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Electromagnetic Reciprocity

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ELECTROMAGNETIC RECIPROCITY

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ABSTRACT

A reciprocity theorem is an explicit mathematical relationship between two different wavefields that can exist within the same space-time configuration. Reciprocity theorems provide the theoretical underpinning for modern full waveform inversion solutions, and also suggest practical strategies for speeding up large-scale numerical modeling of geophysical datasets. In the present work, several previously-developed electromagnetic reciprocity theorems are generalized to accommodate a broader range of medium, source, and receiver types. Reciprocity relations enabling the interchange of various types of point sources and point receivers within a three-dimensional electromagnetic model are derived. Two numerical modeling algorithms in current use are successfully tested for adherence to reciprocity. Finally, the reciprocity theorem forms the point of departure for a lengthy derivation of electromagnetic Fréchet derivatives. These mathematical objects quantify the sensitivity of geophysical electromagnetic data to variations in medium parameters, and thus constitute indispensable tools for solution of the full waveform inverse problem.

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CONTENTS

1.0 INTRODUCTION

A reciprocity theorem is an explicit mathematical relationship between two *different* wavefields that can exist in the *same* space-time configuration. Reciprocity theorems may be developed for both continuum mechanical wavefields and electromagnetic wavefields propagating in a wide variety of media types. More specifically, a reciprocity theorem can be derived for any medium characterized by a constitutive relation (i.e., a relation between the pertaining flux densities and field strengths) that is linear, time-invariant, and local. Within these very broad confines, reciprocity holds for media that are spatially homogeneous or heterogeneous, isotropic or anisotropic, ideal or attenuative/dispersive, etc. Interestingly, de Hoop (1992) states that "field reciprocity theorems can be considered to be the most basic relations that exist in the theory of classical fields and waves". The clear implication is that reciprocity is fundamental to the understanding of classical wavefield phenomena. It particular, reciprocity forms the underpinning for various wavefield representation theorems and Fréchet derivative expressions that are central to modern inverse theory solutions.

The first statement of electromagnetic (EM) reciprocity is commonly credited to Lorentz (1895), who apparently dealt with time-harmonic (i.e., fixed-frequency) EM wavefields. Subsequent investigators (many referenced herein) have improved the understanding of electromagnetic reciprocity, and generalized the reciprocity theorem to apply to more complicated media types and wavefield sourcing conditions. Any study of reciprocity must recognize the significant contributions of Professor Adrianus T. de Hoop of Delft University in The Netherlands. In a series of papers (de Hoop, 1960, 1987, 1988, 1992; de Hoop and Stam, 1988; de Hoop and de Hoop, 2000), A.T. de Hoop and co-workers developed the rigorous derivational procedure that is followed in Chapters 2 and 3 (and Appendices B and C) of this report. Particular emphasis is placed on time-domain (rather than frequency-domain) derivations utilizing the tools of convolution mathematics. The *time-convolution* reciprocity theorem developed in Chapter 3 is a slight generalization of de Hoop's (1992) theorem. In particular, we consider more general constitutive relations and EM body sources, and also incorporate EM surface sources into the formalism. The effect of electric current conductivity, which is inexplicitly omitted from de Hoop's (1998 and 1992) treatments, is also included. Rigorous derivation of a *time-correlation* reciprocity theorem is reserved for Appendix B.

The usual mathematical statements of electromagnetic reciprocity pertain to the physical EM wavefields (i.e., the electric vector $\mathbf{e}(\mathbf{x},t)$ and the magnetic vector $\mathbf{h}(\mathbf{x},t)$). However, it is possible to develop analogous reciprocity theorems for the potential functions (i.e., the vector potential $\mathbf{a}(\mathbf{x},t)$ and the scalar potential $\phi(\mathbf{x},t)$) that are commonly used for practical electromagnetic problem solving. This is subject of Appendix C, where earlier theorems by Welch (1960) and Bojarski (1983) are generalized to accommodate heterogeneous media.

In geophysics, a popular (but only partially correct!) understanding of reciprocity might be encapsulated in the statement that "A recorded time-domain response (i.e., a trace) remains invariant when the positions of a point source and point receiver are interchanged within the same medium". This statement is investigated in the context of EM reciprocity in Chapter 4, where several *reciprocity relations* are developed for different types of sources and receivers. In the simple case of a point source situated in a homogeneous and isotropic wholespace, closed form mathematical formulae for the frequency-domain electromagnetic fields exist (e.g., Loseth, et al., 2006; Aldridge, 2013). These formulae exactly satisfy the reciprocity relations for the various point source / point receiver pairs. Interestingly, we find that specific terms in these expressions (i.e., near-field and far-field terms) also satisfy reciprocity.

Reciprocity has important practical applications for numerical simulation of geophysical datasets. In situations where there are many more source locations than receiver locations (e.g., an electromagnetic scattering experiment where radiation from many scattering loci are recorded by only a few EM receivers) reciprocity may be used to dramatically reduce the computational modeling burden by swapping source and receiver positions. Also, adherence to reciprocity constitutes a strong validation test for a numerical simulation algorithm. In Chapter 5, we confirm that reciprocity (in the sense of swapping point source and point receiver positions) holds for two electromagnetic modeling algorithms in current use. Algorithm EMHOLE (Aldridge, 2013) is a frequency-domain Green function algorithm appropriate for a homogeneous and isotropic EM wholespace. Algorithm FDEM (Aldridge, 2014, *in progress*) is an explicit, time-domain, finite-difference algorithm that solves the governing EM partial differential equations for a heterogeneous and isotropic three-dimensional medium. Both algorithms satisfy reciprocity, for the full range of point source and point receiver types.

Finally, reciprocity has strong theoretical utility. In Chapter 6, we undertake a lengthy derivation of electromagnetic "Fréchet derivatives". The time-convolution reciprocity theorem developed in Chapter 3 forms the point of departure for the analysis. Broadly speaking, Fréchet derivatives are quantitative measures of the sensitivity of synthetic (or predicted, or modeled) geophysical data with respect to various parameters. Commonly, Fréchet derivatives are calculated with respect to medium parameters, as with the permittivity, permeability, and conductivity characterizing an isotropic electromagnetic medium. Such Fréchet derivatives are indispensable tools for solving the full waveform electromagnetic inverse problem. The derived formulae form the basis of a future algorithm for inverting geophysical EM data to quantitatively estimate 3D medium parameter distributions.

2.0 PROBLEM DEFINITION

A logical and systematic framework for the mathematical derivation of reciprocity theorems, of both the time-convolution and time-correlation type, has been described by A.T. de Hoop and co-authors (e.g., 1987, 1988, 1992, 2002). This framework consists of specifying:

1) The partial differential equations governing the wavefield,
2) The pertaining constitutive relations,
3) Interface (or boundary) conditions that hold at discontinuity surfaces in medium parameters,
4) An *interaction quantity* relating two distinct wavefields that may exist in the same space-time configuration.

A description of this approach is given in this chapter, as well as a brief review of the mathematical tools utilized in the derivations. These tools consist of several basic theorems of convolution mathematics, as well as a generalized divergence theorem that can accommodate media with internal interfaces (i.e., discontinuity surfaces). Specialization to overly simplistic geophysical situations (say, homogeneity in medium parameters, or isotropy in constitutive relations) is avoided.

