\sin(m\varphi) + G_{nm}^{o\prime}(z)\cos(m\varphi)\right] \tag{950}$$

The electric field is

$$E_z = \left(\frac{\partial^2}{\partial z^2} + \gamma^2\right)\psi \approx \gamma^2 J_m(\gamma\rho)\left[F_{nm}^e(z)\cos(m\varphi) + F_{nm}^o(z)\sin(m\varphi)\right] \tag{951}$$

$$E_\varphi = -i\omega\mu\frac{\partial\psi_e}{\partial\rho} + \left(\frac{\partial^2}{\rho\partial\varphi\partial z}\right)\psi$$

$$\approx -i\omega\mu\gamma J_m'(\gamma\rho)\left[G_{nm}^e(z)\sin(m\varphi) + G_{nm}^o(z)\cos(m\varphi)\right] - \frac{m}{\rho}J_m(\gamma\rho)\left[F_{nm}^{e\prime}(z)\sin(m\varphi) - F_{nm}^{o\prime}(z)\cos(m\varphi)\right] \tag{952}$$

$$E_\rho = i\omega\mu\frac{1}{\rho}\frac{\partial\psi_e}{\partial\varphi} + \left(\frac{\partial^2}{\partial\rho\partial z}\right)\psi$$

$$\approx i\omega\mu\frac{m}{\rho}J_m(\gamma\rho)\left[G_{nm}^e(z)\cos(m\varphi) - G_{nm}^o(z)\sin(m\varphi)\right] + \gamma J_m'(\gamma\rho)\left[F_{nm}^{e\prime}(z)\cos(m\varphi) + F_{nm}^{o\prime}(z)\sin(m\varphi)\right] \tag{953}$$

For $m = 0$ the total current on a wire segment from $2\pi a H_\varphi(\rho = a)$ is

$$I_n \approx -\sigma\gamma 2\pi a J_0'(\gamma a) F_{n0}^e = \sigma\gamma 2\pi a J_1(\gamma a) F_{n0}^e \tag{954}$$

Because we will take this to not vary on a segment we must have F_{n0}^e be a constant in z. In fact we will take the wires to be insulated from one another and this will be a constant along each wire.

Because of the high level of wire conductivity we need to set the normal component of the current at the surface equal to zero

$$J_\rho(\rho = a) = \sigma E_\rho(\rho = a) \approx 0 \tag{955}$$

or

$$-i\omega\mu m J_m(\gamma a)\left[G^e_{nm}(z)\cos(m\varphi) - G^o_{nm}(z)\sin(m\varphi)\right] = \gamma a J'_m(\gamma a)\left[F^{e\prime}_{nm}(z)\cos(m\varphi) + F^{o\prime}_{nm}(z)\sin(m\varphi)\right] \tag{956}$$

and thus

$$-i\omega\mu m J_m(\gamma a)\, G^e_{nm}(z) = \gamma a J'_m(\gamma a)\, F^{e\prime}_{nm}(z) \tag{957}$$

$$i\omega\mu m J_m(\gamma a)\, G^o_{nm}(z) = \gamma a J'_m(\gamma a)\, F^{o\prime}_{nm}(z) \tag{958}$$

The preceding two equations connect the axial derivatives of the axial coefficients with the value of the transverse coefficients. This makes sense because to have axial variations we would expect transverse currents to exist.

If it were desirable to include the perturbing effects of the slow meander of the braid wires on the interior fields we may be able to use some previous treatments of bent waveguides [16]. At present (in the next sections) we assume local cylindrical coordinate systems are aligned with each of the n segments.

5.7 Combination Of External Scalar And Filament Vector Potentials With Internal Hertz Potentials

As discussed in the perfectly conducting case, it may be more straightforward externally to the braid wires to use the field representation involving a filament vector potential from each wire carrying the net current plus a scalar magnetic potential to match the boundary conditions

$$\underline{B} = \mu_0 \underline{H} = -\mu_0 \nabla\phi_m + \nabla \times \underline{A}_f \tag{959}$$

The boundary continuity relations for the magnetic field at $\rho = a$ are then

$$H_z \approx \gamma^2 J_m(\gamma a)\left[G^e_{nm}(z)\sin(m\varphi) + G^o_{nm}(z)\cos(m\varphi)\right]$$

$$= \left[-\frac{\partial\phi_m}{\partial z} + \frac{1}{\mu_0}\left(\nabla \times \underline{A}_f\right)\cdot \underline{e}_z\right]_m \tag{960}$$

$$H_\varphi \approx -\sigma\gamma J'_m(\gamma a)\left[F^e_{nm}(z)\cos(m\varphi) + F^o_{nm}(z)\sin(m\varphi)\right] + \frac{m}{a} J_m(\gamma a)\left[G^{e\prime}_{nm}\cos(m\varphi) - G^{o\prime}_{nm}(z)\sin(m\varphi)\right]$$

$$= \left[-\frac{1}{\rho}\frac{\partial\phi_m}{\partial\varphi} + \frac{1}{\mu_0}\left(\nabla \times \underline{A}_f\right)\cdot \underline{e}_\varphi\right]_m \tag{961}$$