2.1 Electromagnetic Field Equations

Only two of Maxwell's electromagnetic equations are directly used in the derivation of a reciprocity theorem. These are the Faraday law and the Ampere-Maxwell law. In Cartesian coordinate indicial notation (i.e., i,j,k = 1,2,3) these are

$$\frac{\partial b_i(\mathbf{x},t)}{\partial t} + \varepsilon_{ijk}\frac{\partial e_k(\mathbf{x},t)}{\partial x_j} = 0\,, \qquad \text{(Faraday law)} \qquad (2.1a)$$

$$\frac{\partial d_i(\mathbf{x},t)}{\partial t} + j_i(\mathbf{x},t) - \varepsilon_{ijk}\frac{\partial h_k(\mathbf{x},t)}{\partial x_j} = 0\,, \qquad \text{(Ampere – Maxwell law)} \qquad (2.1b)$$

where ε_{ijk} is the permuting symbol and summation over repeated subscripts is assumed. Dependent variables in these first-order (in space $\mathbf{x}$ and time t) partial differential equations are

$e_i(\mathbf{x},t)$: electric field strength (SI unit: V/m),

$h_i(\mathbf{x},t)$: magnetic field strength (A/m),

$b_i(\mathbf{x},t)$: magnetic flux density ((V-s)/m^2),

$d_i(\mathbf{x},t)$: electric flux density ((A-s)/m^2)),

$j_i(\mathbf{x},t)$: conduction current density (A/m^2).

We adopt a point of view, different from Crowley (1954), Welch (1960), de Hoop (1987, 1992), Loseth et al. (2006), and many others, that body sources of electromagnetic fields do *not* appear in Maxwell's equations. Rather, sources are introduced via media-specific constitutive relations. Although this approach is subtly different from a physical standpoint, it is mathematically equivalent for the goal of deriving a reciprocity theorem. Expressions (2.1a and b) hold within subdomains of the three-dimensional volume supporting electromagnetic fields where both medium properties and body sources vary continuously with position.

2.2 Electromagnetic Constitutive Relations

The two flux density vectors and the current density vector in Maxwell's equations (2.1a and b) are related to the electromagnetic fields strengths via constitutive relations. For media that are *linear*, *time-invariant*, and *locally-reacting*, reasonably general constitutive relations are given by

$$b_i(\mathbf{x},t) = \varphi_{ij}(\mathbf{x},t) * h_j(\mathbf{x},t) + b_i^s(\mathbf{x},t), \tag{2.2a}$$

$$d_i(\mathbf{x},t) = \psi_{ij}(\mathbf{x},t) * e_j(\mathbf{x},t) + d_i^s(\mathbf{x},t), \tag{2.2b}$$

$$j_i(\mathbf{x},t) = \eta_{ij}(\mathbf{x},t) * e_j(\mathbf{x},t) + j_i^s(\mathbf{x},t). \tag{2.2c}$$

Summation over repeated subscripts is assumed, the asterisks denote convolution with respect to the time variable t, and terms with superscript "s" are body sources of the pertaining quantity. Appendix A provides a development of these constitutive relations from the fundamental assumptions of linearity, time-invariance, and locality.

The three second-rank tensors in equations (2.2a,b,c) are referred to herein as *response functions*:

$\varphi_{ij}(\mathbf{x},t)$: magnetic permeability response function (SI unit: V/A),

$\psi_{ij}(\mathbf{x},t)$: electric permittivity response function (SI unit: A/V),

$\eta_{ij}(\mathbf{x},t)$: current conductivity response function (SI unit: (A/V)/(m-s) = (S/m)/s).

For physically realistic electromagnetic media, these response functions are considered causal functions of time (i.e., they vanish for $t < 0$). However, it is permissible for a "computational" or "numerical" medium to be non-causal. The development of a reciprocity theorem does not pre-suppose either condition. Moreover, for the present, the tensors are not restricted to be symmetric in indices i and j. Tensors $\varphi_{ij}(\mathbf{x},t)$ and $\psi_{ij}(\mathbf{x},t)$ are analogous to (but not identical to) the magnetic and electric "relaxation functions" in de Hoop (1987, 1992). Interestingly, de Hoop's treatment does not include a current conductivity relaxation function analogous to $\eta_{ij}(\mathbf{x},t)$. [However, a cryptic note on the bottom of page 675 in Appendix A of de Hoop and de Hoop (2000) suggests that the conductivity response may be incorporated into the permittivity response in a manner similar to our development below.]

Particular cases of the general constitutive relations (2.2) that are popular in geophysics include *anisotropic* media:

$$\varphi_{ij}(\mathbf{x},t) = \mu_{ij}(\mathbf{x})\delta(t), \qquad \psi_{ij}(\mathbf{x},t) = \varepsilon_{ij}(\mathbf{x})\delta(t), \qquad \eta_{ij}(\mathbf{x},t) = \sigma_{ij}(\mathbf{x})\delta(t),$$

where $\delta(t)$ is the temporal Dirac delta function (SI unit: 1/s), and $\mu_{ij}(\mathbf{x})$, $\varepsilon_{ij}(\mathbf{x})$, and $\sigma_{ij}(\mathbf{x})$ are the well-known magnetic permeability, electric permittivity, and current conductivity tensors, respectively. De Hoop (1987, 1992) refers to a medium with response functions proportional to a temporal Dirac impulse as being "instantaneously reacting". For *isotropic* media, the response function tensors reduce to

$$\mu_{ij}(\mathbf{x}) = \mu(\mathbf{x})\delta_{ij}, \qquad \varepsilon_{ij}(\mathbf{x}) = \varepsilon(\mathbf{x})\delta_{ij}, \qquad \sigma_{ij}(\mathbf{x}) = \sigma(\mathbf{x})\delta_{ij},$$

where δ_{ij} is the Kronecker delta symbol. Space-dependent scalars $\mu(\mathbf{x})$ (SI unit: (V/m)/(A/s) = H/m), $\varepsilon(\mathbf{x})$ ((A/m)/(V/s) = F/m), and $\sigma(\mathbf{x})$ ((A/V)/m = S/m) are the magnetic permeability, electric permittivity, and current conductivity of the isotropic medium.

In view of the straightforward reductions to forms that are common in electromagnetic geophysics, we refer to the constitutive relations (2.2a,b,c) as *conventional* constitutive relations. However, following de Hoop and de Hoop (2000), even more general relations can be stated whereby *each* flux density vector depends on *all* of the field strengths:

$$b_i(\mathbf{x},t) = \varphi_{ij}^h(\mathbf{x},t) * h_j(\mathbf{x},t) + \varphi_{ij}^e(\mathbf{x},t) * e_j(\mathbf{x},t) + b_i^s(\mathbf{x},t)\,, \tag{2.3a}$$

$$d_i(\mathbf{x},t) = \psi_{ij}^e(\mathbf{x},t) * e_j(\mathbf{x},t) + \psi_{ij}^h(\mathbf{x},t) * h_j(\mathbf{x},t) + d_i^s(\mathbf{x},t)\,, \tag{2.3b}$$

$$j_i(\mathbf{x},t) = \eta_{ij}^e(\mathbf{x},t) * e_j(\mathbf{x},t) + \eta_{ij}^h(\mathbf{x},t) * h_j(\mathbf{x},t) + j_i^s(\mathbf{x},t)\,. \tag{2.3c}$$

Identifying superscripts "*h*" and "*e*" are used to distinguish two classes of response functions. If the additional three tensors $\varphi_{ij}^e(\mathbf{x},t) = \psi_{ij}^h(\mathbf{x},t) = \eta_{ij}^h(\mathbf{x},t) = 0$, then these *unconventional* constitutive relations reduce to the conventional forms (2.2). Interestingly, tensors $\varphi_{ij}^e(\mathbf{x},t)$ and $\psi_{ij}^h(\mathbf{x},t)$ are dimensionless, whereas tensor $\eta_{ij}^h(\mathbf{x},t)$ has SI unit 1/(m-s). In electromagnetic geophysics, it is probable that these tensors have small magnitudes compared to their conventional counterparts. de Hoop and de Hoop (2000) indicate that these tensors represent "exotic effects, like the magnetoelectric effect in chiral media". Kong (1971) describes four electromagnetic media types (using the constitutive relation for electric flux as an example) as:

isotropic: $d_i = \varepsilon e_i$,	anisotropic: $d_i = \varepsilon_{ij} e_j$,
bi-isotropic: $d_i = \varepsilon e_i + \chi h_i$,	bi-anisotropic: $d_i = \varepsilon_{ij} e_j + \chi_{ij} h_j$.

For simplicity, subsequent derivations of reciprocity theorems will utilize the conventional constitutive relations (2.2). However, the results will also be stated for the unconventional constitutive relations (2.3).

The introduction of body source terms into the constitutive relations (2.2) and (2.3) follows a practice pioneered in seismology by Backus and Mulcahy (1976). As indicated above, the same reciprocity theorem is developed if the body sources are included in Maxwell's equations (2.1). The most common body source in electromagnetic geophysics is an electric current represented by the current density vector $j_i^s(\mathbf{x},t)$. However, a body source of magnetic induction $b_i^s(\mathbf{x},t)$ is not unusual, although it often shows up in the time-differentiated form $k_i^s(\mathbf{x},t) = \partial j_i^s(\mathbf{x},t)/\partial t$ (e.g., equation (3.6a)) and is referred to as a *magnetic current*. Finally, the body source of electric displacement $d_i^s(\mathbf{x},t)$ (or *displacement current* $l_i^s(\mathbf{x},t) = \partial d_i^s(\mathbf{x},t)/\partial t$ as in equation (3.6b)) appears to be novel. The main reason for including it here is consistency and completeness in the constitutive relations (2.2) and (2.3).

2.3 Electromagnetic Interface Conditions

The response function tensors in constitutive relations (2.2) and (2.3) are assumed to be piecewise continuous functions of position $\mathbf{x}$ within a three-dimensional volume V bounded by a surface S. At an interface between two dissimilar media, one or more of the tensors may possess a finite jump discontinuity. *Interface conditions* (or *boundary conditions*) for the various electromagnetic field quantities hold on this discontinuity surface. Let **n**(**x**) be a unit normal vector to an internal interface at position **x**. Also, let the paired set of square brackets **[ ]** refers to the jump discontinuity in a quantity q as

$$\left[q(\mathbf{x})\right] = q^{+}(\mathbf{x}) - q^{-}(\mathbf{x}) ,$$

where + and – signs are relative to the unit vector **n**(**x**). Then, in indicial notation, interface conditions for the electric and magnetic field strengths are

$$\varepsilon_{ijk} n_j(\mathbf{x}) \left[e_k(\mathbf{x},t)\right] = 0 , \tag{2.4a}$$

$$\varepsilon_{ijk} n_j(\mathbf{x}) \left[h_k(\mathbf{x},t)\right] = s_i(\mathbf{x},t) , \tag{2.4b}$$

where **s**(**x**,t) is the surface current density (SI unit: A/m) at **x** on the interface. Equation (2.4a) indicates that the tangential components of the electric vector are continuous across the interface, whereas (2.4b) implies that the tangential components of the magnetic vector may be discontinuous if a surface current exists.

Only the two interface conditions (2.4a and b) above are utilized in the derivation of an electromagnetic reciprocity theorem. However, for completeness, we also state the analogous interface conditions for the two flux density vectors and the current density vector (e.g., Costen and Adamson, 1965):

$$n_i(\mathbf{x}) \left[b_i(\mathbf{x},t)\right] = 0 , \tag{2.4c}$$

$$n_i(\mathbf{x}) \left[d_i(\mathbf{x},t)\right] = \chi(\mathbf{x},t) , \tag{2.4d}$$

$$n_i(\mathbf{x}) \left[j_i(\mathbf{x},t)\right] = -n_i(\mathbf{x}) \varepsilon_{ijk} \frac{\partial}{\partial x_j} \left(\varepsilon_{klm} n_l(\mathbf{x}) s_m(\mathbf{x},t)\right) - \frac{\partial \chi(\mathbf{x},t)}{\partial t} . \tag{2.4e}$$

Here χ(**x**,t) is the surface charge density (SI unit: C/m^2) at position **x** on the interface. Clearly, these interface conditions pertain to the normal components of the **b**, **d**, and **j** vectors.

2.4 Convolution Mathematics

The derivation and analysis of a reciprocity relation utilize several well-known theorems of convolution mathematics. These theorems are summarized here without proof. Convolution is a binary operation involving two functions of time, and is defined by the integral

$$x(t) * y(t) \equiv \int_{-\infty}^{+\infty} x(\tau) y(t-\tau)\, d\tau . \tag{2.5}$$

Although functions $x(t)$ and $y(t)$ may also depend on the spatial position **x**, for simplicity this dependence is suppressed in the notation immediately below. The functions may be distributions (i.e., including the temporal Dirac delta function and/or its derivatives). From the above definition, the following theorems are readily demonstrated.

1) Convolution is commutative:

$$x(t) * y(t) = y(t) * x(t) .$$

2) Convolution distributes over addition:

$$x(t) * \big(y(t) + z(t)\big) = x(t) * y(t) + x(t) * z(t) .$$

3) Convolution is associative:

$$x(t) * y(t) * z(t) = \big(x(t) * y(t)\big) * z(t) = x(t) * \big(y(t) * z(t)\big) .$$

4) The derivative of a convolution may be expressed as a convolution involving the derivative of one (or the other) operand:

$$\frac{d}{dt}\big(x(t) * y(t)\big) = \frac{dx(t)}{dt} * y(t) = x(t) * \frac{dy(t)}{dt} .$$

5) The integral of a convolution may be expressed as a convolution involving the integral of one (or the other) operand:

$$\int_{-\infty}^{t} x(\tau) * y(\tau)\, d\tau = \left[\int_{-\infty}^{t} x(\tau)\, d\tau\right] * y(t) = x(t) * \left[\int_{-\infty}^{t} y(\tau)\, d\tau\right] .$$

6) The time-reverse of a convolution equals the convolution of the time-reversed operands:

$$\overline{x(t) * y(t)} = \bar{x}(t) * \bar{y}(t) ,$$

where the overbar notation denotes time reversal: $\bar{x}(t) \equiv x(-t)$. As a corollary, it is easy to show

$$\overline{x(t) * \bar{y}(t)} = \bar{x}(t) * y(t) ,$$

since $\bar{\bar{y}}(t) \equiv y(t)$.