$$B_\rho \approx -\mu\sigma\frac{m}{a} J_m(\gamma a)\left[F^e_{nm}(z)\sin(m\varphi) - F^o_{nm}(z)\cos(m\varphi)\right] + \mu\gamma J'_m(\gamma a)\left[G^{e\prime}_{nm}(z)\sin(m\varphi) + G^{o\prime}_{nm}(z)\cos(m\varphi)\right]$$

$$= \left[-\mu_0 \frac{\partial \phi_m}{\partial \rho} + \left(\nabla \times \underline{A}_f \right) \cdot \underline{e}_\rho \right]_m \tag{962}$$

where the subscript m on the brackets means that only the $\sin(m\varphi)$ and $\cos(m\varphi)$ variations of the quantity is retained. Solving for the coefficients from continuity of H_z gives

$$G^e_{nm}(z) = \frac{1}{\pi \gamma^2 J_m(\gamma a)} \int_0^{2\pi} \left[-\frac{\partial \phi_m}{\partial z} + \frac{1}{\mu_0} \left(\nabla \times \underline{A}_f \right) \cdot \underline{e}_z \right] \sin(m\varphi)\, d\varphi \tag{963}$$

$$G^o_{nm}(z) = \frac{\varepsilon_m}{2\pi \gamma^2 J_m(\gamma a)} \int_0^{2\pi} \left[-\frac{\partial \phi_m}{\partial z} + \frac{1}{\mu_0} \left(\nabla \times \underline{A}_f \right) \cdot \underline{e}_z \right] \cos(m\varphi)\, d\varphi \tag{964}$$

and continuity of H_φ gives

$$G^{e\prime}_{nm}(z) = \sigma\gamma (a/m) \frac{J'_m(\gamma a)}{J_m(\gamma a)} F^e_{nm}(z) + \frac{\varepsilon_m a/m}{2\pi J_m(\gamma a)} \int_0^{2\pi} \left[-\frac{1}{a} \frac{\partial \phi_m}{\partial \varphi} + \frac{1}{\mu_0} \left(\nabla \times \underline{A}_f \right) \cdot \underline{e}_\varphi \right] \cos(m\varphi)\, d\varphi \tag{965}$$

$$G^{o\prime}_{nm}(z) = -\sigma\gamma (a/m) \frac{J'_m(\gamma a)}{J_m(\gamma a)} F^o_{nm}(z) - \frac{a/m}{\pi J_m(\gamma a)} \int_0^{2\pi} \left[-\frac{1}{a} \frac{\partial \phi_m}{\partial \varphi} + \frac{1}{\mu_0} \left(\nabla \times \underline{A}_f \right) \cdot \underline{e}_\varphi \right] \sin(m\varphi)\, d\varphi \tag{966}$$

Using the preceding $E_\rho(\rho = a) = 0$ expressions

$$-i\omega\mu m J_m(\gamma a)\, G^e_{nm}(z) = \gamma a J'_m(\gamma a)\, F^{e\prime}_{nm}(z) \tag{967}$$

$$i\omega\mu m J_m(\gamma a)\, G^o_{nm}(z) = \gamma a J'_m(\gamma a)\, F^{o\prime}_{nm}(z) \tag{968}$$

or to use these in (965) and (966)

$$G^{e\prime}_{nm}(z) = -\frac{1}{i\omega\mu} \frac{\gamma a J'_m(\gamma a)}{m J_m(\gamma a)} F^{e\prime\prime}_{nm}(z) \tag{969}$$

$$G^{o\prime}_{nm}(z) = \frac{1}{i\omega\mu} \frac{\gamma a J'_m(\gamma a)}{m J_m(\gamma a)} F^{o\prime\prime}_{nm}(z) \tag{970}$$

we can write the magnetic field continuity conditions as

$$H_z \approx \frac{\gamma a}{i\omega\mu m} \gamma^2 J'_m(\gamma a) \left[-F^{e\prime}_{nm}(z) \sin(m\varphi) + F^{o\prime}_{nm}(z) \cos(m\varphi) \right]$$

$$= \left[-\frac{\partial \phi_m}{\partial z} + \frac{1}{\mu_0} \left(\nabla \times \underline{A}_f \right) \cdot \underline{e}_z \right]_m \tag{971}$$

$$H_\varphi \approx -\sigma\gamma J'_m(\gamma a) \left[\left\{ F^e_{nm}(z) + \frac{F^{e\prime\prime}_{nm}}{i\omega\mu\sigma} \right\} \cos(m\varphi) + \left\{ F^o_{nm}(z) + \frac{F^{o\prime\prime}_{nm}(z)}{i\omega\mu\sigma} \right\} \sin(m\varphi) \right]$$

$$\approx -\sigma\gamma J'_m(\gamma a) \left(1 + \frac{1}{\gamma^2} \frac{\partial^2}{\partial z^2} \right) \left[F^e_{nm}(z) \cos(m\varphi) + F^o_{nm}(z) \sin(m\varphi) \right]$$

$$= \left[-\frac{1}{\rho}\frac{\partial \phi_m}{\partial \varphi} + \frac{1}{\mu_0}\left(\nabla \times \underline{A}_f\right) \cdot \underline{e}_\varphi \right]_m \tag{972}$$

$$B_\rho \approx -\mu\sigma \frac{m}{a} J_m(\gamma a)\left[F_{nm}^e(z)\sin(m\varphi) - F_{nm}^o(z)\cos(m\varphi)\right] + \mu\gamma J_m'(\gamma a)\left[G_{nm}^{e\prime}(z)\sin(m\varphi) + G_{nm}^{o\prime}(z)\cos(m\varphi)\right]$$

$$\approx -\mu\sigma \frac{m}{a} J_m(\gamma a)\left[1 + \left\{\frac{\gamma a J_m'(\gamma a)}{m J_m(\gamma a)}\right\}^2 \frac{1}{\gamma^2}\frac{\partial^2}{\partial z^2}\right]\left[F_{nm}^e(z)\sin(m\varphi) - F_{nm}^o(z)\cos(m\varphi)\right]$$

$$= \left[-\mu_0 \frac{\partial \phi_m}{\partial \rho} + \left(\nabla \times \underline{A}_f\right) \cdot \underline{e}_\rho \right]_m \tag{973}$$