7) Convolution and multiplication operations may be exchanged in an integrand, accompanied by time-reversal of the central operand:

$$\int_{-\infty}^{+\infty} \big[x(t) * y(t)\big]\, z(t)\, dt = \int_{-\infty}^{+\infty} x(t)\, \big[\bar{y}(t) * z(t)\big]\, dt .$$

This expression is utilized in the development of a time-invariant sensitivity equation for the full waveform electromagnetic inverse problem. It may be expressed in the alternative form

$$\int_{-\infty}^{+\infty} \big[x(t) * y(t)\big]\, z(t)\, dt = \int_{-\infty}^{+\infty} \bar{x}(t)\, \big[y(t) * \bar{z}(t)\big]\, dt ,$$

by making the change of integration variable $t = -\tau$ and using the above convolution time-reversal theorem.

Equation (2.5) above is a general definition of one-dimensional convolution. However, in special circumstances, the temporal integration limits may be altered. If one function [say $y(t)$] entering the convolution is causal (i.e., $y(t) = 0$ for $t < 0$), then the convolution integral may be written as

$$x(t) * y(t) = \int_{-\infty}^{t} x(\tau) y(t-\tau)\, d\tau \,.$$

If both functions are causal, then the convolution integral becomes

$$x(t) * y(t) = \int_{0}^{t} x(\tau) y(t-\tau)\, d\tau \,,$$

for $t > 0$, and $x(t)*y(t) = 0$ for $t \le 0$. Finally, assume $x(t)$ and $y(t)$ are finite-duration functions contained within the time intervals $[t_{\text{xmin}}, t_{\text{xmax}}]$ and $[t_{\text{ymin}}, t_{\text{ymax}}]$, respectively. That is

$$x(t) = 0 \text{ for } t < t_{\text{xmin}} \text{ and } t > t_{\text{xmax}} \,, \qquad y(t) = 0 \text{ for } t < t_{\text{ymin}} \text{ and } t > t_{\text{ymax}} \,.$$

Then, the convolution $z(t) = x(t)*y(t)$ is a finite-duration function wholly contained within the time interval $[t_{\text{zmin}}, t_{\text{zmax}}] = [t_{\text{xmin}} + t_{\text{ymin}}, t_{\text{xmax}} + t_{\text{ymax}}]$, and is given by

$$x(t) * y(t) = \int_{\tau_{\min}}^{\tau_{\max}} x(\tau) y(t-\tau)\, d\tau \,,$$

where the lower and upper temporal integration limits are

$$\tau_{\min} = \max\left[t_{\text{xmin}}, t - t_{\text{ymax}}\right], \qquad \tau_{\max} = \min\left[t_{\text{xmax}}, t - t_{\text{ymin}}\right].$$

These are special cases of the "double causal" convolution integral limits. For example, taking $[t_{\text{xmin}}, t_{\text{xmax}}] = [t_{\text{ymin}}, t_{\text{ymax}}] = [0, +\infty]$ gives $[\tau_{\min}, \tau_{\max}] = [0, t]$.

The *cross-correlation* of two functions $x(t)$ and $y(t)$ is defined as

$$x(t) \otimes y(t) \equiv \int_{-\infty}^{+\infty} x(\tau) y(\tau - t)\, d\tau \,. \qquad (2.6)$$

It is straightforward to demonstrate that cross-correlation is related to convolution via

$$x(t) \otimes y(t) = x(t) * \bar{y}(t) \,.$$

That is, cross-correlation is equivalent to convolution, with the second operand reversed in time. From the integral definition of cross-correlation, it is easy to show

$$x(t) \otimes y(t) = \overline{y(t) \otimes x(t)} \, .$$

Hence, unlike convolution, cross-correlation is not commutative in the operands.

Finally, it is worthwhile to note the time-reversal differentiation rule $\frac{d\overline{x}(t)}{dt} = -\overline{\frac{dx(t)}{dt}}$ (note the negative sign out front!). This property is used repeatedly in the derivation of a time-correlation reciprocity theorem.

2.5 Generalized Divergence Theorem

The final mathematical tool required for the derivation is a divergence theorem applicable to media containing interfaces, or surfaces of discontinuity in material parameters. Let $\varphi_i(\mathbf{x},t)$ be a piecewise-continuous function (i.e., a component of a vector) defined within a three-dimensional volume *V* bounded by outer surface *S*, and let $\mathbf{m}(\mathbf{x})$ be the outward-directed unit normal to *S* at position $\mathbf{x}$. At a finite number *N* of interfaces S_n (n = 1,2,3,…N) contained within *V*, the function $\varphi_i(\mathbf{x},t)$ possesses jump discontinuities. Then, the generalized divergence theorem for the field $\varphi_i(\mathbf{x},t)$ is

$$\int_V \frac{\partial \varphi_i(\mathbf{x},t)}{\partial x_i} \, dV(\mathbf{x}) = \int_S \varphi_i(\mathbf{x},t) m_i(\mathbf{x}) \, dS(\mathbf{x}) - \sum_{n=1}^{N} \int_{S_n} \left[\varphi_i(\mathbf{x},t) \right] n_i(\mathbf{x}) \, dS(\mathbf{x}) \, , \tag{2.7}$$

(Eringen and Suhubi, 1975, equation (1.11.2) on page 26). Here, $\mathbf{n}(\mathbf{x})$ is a unit normal to interface S_n, and the bracket notation [ ] refers to the jump discontinuity of $\varphi_i(\mathbf{x},t)$ at $\mathbf{x}$ on S_n. It is easy to construct a non-rigorous "geometric proof" of this generalized divergence theorem.

2.6 Interaction Quantities

As indicated previously, a reciprocity theorem is an explicit mathematical relationship between two *different* electromagnetic fields that can exist in the *same* domain in space and time. We use superscripts "A" and "B" on all mathematical symbols associated with these two electromagnetic fields (i.e., field strengths, flux densities, response functions, body sources). The two distinct fields are often referred to as "states" existing within the same space-time configuration. Commonly, the two fields may arise from different body source distributions (i.e., electric and/or magnetic currents). However, the medium properties (represented by the response function tensors) in the two states may also differ, leading to differing electromagnetic fields even if the source distributions are identical. The only requirement is that the two states occupy the same time-invariant spatial volume.

The point of departure in the development of a reciprocity theorem is the definition of a mathematical *interaction quantity* that inter-relates the electromagnetic fields of states A and B. de Hoop (1987) distinguishes interaction quantities of *time-convolution* type and *time-correlation* type. Our interaction quantities are slightly different from those in de Hoop (1987, 1992), and are defined as follows:

The interaction quantity of the time-convolution type is defined as

$$\Phi(\mathbf{x},t) \equiv \frac{\partial}{\partial x_i} \left\{ \varepsilon_{ijk} \left(e_j^A(\mathbf{x},t) * h_k^B(\mathbf{x},t) - e_j^B(\mathbf{x},t) * h_k^A(\mathbf{x},t) \right) \right\}, \tag{2.8}$$

where ε_{ijk} is the permuting symbol, and the asterisk denotes convolution. Derivation of a reciprocity theorem appropriate for this interaction quantity is the subject of chapter 3. The analogous interaction quantity of the time-correlation type is

$$\Psi(\mathbf{x},t) \equiv \frac{\partial}{\partial x_i}\left\{\varepsilon_{ijk}\left(e_j^A(\mathbf{x},t)\otimes h_k^B(\mathbf{x},t)+\overline{e_j^B(\mathbf{x},t)\otimes h_k^A(\mathbf{x},t)}\right)\right\}, \tag{2.9}$$

where $\otimes$ denotes cross-correlation, and the overbar designates time reversal: $\bar{x}(t) = x(-t)$. The development of the associated reciprocity theorem in contained in Appendix B.

Each interaction quantity is the divergence of a vector, and has physical dimension "energy/volume" (i.e., energy density, equal to pressure) and SI unit: J/m^3 = N/m^2 = P. If the two electromagnetic wavefields A and B are identical (say, because they have identical body source distributions situated within the same medium) then the interaction quantity of the time-convolution type vanishes, whereas the interaction quantity of the time-correlation type does not.

There appear to been a variety of electromagnetic interaction quantities in historical use, which apparently prompted Crowley (1954) to state that the proof a particular reciprocity relation "depends on knowing the form of the theorem in order to carry out the proof". de Hoop (1987) credits Bojarski (1983) as the first to clearly distinguish between the time-convolution and time-correlation reciprocity relations. Welch (1960) states the original electromagnetic reciprocity theorem due to Lorentz (1895) (Welch's first un-numbered equation) and indicates that it is appropriate for time-harmonic fields. Our analysis in chapter 3 implies that Lorentz's interaction quantity is of the of the time-convolution type. This interaction quantity was later denoted by Rumsey (1954) as the "reaction" between the sources and fields of the two states.

3.0 RECIPROCITY THEOREM DERIVATION

A rigorous mathematical derivation of an electromagnetic reciprocity theorem is developed in this section. The analysis is similar to that described in de Hoop (1987, 1992), although here we explicitly account for current conductivity (represented by the constitutive relation response functions $\eta_{ij}(\mathbf{x},t)$), and body sources of displacement current (represented by the source electric displacement vector $d_i^s(\mathbf{x},t)$). Moreover, our interface conditions appear to be more general than those utilized by de Hoop (1987, 1992) and de Hoop and de Hoop (2000). For simplicity, the derivation utilizes the *conventional* electromagnetic constitutive relations (2.2). We merely state the analogous results for the *unconventional* constitutive relations (2.3).

3.1 Local Reciprocity Theorem

Following de Hoop (1987, 1992) and de Hoop and de Hoop (2000), a *local interaction quantity of the time-convolution type* is defined as

$$\Phi(\mathbf{x},t) \equiv \frac{\partial}{\partial x_i}\left\{\varepsilon_{ijk}\left(e_j^A(\mathbf{x},t) * h_k^B(\mathbf{x},t) - e_j^B(\mathbf{x},t) * h_k^A(\mathbf{x},t)\right)\right\}, \tag{3.1}$$

where the asterisks denote temporal convolution, and summation over repeated subscripts is assumed. Scalar quantity $\Phi(\mathbf{x},t)$ is the divergence of a vector, and has physical dimension "energy/volume" (i.e., energy density, equal to pressure) and SI unit: J/m^3 = N/m^2 = Pa. Note that if the two electromagnetic wavefields A and B are identical, then this particular interaction quantity vanishes.

The spatial differentiations in (3.1) distribute over the temporal convolutions, yielding

$$\begin{aligned}\Phi(\mathbf{x},t) = {}&\varepsilon_{ijk}\left\{\frac{\partial e_j^A(\mathbf{x},t)}{\partial x_i} * h_k^B(\mathbf{x},t) + e_j^A(\mathbf{x},t) * \frac{\partial h_k^B(\mathbf{x},t)}{\partial x_i}\right\}\\ &- \varepsilon_{ijk}\left\{\frac{\partial e_j^B(\mathbf{x},t)}{\partial x_i} * h_k^A(\mathbf{x},t) + e_j^B(\mathbf{x},t) * \frac{\partial h_k^A(\mathbf{x},t)}{\partial x_i}\right\}.\end{aligned}$$

Regrouping terms gives the equivalent expression:

$$\begin{aligned}\Phi(\mathbf{x},t) = {}&\left\{\varepsilon_{ijk}\frac{\partial e_j^A(\mathbf{x},t)}{\partial x_i} * h_k^B(\mathbf{x},t) - \varepsilon_{ijk}\frac{\partial e_j^B(\mathbf{x},t)}{\partial x_i} * h_k^A(\mathbf{x},t)\right\}\\ &+ \left\{e_j^A(\mathbf{x},t) * \varepsilon_{ijk}\frac{\partial h_k^B(\mathbf{x},t)}{\partial x_i} - e_j^B(\mathbf{x},t) * \varepsilon_{ijk}\frac{\partial h_k^A(\mathbf{x},t)}{\partial x_i}\right\}.\end{aligned}$$

If any two indices of the permuting symbol are swapped, then its value changes sign: $\varepsilon_{ijk} = -\varepsilon_{jik}$, $\varepsilon_{ijk} = -\varepsilon_{ikj}$, and $\varepsilon_{ijk} = -\varepsilon_{kji}$. Hence, two successive swaps preserves the sign. Thus, the interaction quantity is put into the form

$$\Phi(\mathbf{x},t) = \left\{ \varepsilon_{kij} \frac{\partial e_j^A(\mathbf{x},t)}{\partial x_i} * h_k^B(\mathbf{x},t) - \varepsilon_{kij} \frac{\partial e_j^B(\mathbf{x},t)}{\partial x_i} * h_k^A(\mathbf{x},t) \right\}$$
$$- \left\{ e_j^A(\mathbf{x},t) * \varepsilon_{jik} \frac{\partial h_k^B(\mathbf{x},t)}{\partial x_i} - e_j^B(\mathbf{x},t) * \varepsilon_{jik} \frac{\partial h_k^A(\mathbf{x},t)}{\partial x_i} \right\}. \tag{3.2}$$

In this form, the spatial derivatives are easily eliminated in favor of temporal derivatives by substituting from the electromagnetic field equations (2.1a,b):

$$\Phi(\mathbf{x},t) = -\left\{ \frac{\partial b_k^A(\mathbf{x},t)}{\partial t} * h_k^B(\mathbf{x},t) - \frac{\partial b_k^B(\mathbf{x},t)}{\partial t} * h_k^A(\mathbf{x},t) \right\}$$
$$- \left\{ e_j^A(\mathbf{x},t) * \left(\frac{\partial d_j^B(\mathbf{x},t)}{\partial t} + j_j^B(\mathbf{x},t) \right) - e_j^B(\mathbf{x},t) * \left(\frac{\partial d_j^A(\mathbf{x},t)}{\partial t} + j_j^A(\mathbf{x},t) \right) \right\}. \tag{3.3}$$