The expressions resulting from continuity of H_z can be written in terms of $F_{nm}^e(z)$ and $F_{nm}^o(z)$ alone

$$F_{nm}^{e\prime}(z) = -\frac{m/a}{\sigma\gamma J_m'(\gamma a)}\frac{1}{\pi}\int_0^{2\pi}\left[-\frac{\partial \phi_m}{\partial z} + \frac{1}{\mu_0}\left(\nabla \times \underline{A}_f\right)\cdot \underline{e}_z\right]\sin(m\varphi)\,d\varphi \tag{974}$$

$$F_{nm}^{o\prime}(z) = \frac{m/a}{\sigma\gamma J_m'(\gamma a)}\frac{\varepsilon_m}{2\pi}\int_0^{2\pi}\left[-\frac{\partial \phi_m}{\partial z} + \frac{1}{\mu_0}\left(\nabla \times \underline{A}_f\right)\cdot \underline{e}_z\right]\cos(m\varphi)\,d\varphi \tag{975}$$

and the expressions resulting from continuity of H_φ can be written in terms of $F_{nm}^e(z)$ and $F_{nm}^o(z)$ alone

$$\left[1 + \frac{1}{\gamma^2}\frac{\partial^2}{\partial z^2}\right] F_{nm}^e(z) = -\frac{1}{\sigma\gamma J_m'(\gamma a)}\frac{\varepsilon_m}{2\pi}\int_0^{2\pi}\left[-\frac{1}{a}\frac{\partial \phi_m}{\partial \varphi} + \frac{1}{\mu_0}\left(\nabla \times \underline{A}_f\right)\cdot \underline{e}_\varphi\right]\cos(m\varphi)\,d\varphi \tag{976}$$

$$\left[1 + \frac{1}{\gamma^2}\frac{\partial^2}{\partial z^2}\right] F_{nm}^o(z) = -\frac{1}{\sigma\gamma J_m'(\gamma a)}\frac{1}{\pi}\int_0^{2\pi}\left[-\frac{1}{a}\frac{\partial \phi_m}{\partial \varphi} + \frac{1}{\mu_0}\left(\nabla \times \underline{A}_f\right)\cdot \underline{e}_\varphi\right]\sin(m\varphi)\,d\varphi \tag{977}$$

The preceding B_ρ continuity condition can be grouped as

$$-\mu\sigma \frac{m}{a} J_m(\gamma a)\left(1 + \frac{1}{\gamma^2}\frac{\partial^2}{\partial z^2}\right)\left[F_{nm}^e(z)\sin(m\varphi) - F_{nm}^o(z)\cos(m\varphi)\right]$$

$$-\mu\sigma \frac{m}{a} J_m(\gamma a)\left[\left\{\frac{\gamma a J_m'(\gamma a)}{m J_m(\gamma a)}\right\}^2 - 1\right]\frac{1}{\gamma^2}\frac{\partial^2}{\partial z^2}\left[F_{nm}^e(z)\sin(m\varphi) - F_{nm}^o(z)\cos(m\varphi)\right]$$

$$= \left[-\mu_0 \frac{\partial \phi_m}{\partial \rho} + \left(\nabla \times \underline{A}_f\right) \cdot \underline{e}_\rho \right]_m \tag{978}$$

or

$$\mu\sigma \frac{m}{a} J_m(\gamma a)\left(1 + \frac{1}{\gamma^2}\frac{\partial^2}{\partial z^2}\right) F_{nm}^o(z)$$

$$+\mu\sigma \frac{m}{a} J_m(\gamma a)\left[\left\{\frac{\gamma a J_m'(\gamma a)}{m J_m(\gamma a)}\right\}^2 - 1\right]\frac{1}{\gamma^2}\frac{\partial^2}{\partial z^2} F_{nm}^o(z)$$

$$= \frac{\varepsilon_m}{2\pi} \int_{-\pi}^{\pi} \left[-\mu_0 \frac{\partial \phi_m}{\partial \rho} + \left(\nabla \times \underline{A}_f \right) \cdot \underline{e}_\rho \right] \cos(m\varphi)\, d\varphi \tag{979}$$

and

$$\mu\sigma \frac{m}{a} J_m(\gamma a) \left(1 + \frac{1}{\gamma^2} \frac{\partial^2}{\partial z^2} \right) F_{nm}^e(z)$$

$$+\mu\sigma \frac{m}{a} J_m(\gamma a) \left[\left\{ \frac{\gamma a J_m'(\gamma a)}{m J_m(\gamma a)} \right\}^2 - 1 \right] \frac{1}{\gamma^2} \frac{\partial^2}{\partial z^2} F_{nm}^e(z)$$

$$= -\frac{1}{\pi} \int_{-\pi}^{\pi} \left[-\mu_0 \frac{\partial \phi_m}{\partial \rho} + \left(\nabla \times \underline{A}_f \right) \cdot \underline{e}_\rho \right] \sin(m\varphi)\, d\varphi \tag{980}$$

These final expressions provide the connection between the Fourier coefficients associated with the tangential derivatives of the external potential and the Fourier coefficients associated with the radial derivative of the external potential.

5.7.1 Symmetric Mode Currents

As in the vector potential treatment above we do not expect the net wire currents to be determined from the preceding matching. Instead these net currents must be found from the total injected current of the braid and a condition of no net voltage difference between wires over a braid period. Again from Faraday's law on a contour including the surfaces of two wires over a braid period

$$\oint_{C_F} \underline{E} \cdot \underline{d\ell} = i\omega \int_{S_F} \underline{B} \cdot \underline{n} dS \tag{981}$$

or

$$\sum_n \left(E_{z_n}^1 - E_{z_n}^2 \right) \Delta z_n + \Delta V = i\omega \Phi_{12} \tag{982}$$

where the two electric fields are those on the surface of the two wires summed over the segments making up a braid period, and the magnetic flux enclosed is Φ_{12}. This can be applied with $\Delta V = 0$ to all the braid wires in a carrier strip to give $N_w - 1$ conditions on the currents I_n. In the symmetrical case the total strip current is constrained by the total braid current divided by the number of carrier strips

$$\sum_{n=1}^{N_w} I_n = I/N_s \tag{983}$$

In this case where we have an accurate treatment of the internal electric field inside the wires, we can alternatively apply the condition inside the wires over an axial period. Hence we can apply continuity of the internal voltage over a period

$$\int_{C_{pj}} \underline{E}^j \cdot \underline{d\ell} = \int_{C_{p1}} \underline{E}^1 \cdot \underline{d\ell}\ ,\ j = 2, ..., N_w \tag{984}$$

where the contour C_{pj} extends over an axial period inside of the jth wire.

5.7.2 Limiting Cases Of Conductivity

If the braid wires are electrically thin, with $\gamma a << 1$, then

$$\frac{\gamma a J_m'(\gamma a)}{m J_m(\gamma a)} \sim 1 \tag{985}$$

and the bracketed factor of the second terms of the preceding expressions tend to zero. We can then write

$$-\mu \frac{1}{\pi} \int_0^{2\pi} \left[-\frac{1}{a} \frac{\partial \phi_m}{\partial \varphi} + \frac{1}{\mu_0} \left(\nabla \times \underline{A}_f \right) \cdot \underline{e}_\varphi \right] \sin(m\varphi)\, d\varphi$$

$$\sim \frac{\varepsilon_m}{2\pi} \int_{-\pi}^{\pi} \left[-\mu_0 \frac{\partial \phi_m}{\partial \rho} + \left(\nabla \times \underline{A}_f \right) \cdot \underline{e}_\rho \right] \cos(m\varphi)\, d\varphi \tag{986}$$

and

$$-\mu \frac{\varepsilon_m}{2\pi} \int_0^{2\pi} \left[-\frac{1}{a} \frac{\partial \phi_m}{\partial \varphi} + \frac{1}{\mu_0} \left(\nabla \times \underline{A}_f \right) \cdot \underline{e}_\varphi \right] \cos(m\varphi)\, d\varphi$$

$$\sim -\frac{1}{\pi} \int_{-\pi}^{\pi} \left[-\mu_0 \frac{\partial \phi_m}{\partial \rho} + \left(\nabla \times \underline{A}_f \right) \cdot \underline{e}_\rho \right] \sin(m\varphi)\, d\varphi \tag{987}$$

This implies that on the external surface of the wire

$$-\mu H_\varphi \,(\text{odd in } \varphi) = \mu_0 H_\rho \,(\text{even in } \varphi) \tag{988}$$

$$\mu H_\varphi \,(\text{even in } \varphi) = \mu_0 H_\rho \,(\text{odd in } \varphi) \tag{989}$$

Noting that the static fields about a wire driven by a plane field with permeability μ having odd parity for H_φ are

$$H_\varphi = -\frac{2}{1 + \mu/\mu_0} H_0 \sin\varphi \tag{990}$$

$$H_\rho = \frac{2}{1 + \mu/\mu_0} H_0 \cos\varphi \ , \ \text{internal}$$

$$= \frac{\mu}{\mu_0} \frac{2}{1 + \mu/\mu_0} H_0 \cos\varphi \ , \ \text{external} \tag{991}$$

and that the fields for even parity for H_φ are

$$H_\varphi = \frac{2}{1 + \mu/\mu_0} H_0 \cos\varphi \tag{992}$$

$$H_\rho = \frac{2}{1 + \mu/\mu_0} H_0 \sin\varphi \ , \ \text{internal}$$

$$= \frac{\mu}{\mu_0}\frac{2}{1+\mu/\mu_0}H_0 \sin\varphi \ , \ \text{external} \tag{993}$$