Using the convolution-differentiation theorem $d/dt\big(x(t) * y(t)\big) = x'(t) * y(t) = x(t) * y'(t)$ (where a prime indicates differentiation with respect to the argument), the interaction quantity is put into the form:

$$\Phi(\mathbf{x},t) = -\frac{\partial}{\partial t} \left\{ b_k^A(\mathbf{x},t) * h_k^B(\mathbf{x},t) - b_k^B(\mathbf{x},t) * h_k^A(\mathbf{x},t) \right\}$$
$$-\frac{\partial}{\partial t} \left\{ e_j^A(\mathbf{x},t) * \big(d_j^B(\mathbf{x},t) + j_j^B(\mathbf{x},t) * H(t) \big) - e_j^B(\mathbf{x},t) * \big(d_j^A(\mathbf{x},t) + j_j^A(\mathbf{x},t) * H(t) \big) \right\}, \tag{3.4}$$

where $H(t)$ is the Heaviside unit step function. Recall that $dH(t)/dt = \delta(t)$ where $\delta(t)$ is the temporal Dirac delta function.

Expression (3.4) contains the two electromagnetic flux density vectors $\mathbf{b}(\mathbf{x},t)$ and $\mathbf{d}(\mathbf{x},t)$, as well as the current density vector $\mathbf{j}(\mathbf{x},t)$. These vector components are eliminated in favor of the field variables $\mathbf{e}(\mathbf{x},t)$ and $\mathbf{h}(\mathbf{x},t)$ by substituting from the (conventional) constitutive relations (2.2a,b,c):

$$\Phi(\mathbf{x},t) = -\frac{\partial}{\partial t} \left\{ \big(\varphi_{kj}^A(\mathbf{x},t) * h_j^A(\mathbf{x},t) + b_k^{A-s}(\mathbf{x},t) \big) * h_k^B(\mathbf{x},t) - \big(\varphi_{kj}^B(\mathbf{x},t) * h_j^B(\mathbf{x},t) + b_k^{B-s}(\mathbf{x},t) \big) * h_k^A(\mathbf{x},t) \right\}$$

$$-\frac{\partial}{\partial t} \left\{ e_j^A(\mathbf{x},t) * \left\{ \big(\psi_{jk}^B(\mathbf{x},t) * e_k^B(\mathbf{x},t) + d_j^{B-s}(\mathbf{x},t) \big) + \big(\eta_{jk}^B(\mathbf{x},t) * e_k^B(\mathbf{x},t) + j_j^{B-s}(\mathbf{x},t) \big) * H(t) \right\} \right.$$
$$\left. - e_j^B(\mathbf{x},t) * \left\{ \big(\psi_{jk}^A(\mathbf{x},t) * e_k^A(\mathbf{x},t) + d_j^{A-s}(\mathbf{x},t) \big) + \big(\eta_{jk}^A(\mathbf{x},t) * e_k^A(\mathbf{x},t) + j_j^{A-s}(\mathbf{x},t) \big) * H(t) \right\} \right\}.$$

Separate the terms into two major groups, the first including the medium-dependent response functions, and the second including the electromagnetic wavefield source functions:

$$
\begin{aligned}
\Phi(\mathbf{x},t) = &-\frac{\partial}{\partial t}\left\{ \varphi^A_{kj}(\mathbf{x},t) * h^A_j(\mathbf{x},t) * h^B_k(\mathbf{x},t) - \varphi^B_{kj}(\mathbf{x},t) * h^A_k(\mathbf{x},t) * h^B_j(\mathbf{x},t) \right\} \\
&-\frac{\partial}{\partial t}\left\{ \psi^B_{jk}(\mathbf{x},t) * e^A_j(\mathbf{x},t) * e^B_k(\mathbf{x},t) - \psi^A_{jk}(\mathbf{x},t) * e^A_k(\mathbf{x},t) * e^B_j(\mathbf{x},t) \right. \\
&\quad \left. + \eta^B_{jk}(\mathbf{x},t) * H(t) * e^A_j(\mathbf{x},t) * e^B_k(\mathbf{x},t) - \eta^A_{jk}(\mathbf{x},t) * H(t) * e^A_k(\mathbf{x},t) * e^B_j(\mathbf{x},t) \right\} \\
&\quad - \left\{ e^A_j(\mathbf{x},t) * j^{B-s}_j(\mathbf{x},t) - e^B_j(\mathbf{x},t) * j^{A-s}_j(\mathbf{x},t) \right\} \\
&\quad + \left\{ h^A_k(\mathbf{x},t) * \frac{\partial b^{B-s}_k(\mathbf{x},t)}{\partial t} - h^B_k(\mathbf{x},t) * \frac{\partial b^{A-s}_k(\mathbf{x},t)}{\partial t} \right\} \\
&\quad - \left\{ e^A_j(\mathbf{x},t) * \frac{\partial d^{B-s}_j(\mathbf{x},t)}{\partial t} - e^B_j(\mathbf{x},t) * \frac{\partial d^{A-s}_j(\mathbf{x},t)}{\partial t} \right\}. \qquad (3.5)
\end{aligned}
$$

Note that the time-differentiations are brought back into the convolutions in the second group of terms. This motivates defining a *source magnetic current density* (SI unit: T/s = V/m^2) and *source displacement current density* (SI unit: (C/m^2)/s = A/m^2) as

$$
k^s_i(\mathbf{x},t) \equiv \frac{\partial b^s_i(\mathbf{x},t)}{\partial t}, \qquad l^s_i(\mathbf{x},t) \equiv \frac{\partial d^s_i(\mathbf{x},t)}{\partial t}, \qquad (3.6a,b)
$$

respectively. Clearly, the source displacement current has the same dimension (and SI unit) as the *source conduction current density* $j^s_i(\mathbf{x},t)$. Then, engaging in some re-indexing, equation (3.5) is put into the more compact form

$$
\begin{aligned}
\Phi(\mathbf{x},t) = &-\frac{\partial}{\partial t}\left\{ \left(\varphi^A_{ji}(\mathbf{x},t) - \varphi^B_{ij}(\mathbf{x},t)\right) * h^A_i(\mathbf{x},t) * h^B_j(\mathbf{x},t) \right\} \\
&+\frac{\partial}{\partial t}\left\{ \left\{ \left(\psi^A_{ji}(\mathbf{x},t) - \psi^B_{ij}(\mathbf{x},t)\right) + \left(\eta^A_{ji}(\mathbf{x},t) - \eta^B_{ij}(\mathbf{x},t)\right) * H(t) \right\} * e^A_i(\mathbf{x},t) * e^B_j(\mathbf{x},t) \right\} \\
&\quad - \left\{ e^A_i(\mathbf{x},t) * j^{B-s}_i(\mathbf{x},t) - e^B_i(\mathbf{x},t) * j^{A-s}_i(\mathbf{x},t) \right\} \\
&\quad + \left\{ h^A_i(\mathbf{x},t) * k^{B-s}_i(\mathbf{x},t) - h^B_i(\mathbf{x},t) * k^{A-s}_i(\mathbf{x},t) \right\} \\
&\quad - \left\{ e^A_i(\mathbf{x},t) * l^{B-s}_i(\mathbf{x},t) - e^B_i(\mathbf{x},t) * l^{A-s}_i(\mathbf{x},t) \right\}. \qquad (3.7)
\end{aligned}
$$