we see that (988) and (989) are satisfied. For $\mu = \mu_0$ these become

$$\frac{1}{\pi}\int_0^{2\pi}\left[-\frac{1}{a}\frac{\partial\phi_m}{\partial\varphi} + \frac{1}{\mu_0}\left(\nabla\times\underline{A}_f\right)\cdot\underline{e}_\varphi\right]\sin\left(m\varphi\right)d\varphi$$

$$= -\frac{\varepsilon_m}{2\pi}\int_0^{2\pi}\left[-\frac{\partial\phi_m}{\partial\rho} + \frac{1}{\mu_0}\left(\nabla\times\underline{A}_f\right)\cdot\underline{e}_\rho\right]\cos\left(m\varphi\right)d\varphi \tag{994}$$

and

$$\frac{\varepsilon_m}{2\pi}\int_0^{2\pi}\left[-\frac{1}{a}\frac{\partial\phi_m}{\partial\varphi} + \frac{1}{\mu_0}\left(\nabla\times\underline{A}_f\right)\cdot\underline{e}_\varphi\right]\cos\left(m\varphi\right)d\varphi$$

$$= \frac{1}{\pi}\int_0^{2\pi}\left[-\frac{\partial\phi_m}{\partial\rho} + \frac{1}{\mu_0}\left(\nabla\times\underline{A}_f\right)\cdot\underline{e}_\rho\right]\sin\left(m\varphi\right)d\varphi \tag{995}$$

or at $\rho = a$ but for the $m \neq 0$ part of the field (the filament generated field from the net current in the self wire is not included)

$$-H_\varphi\left(\text{odd in }\varphi\right) = H_\rho\left(\text{even in }\varphi\right) \tag{996}$$

$$H_\varphi\left(\text{even in }\varphi\right) = H_\rho\left(\text{odd in }\varphi\right) \tag{997}$$

Noting that

$$\underline{e}_\rho = \underline{e}_x\cos\varphi + \underline{e}_y\sin\varphi \tag{998}$$

$$\underline{e}_\varphi = -\underline{e}_x\sin\varphi + \underline{e}_y\cos\varphi \tag{999}$$

and with

$$\underline{H} = H_x\underline{e}_x + H_y\underline{e}_y$$

$$= \left(H_x\cos\varphi + H_y\sin\varphi\right)\underline{e}_\rho + \left(-H_x\sin\varphi + H_y\cos\varphi\right)\underline{e}_\varphi \tag{1000}$$

we can see that this limit corresponds to a uniform field inside the wire, which makes sense for slowly varying external drives because for $\mu = \mu_0$ it is nonmagnetic and will not distort the locally plane field in this limit.

If the wire is electrically thick, with $\gamma a >> 1$, or $a/\delta \to \infty$ with $\gamma = \left(1+i\right)/\delta$ and skin depth

$$\delta = \sqrt{2/\left(\omega\mu\sigma\right)} \tag{1001}$$

we have

$$\frac{J_m'(\gamma a)}{J_m(\gamma a)} \sim \frac{-\sin(\gamma a - m\pi/2 - \pi/4)}{\cos(\gamma a - m\pi/2 - \pi/4)} \sim -i \tag{1002}$$

and again from the B_ρ continuity equation

$$-\mu\sigma J_m'(\gamma a)\frac{m}{a}i\left(1+\frac{1}{\gamma^2}\frac{\partial^2}{\partial z^2}\right)\left[F_{nm}^e(z)\sin(m\varphi) - F_{nm}^o(z)\cos(m\varphi)\right]$$

$$+\mu\sigma J_m'(\gamma a)\, i\left(\frac{a}{m}\right)\frac{\partial^2}{\partial z^2}\left[F_{nm}^e(z)\sin(m\varphi) - F_{nm}^o(z)\cos(m\varphi)\right]$$

$$= \left[-\mu_0\frac{\partial\phi_m}{\partial\rho} + \left(\nabla\times\underline{A}_f\right)\cdot\underline{e}_\rho\right]_m \tag{1003}$$

with

$$\left[1+\frac{1}{\gamma^2}\frac{\partial^2}{\partial z^2}\right]F_{nm}^e(z) = -\frac{1}{\sigma\gamma J_m'(\gamma a)}\frac{\varepsilon_m}{2\pi}\int_0^{2\pi}\left[-\frac{1}{a}\frac{\partial\phi_m}{\partial\varphi} + \frac{1}{\mu_0}\left(\nabla\times\underline{A}_f\right)\cdot\underline{e}_\varphi\right]\cos(m\varphi)\,d\varphi \tag{1004}$$

$$\left[1+\frac{1}{\gamma^2}\frac{\partial^2}{\partial z^2}\right]F_{nm}^o(z) = -\frac{1}{\sigma\gamma J_m'(\gamma a)}\frac{1}{\pi}\int_0^{2\pi}\left[-\frac{1}{a}\frac{\partial\phi_m}{\partial\varphi} + \frac{1}{\mu_0}\left(\nabla\times\underline{A}_f\right)\cdot\underline{e}_\varphi\right]\sin(m\varphi)\,d\varphi \tag{1005}$$

The left side of (1003) using (1004) and (1005) appears to be $O\left(\frac{1}{\gamma a}\right)$ of the right side of (1003) and thus this becomes, for $m \neq 0$

$$\left[-\frac{\partial\phi_m}{\partial\rho} + \frac{1}{\mu_0}\left(\nabla\times\underline{A}_f\right)\cdot\underline{e}_\rho\right]_m = [H_n]_m \to 0 \tag{1006}$$

5.7.3 Axisymmetric Term

For $m = 0$ the current on the nth wire is connected to the coefficient F_{n0}^e by means of

$$I_n \approx -\sigma\gamma 2\pi a J_0'(\gamma a)\,F_{n0}^e = \sigma\gamma 2\pi a J_1(\gamma a)\,F_{n0}^e \tag{1007}$$

The $m = 0$ magnetic fields at the surface are

$$H_z \approx \gamma^2 J_0(\gamma\rho)\,G_{n0}^o(z)$$

$$= \left[-\frac{\partial\phi_m}{\partial z} + \frac{1}{\mu_0}\left(\nabla\times\underline{A}_f\right)\cdot\underline{e}_z\right]_0 \tag{1008}$$

$$H_\varphi = \frac{I_n}{2\pi a} \approx -\sigma\gamma J_0'(\gamma a)\,F_{n0}^e(z)$$

$$= \left[-\frac{1}{\rho}\frac{\partial\phi_m}{\partial\varphi} + \frac{1}{\mu_0}\left(\nabla\times\underline{A}_f\right)\cdot\underline{e}_\varphi\right]_0 \tag{1009}$$

$$B_\rho \approx \mu\gamma J_0'(\gamma a)\, G_{n0}^{o\prime}(z)$$

$$= \left[-\mu_0 \frac{\partial \phi_m}{\partial \rho} + \left(\nabla \times \underline{A}_f\right) \cdot \underline{e}_\rho\right]_0 \tag{1010}$$

Ordinarily we would expect that $[H_\rho]_0 = 0$ due to the absence of magnetic charge. However this net flux can arise as a result of net flux entering or leaving neighboring segments.

5.7.4 Redundancy Of Continuity Of The Axial Magnetic Field

If we match the axial magnetic field rather than the radial component (retaining the second derivative in z)

$$H_z \approx J_m(\gamma a)\left(\frac{\partial^2}{\partial z^2} + \gamma^2\right)\left[G_{nm}^e(z)\sin(m\varphi) + G_{nm}^o(z)\cos(m\varphi)\right]$$

$$= \left[-\frac{\partial \phi_m}{\partial z} + \frac{1}{\mu_0}\left(\nabla \times \underline{A}_f\right) \cdot \underline{e}_z\right]_m \tag{1011}$$

using the preceding continuity of H_φ expressions

$$\left[1 + \frac{1}{\gamma^2}\frac{\partial^2}{\partial z^2}\right] F_{nm}^e(z) = -\frac{1}{\sigma\gamma J_m'(\gamma a)}\frac{\varepsilon_m}{2\pi}\int_0^{2\pi}\left[-\frac{1}{a}\frac{\partial \phi_m}{\partial \varphi} + \frac{1}{\mu_0}\left(\nabla \times \underline{A}_f\right) \cdot \underline{e}_\varphi\right]\cos(m\varphi)\, d\varphi \tag{1012}$$

$$\left[1 + \frac{1}{\gamma^2}\frac{\partial^2}{\partial z^2}\right] F_{nm}^o(z) = -\frac{1}{\sigma\gamma J_m'(\gamma a)}\frac{1}{\pi}\int_0^{2\pi}\left[-\frac{1}{a}\frac{\partial \phi_m}{\partial \varphi} + \frac{1}{\mu_0}\left(\nabla \times \underline{A}_f\right) \cdot \underline{e}_\varphi\right]\sin(m\varphi)\, d\varphi \tag{1013}$$

Using the expressions from $E_\rho(\rho = a) = 0$

$$-i\omega\mu m J_m(\gamma a)\, G_{nm}^e(z) = \gamma a J_m'(\gamma a)\, F_{nm}^{e\prime}(z) \tag{1014}$$

$$i\omega\mu m J_m(\gamma a)\, G_{nm}^o(z) = \gamma a J_m'(\gamma a)\, F_{nm}^{o\prime}(z) \tag{1015}$$

we can write the H_z expression as

$$H_z \approx \frac{\gamma a}{i\omega\mu m} J_m'(\gamma a)\left(\frac{\partial^2}{\partial z^2} + \gamma^2\right)\left[-F_{nm}^{e\prime}(z)\sin(m\varphi) + F_{nm}^{o\prime}(z)\cos(m\varphi)\right]$$

$$= \left[-\frac{\partial \phi_m}{\partial z} + \frac{1}{\mu_0}\left(\nabla \times \underline{A}_f\right) \cdot \underline{e}_z\right]_m \tag{1016}$$