Finally, inserting the definition (3.1) of the interaction quantity on the left-hand-side yields the *local form* of the time-convolution reciprocity theorem (next page):

$$\frac{\partial}{\partial x_i}\left\{\varepsilon_{ijk}\left(e_j^A(\mathbf{x},t)*h_k^B(\mathbf{x},t)-e_j^B(\mathbf{x},t)*h_k^A(\mathbf{x},t)\right)\right\}=$$

$$\begin{aligned}
&-\frac{\partial}{\partial t}\left\{\left(\varphi_{ji}^A(\mathbf{x},t)-\varphi_{ij}^B(\mathbf{x},t)\right)*h_i^A(\mathbf{x},t)*h_j^B(\mathbf{x},t)\right\}\\
&+\frac{\partial}{\partial t}\left\{\left\{\left(\psi_{ji}^A(\mathbf{x},t)-\psi_{ij}^B(\mathbf{x},t)\right)+\left(\eta_{ji}^A(\mathbf{x},t)-\eta_{ij}^B(\mathbf{x},t)\right)*H(t)\right\}*e_i^A(\mathbf{x},t)*e_j^B(\mathbf{x},t)\right\}\\
&-\left\{e_i^A(\mathbf{x},t)*j_i^{B-s}(\mathbf{x},t)-e_i^B(\mathbf{x},t)*j_i^{A-s}(\mathbf{x},t)\right\}\\
&+\left\{h_i^A(\mathbf{x},t)*k_i^{B-s}(\mathbf{x},t)-h_i^B(\mathbf{x},t)*k_i^{A-s}(\mathbf{x},t)\right\}\\
&-\left\{e_i^A(\mathbf{x},t)*l_i^{B-s}(\mathbf{x},t)-e_i^B(\mathbf{x},t)*l_i^{A-s}(\mathbf{x},t)\right\}.
\end{aligned} \tag{3.8}$$

This is consistent with equation (23) in de Hoop (1987) and equation (16) in de Hoop (1992), when conductivity response functions η_{ij} and displacement current sources l_i are neglected.

The first group of terms on the right-hand-side of (3.8) depends on electromagnetic medium properties in the two states A and B, via the three constitutive relation response functions. If the three conditions

$$\varphi_{ij}^A(\mathbf{x},t)=\varphi_{ji}^B(\mathbf{x},t)\,,\qquad \psi_{ij}^A(\mathbf{x},t)=\psi_{ji}^B(\mathbf{x},t)\,,\qquad \eta_{ij}^A(\mathbf{x},t)=\eta_{ji}^B(\mathbf{x},t)\,, \tag{3.9a,b,c}$$

hold, then the first group of terms vanishes. In this case media A and B are referred to as *adjoints* of each other. In particular, if the response functions for medium A are causal, then so are those for medium B. Since the response functions for physically realistic electromagnetic media must be causal, we conclude that the reciprocity theorem of the time-convolution type is appropriate for physical (as opposed to numerical or computational) media. If media A and B are identical (in which case the state superscripts are omitted), then the adjoint condition implies that the response function tensors are symmetric:

$$\varphi_{ij}(\mathbf{x},t)=\varphi_{ji}(\mathbf{x},t)\,,\qquad \psi_{ij}(\mathbf{x},t)=\psi_{ji}(\mathbf{x},t)\,,\qquad \eta_{ij}(\mathbf{x},t)=\eta_{ji}(\mathbf{x},t)\,. \tag{3.10a,b,c}$$

The medium is referred to as *self-adjoint*.

The second group of terms in the local reciprocity theorem (3.8) depends on electromagnetic body sources. Interestingly, these terms involve time convolutions of the "A" electromagnetic wavefield with the "B" electromagnetic body sources, and vice versa.

What is the situation when the *unconventional* electromagnetic constitutive relations (2.3a,b,c) (i.e., containing response functions with identifying superscripts) are substituted into the local interaction quantity expression (3.4)? *Three* major groups of terms are obtained, as follows (next page):

$$\Phi(\mathbf{x},t) = -\frac{\partial}{\partial t}\left\{\left(\varphi_{ji}^{A-h}(\mathbf{x},t) - \varphi_{ij}^{B-h}(\mathbf{x},t)\right) * h_i^A(\mathbf{x},t) * h_j^B(\mathbf{x},t)\right\}$$
$$+\frac{\partial}{\partial t}\left\{\left\{\left(\psi_{ji}^{A-e}(\mathbf{x},t) - \psi_{ij}^{B-e}(\mathbf{x},t)\right) + \left(\eta_{ji}^{A-e}(\mathbf{x},t) - \eta_{ij}^{B-e}(\mathbf{x},t)\right) * H(t)\right\} * e_i^A(\mathbf{x},t) * e_j^B(\mathbf{x},t)\right\}$$

$$-\frac{\partial}{\partial t}\left\{\left(\varphi_{ji}^{A-e}(\mathbf{x},t) + \psi_{ij}^{B-h}(\mathbf{x},t) + \eta_{ij}^{B-h}(\mathbf{x},t) * H(t)\right) * e_i^A(\mathbf{x},t) * h_j^B(\mathbf{x},t)\right\}$$
$$+\frac{\partial}{\partial t}\left\{\left(\psi_{ji}^{A-h}(\mathbf{x},t) + \eta_{ji}^{A-h}(\mathbf{x},t) * H(t) + \varphi_{ij}^{B-e}(\mathbf{x},t)\right) * h_i^A(\mathbf{x},t) * e_j^B(\mathbf{x},t)\right\}$$

$$\begin{aligned}
&-\left\{e_i^A(\mathbf{x},t) * j_i^{B-s}(\mathbf{x},t) - e_i^B(\mathbf{x},t) * j_i^{A-s}(\mathbf{x},t)\right\} \\
&+\left\{h_i^A(\mathbf{x},t) * k_i^{B-s}(\mathbf{x},t) - h_i^B(\mathbf{x},t) * k_i^{A-s}(\mathbf{x},t)\right\} \\
&-\left\{e_i^A(\mathbf{x},t) * l_i^{B-s}(\mathbf{x},t) - e_i^B(\mathbf{x},t) * l_i^{A-s}(\mathbf{x},t)\right\}.
\end{aligned} \tag{3.11}$$

The first group of terms involves convolutions between the magnetic fields of states A and B, as well as convolutions between the electric fields of states A and B. These terms are identical to the corresponding terms in equation (3.8). The third group of terms, involving convolutions between electromagnetic wavefields and body sources (of opposing states) is also identical to equation (3.8). The second group of terms is new, and involves convolutions between the electric field of state A and the magnetic field of state B, and vice versa. These terms vanish if the *additional* adjoint conditions hold:

$$\varphi_{ij}^{A-e}(\mathbf{x},t) = -\left(\psi_{ji}^{B-h}(\mathbf{x},t) + \eta_{ji}^{B-h}(\mathbf{x},t) * H(t)\right), \tag{3.12a}$$

$$\left(\psi_{ij}^{A-h}(\mathbf{x},t) + \eta_{ij}^{A-h}(\mathbf{x},t) * H(t)\right) = -\varphi_{ji}^{B-e}(\mathbf{x},t)\,. \tag{3.12b}$$

If the two media are identical, then expressions (3.12 a and b) reduce to the single self-adjoint condition

$$\varphi_{ij}^{e}(\mathbf{x},t) = -\left(\psi_{ji}^{h}(\mathbf{x},t) + \eta_{ji}^{h}(\mathbf{x},t) * H(t)\right). \tag{3.12c}$$

The above adjoint conditions (3.9) and (3.12) appear to be consistent with those given in the more general treatment of wavefield reciprocity given by de Hoop and de Hoop (2000) (i.e., their equation (8.10)).