Then we obtain

$$\frac{\gamma^3 a}{i\omega\mu m} J_m'(\gamma a)\left(\frac{1}{\gamma^2}\frac{\partial^2}{\partial z^2} + 1\right) F_{nm}^{o\prime}(z)$$

$$= \frac{\varepsilon_m}{2\pi} \int_0^{2\pi} \left[-\frac{\partial \phi_m}{\partial z} + \frac{1}{\mu_0} \left(\nabla \times \underline{A}_f \right) \cdot \underline{e}_z \right]_m \cos(m\varphi)\, d\varphi \tag{1017}$$

$$\frac{\gamma^3 a}{i\omega\mu m} J'_m(\gamma a) \left(\frac{1}{\gamma^2} \frac{\partial^2}{\partial z^2} + 1 \right) F^{e\prime}_{nm}(z)$$

$$= -\frac{1}{\pi} \int_0^{2\pi} \left[-\frac{\partial \phi_m}{\partial z} + \frac{1}{\mu_0} \left(\nabla \times \underline{A}_f \right) \cdot \underline{e}_z \right]_m \sin(m\varphi)\, d\varphi \tag{1018}$$

or using (1013) and (1017)

$$-\frac{a}{m} \frac{1}{\pi} \frac{\partial}{\partial z} \int_0^{2\pi} \left[-\frac{1}{a} \frac{\partial \phi_m}{\partial \varphi} + \frac{1}{\mu_0} \left(\nabla \times \underline{A}_f \right) \cdot \underline{e}_\varphi \right] \sin(m\varphi)\, d\varphi$$

$$= \frac{\varepsilon_m}{2\pi} \int_0^{2\pi} \left[-\frac{\partial \phi_m}{\partial z} + \frac{1}{\mu_0} \left(\nabla \times \underline{A}_f \right) \cdot \underline{e}_z \right] \cos(m\varphi)\, d\varphi \tag{1019}$$

and using (1012) and (1018)

$$\frac{a}{m} \frac{\varepsilon_m}{2\pi} \frac{\partial}{\partial z} \int_0^{2\pi} \left[-\frac{1}{a} \frac{\partial \phi_m}{\partial \varphi} + \frac{1}{\mu_0} \left(\nabla \times \underline{A}_f \right) \cdot \underline{e}_\varphi \right] \cos(m\varphi)\, d\varphi$$

$$= \frac{1}{\pi} \int_0^{2\pi} \left[-\frac{\partial \phi_m}{\partial z} + \frac{1}{\mu_0} \left(\nabla \times \underline{A}_f \right) \cdot \underline{e}_z \right] \sin(m\varphi)\, d\varphi \tag{1020}$$

These enforce

$$\frac{\partial}{\partial z} \left[H_\varphi(\text{odd}) \right]_m = -\frac{m}{a} \left[H_z(\text{even}) \right]_m \tag{1021}$$

$$\frac{\partial}{\partial z} \left[H_\varphi(\text{even}) \right]_m = \frac{m}{a} \left[H_z(\text{odd}) \right]_m \tag{1022}$$

or in terms of the potential (if it were entirely represented in terms of ϕ_m)

$$\frac{\partial}{\partial z} \left[\frac{1}{a} \frac{\partial}{\partial \varphi} \phi_m(\text{odd}) \right]_m = -\frac{m}{a} \left[\frac{\partial}{\partial z} \phi_m(\text{even}) \right]_m \tag{1023}$$

$$\frac{\partial}{\partial z} \left[\frac{1}{a} \frac{\partial}{\partial \varphi} \phi_m(\text{even}) \right]_m = \frac{m}{a} \left[\frac{\partial}{\partial z} \phi_m(\text{odd}) \right]_m \tag{1024}$$

or removing the extra z derivative

$$\left[\frac{\partial}{\partial \varphi} \phi_m(\text{odd}) \right]_m = -m \left[\phi_m(\text{even}) \right]_m \tag{1025}$$

$$\left[\frac{\partial}{\partial \varphi} \phi_m(\text{even}) \right]_m = m \left[\phi_m(\text{odd}) \right]_m \tag{1026}$$

These are simply the behaviors of $\sin(m\varphi)$ and $\cos(m\varphi)$, and thus we have not gained anything new from this enforcement of continuity of the second tangential field component.

6 CONCLUSIONS

This report discusses the formulation of the electromagnetic cable braid penetration and propagation problem. Basic energy and reciprocity formulas are used to define the immittances. Integral forms for the transfer immittances are also derived by use of a simplified form of reciprocity and applied to nonuniform geometries such as an exterior ground plane and an interior eccentric coax.

The detailed solution of the boundary value problems involved in the wire braid shield are set up using a basis of line multipoles along the braid wires. This approach leads to an efficient formulation of the periodic cell of the braid. In the electric problem images are used to treat adjacent dielectric boundaries. In the magnetic field problem we investigate an approximate approach using only the vector potential components in the direction along the braid wires, and a more accurate approach using a combination of Hertz potentials inside the wires and a combination of a magnetic scalar potential and filament vector potential (to carry the net wire current) on the outside of the wires. In this way the porpoising and hole penetration characteristics of the braid penetration arise in a self consistent way. We further simplify the braid geometry by mapping it to a planar surface so that the penetration, as well as the self immittance characteristics are reduced to certain intrinsic constants linked to the basic geometry of the braid wires.

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Appendix A. Orthogonality Relations

Implementation of the orthogonality relations

$$\int_0^{2\pi} \cos(m\varphi)\cos(m'\varphi)\,d\varphi = \frac{2\pi}{\varepsilon_m}\delta_{mm'} \tag{A-1}$$

$$\int_0^{2\pi} \sin(m\varphi)\cos(m'\varphi)\,d\varphi = 0 \tag{A-2}$$

$$\int_0^{2\pi} \sin(m\varphi)\sin(m'\varphi)\,d\varphi = \pi\delta_{mm'} \tag{A-3}$$

for the preceding multipole moments may be more conveniently done in terms of a summation of discrete values

$$\varphi_j = \pi j/M\ ,\ j = 0, 1, ..., 2M-1 \tag{A-4}$$

Using the results of the geometric series

$$S = \sum_{j=0}^{2M-1} r^j \tag{A-5}$$

$$rS = \sum_{j=1}^{2M} r^j \tag{A-6}$$

$$(1-r)\,S = 1 - r^{2M} \tag{A-7}$$

$$S = \frac{1-r^{2M}}{1-r} \tag{A-8}$$

We find for $m, m' = 0, 1, ..., M$

$$\sum_{j=0}^{2M-1} \cos(mj\pi/M)\cos(m'j\pi/M) = \frac{1}{4}\sum_{j=0}^{2M-1}\left(e^{imj\pi/M} + e^{-imj\pi/M}\right)\left(e^{im'j\pi/M} + e^{-im'j\pi/M}\right)$$

$$= \frac{1}{4}\sum_{j=0}^{2M-1}\left[e^{i\left(m-m'\right)\pi j/M} + e^{i\left(m+m'\right)\pi j/M} + e^{-i\left(m-m'\right)\pi j/M} + e^{-i\left(m+m'\right)\pi j/M}\right]$$

$$= \frac{1}{4}\left[\frac{1-e^{i\left(m-m'\right)2\pi}}{1-e^{i(m-m')\pi/M}} + \frac{1-e^{i\left(m+m'\right)2\pi}}{1-e^{i(m+m')\pi/M}} + \frac{1-e^{-i\left(m-m'\right)2\pi}}{1-e^{-i(m-m')\pi/M}} + \frac{1-e^{-i\left(m+m'\right)2\pi}}{1-e^{-i(m+m')\pi/M}}\right]$$

$$= \frac{4M}{\varepsilon_m\varepsilon_{M-m}}\delta_{mm'} \tag{A-9}$$

where ε_m is the Neumann number (equal to unity for $m = 0$ and equal to 2 otherwise) and for $m, m' = 1, 2, ..., M-1$

$$\sum_{j=1}^{2M-1} \sin(mj\pi/M)\sin(m'j\pi/M) =$$

$$\sum_{j=0}^{2M-1} \sin(mj\pi/M)\sin(m'j\pi/M) = -\frac{1}{4}\sum_{j=0}^{2M-1}\left(e^{im\pi j/M} - e^{-im\pi j/M}\right)\left(e^{im'\pi j/M} - e^{-im'\pi j/M}\right)$$

$$= -\frac{1}{4}\sum_{j=0}^{2M-1}\left[-e^{i\left(m-m'\right)\pi j/M} + e^{i\left(m+m'\right)\pi j/M} - e^{-i\left(m-m'\right)\pi j/M} + e^{-i\left(m+m'\right)\pi j/M}\right]$$

$$= -\frac{1}{4}\left[-\frac{1-e^{i\left(m-m'\right)2\pi}}{1-e^{i(m-m')\pi/M}} + \frac{1-e^{i\left(m+m'\right)2\pi}}{1-e^{i(m+m')\pi/M}} - \frac{1-e^{-i\left(m-m'\right)2\pi}}{1-e^{-i(m-m')\pi/M}} + \frac{1-e^{-i\left(m+m'\right)2\pi}}{1-e^{-i(m+m')\pi/M}}\right]$$

$$= M\delta_{mm'}\ ,\ m \neq 0, M \tag{A-10}$$

as well as

$$\sum_{j=1}^{2M-1} \sin(mj\pi/M)\cos(m'j\pi/M) =$$

$$\sum_{j=0}^{2M-1} \sin(mj\pi/M)\cos(m'j\pi/M) = \frac{1}{4i}\sum_{j=0}^{2M-1}\left(e^{im\pi j/M} - e^{-im\pi j/M}\right)\left(e^{im'\pi j/M} + e^{-im'\pi j/M}\right)$$

$$= \frac{1}{4i}\sum_{j=0}^{2M-1}\left[e^{i\left(m-m'\right)\pi j/M} + e^{i\left(m+m'\right)\pi j/M} - e^{-i\left(m-m'\right)\pi j/M} - e^{-i\left(m+m'\right)\pi j/M}\right]$$

$$= \frac{1}{4i}\left[\frac{1-e^{i\left(m-m'\right)2\pi}}{1-e^{i(m-m')\pi/M}} + \frac{1-e^{i\left(m+m'\right)2\pi}}{1-e^{i(m+m')\pi/M}} - \frac{1-e^{-i\left(m-m'\right)2\pi}}{1-e^{-i(m-m')\pi/M}} - \frac{1-e^{-i\left(m+m'\right)2\pi}}{1-e^{-i(m+m')\pi/M}}\right]$$

$$= 0 \tag{A-11}$$

Note that in the exponential representation with $m, m' = 0, 1, ..., 2M-1$ the orthogonality relation is

$$\sum_{j=0}^{2M-1} e^{i\left(m-m'\right)j\pi/M} = \frac{1-e^{i\left(m-m'\right)2\pi}}{1-e^{i(m-m')\pi/M}} = 2M\delta_{mm'} \tag{A-12}$$

We see from this that the part of the exponential $i\sin(mj\pi/M)$ vanishes for $j = 0, M$ or $\varphi_j = j\pi/M = 0, \pi$ and hence we do not really sample this part of the function at these points analogous to the preceding sine series.

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