In the sequel, the situation of greatest interest is the case where the medium parameters in states A and B are identical, and the response functions representing these parameters satisfy adjoint conditions (i.e., equations (3.10a,b,c and 3.12a,b). Then, from equation (3.8) or (3.11), the local reciprocity theorem reduces to the particularly simple form

$$\frac{\partial}{\partial x_i}\left\{\varepsilon_{ijk}\left(e_j^A(\mathbf{x},t) * h_k^B(\mathbf{x},t) - e_j^B(\mathbf{x},t) * h_k^A(\mathbf{x},t)\right)\right\} =$$

$$\begin{aligned}
&-\left\{e_i^A(\mathbf{x},t) * j_i^{B-s}(\mathbf{x},t) - e_i^B(\mathbf{x},t) * j_i^{A-s}(\mathbf{x},t)\right\} \\
&+\left\{h_i^A(\mathbf{x},t) * k_i^{B-s}(\mathbf{x},t) - h_i^B(\mathbf{x},t) * k_i^{A-s}(\mathbf{x},t)\right\} \\
&-\left\{e_i^A(\mathbf{x},t) * l_i^{B-s}(\mathbf{x},t) - e_i^B(\mathbf{x},t) * l_i^{A-s}(\mathbf{x},t)\right\},
\end{aligned} \tag{3.13}$$

involving only the body source terms for the two electromagnetic wavefields.

3.2 Global Reciprocity Theorem

A *global* statement of reciprocity is obtained by integrating the local form (3.8) (or subsequently (3.13)) over the three-dimensional volume V bounded by closed surface S supporting electromagnetic wave propagation. Thus, working with equation (3.8) first:

$$\int_V \left\{ e_i^A(\mathbf{x},t) * j_i^{B-s}(\mathbf{x},t) - e_i^B(\mathbf{x},t) * j_i^{A-s}(\mathbf{x},t) \right\} dV(\mathbf{x})$$

$$-\int_V \left\{ h_i^A(\mathbf{x},t) * k_i^{B-s}(\mathbf{x},t) - h_i^B(\mathbf{x},t) * k_i^{A-s}(\mathbf{x},t) \right\} dV(\mathbf{x})$$

$$+\int_V \left\{ e_i^A(\mathbf{x},t) * l_i^{B-s}(\mathbf{x},t) - e_i^B(\mathbf{x},t) * l_i^{A-s}(\mathbf{x},t) \right\} dV(\mathbf{x})$$

$$+\int_V \frac{\partial}{\partial t} \left\{ \left(\varphi_{ji}^A(\mathbf{x},t) - \varphi_{ij}^B(\mathbf{x},t) \right) * h_i^A(\mathbf{x},t) * h_j^B(\mathbf{x},t) \right\} dV(\mathbf{x})$$

$$-\int_V \frac{\partial}{\partial t} \left\{ \left\{ \left(\psi_{ji}^A(\mathbf{x},t) - \psi_{ij}^B(\mathbf{x},t) \right) + \left(\eta_{ji}^A(\mathbf{x},t) - \eta_{ij}^B(\mathbf{x},t) \right) * H(t) \right\} * e_i^A(\mathbf{x},t) * e_j^B(\mathbf{x},t) \right\} dV(\mathbf{x})$$

$$= -\varepsilon_{ijk} \int_V \frac{\partial}{\partial x_i} \left\{ e_j^A(\mathbf{x},t) * h_k^B(\mathbf{x},t) - e_j^B(\mathbf{x},t) * h_k^A(\mathbf{x},t) \right\} dV(\mathbf{x}) . \qquad (3.14)$$

The right-hand-side of equation (3.14) is now simplified by applying the generalized divergence theorem (2.7). Thus

$$-\varepsilon_{ijk} \int_V \frac{\partial}{\partial x_i} \left\{ e_j^A(\mathbf{x},t) * h_k^B(\mathbf{x},t) - e_j^B(\mathbf{x},t) * h_k^A(\mathbf{x},t) \right\} dV(\mathbf{x})$$

$$= -\varepsilon_{ijk} \int_S m_i(\mathbf{x}) \left\{ e_j^A(\mathbf{x},t) * h_k^B(\mathbf{x},t) - e_j^B(\mathbf{x},t) * h_k^A(\mathbf{x},t) \right\} dS(\mathbf{x})$$
$$+ \varepsilon_{ijk} \sum_{n=1}^{N} \int_{S_n} n_i(\mathbf{x}) \left[e_j^A(\mathbf{x},t) * h_k^B(\mathbf{x},t) - e_j^B(\mathbf{x},t) * h_k^A(\mathbf{x},t) \right] dS(\mathbf{x}) , \qquad (3.15)$$

where $\mathbf{m}(\mathbf{x})$ is an outward-directed unit normal vector to the bounding surface S, and S_n (n = 1,2,3…N) refers to the n^{th} parameter discontinuity surface contained within volume V. Recall that paired square brackets refers to the jump discontinuity $[q_i] = q_i^+ - q_i^-$, where + and – signs are relative to the unit vector $\mathbf{n}$ normal to the internal interface.

The summation term above is now simplified. For two functions $a(\mathbf{x},t)$ and $b(\mathbf{x},t)$ of space and time, it is trivial to demonstrate

$$\left[a(\mathbf{x},t)*b(\mathbf{x},t)\right]\equiv\left[\int_{-\infty}^{+\infty}a(\mathbf{x},\tau)b(\mathbf{x},t-\tau)d\tau\right]=\int_{-\infty}^{+\infty}\left[a(\mathbf{x},\tau)b(\mathbf{x},t-\tau)\right]d\tau\,.$$

Next, appeal to the identity (Costen and Adamson, 1965)

$$[Qq]=[Q]\langle q\rangle+\langle Q\rangle[q] \quad \text{with} \quad [Q]\equiv Q^{+}-Q^{-} \quad \text{and} \quad \langle Q\rangle\equiv\frac{1}{2}\left(Q^{+}+Q^{-}\right),$$

which constitutes a "product theorem" for a jump discontinuity. Note that $\langle Q\rangle$ is just the average value of the discontinuous function $Q(\mathbf{x})$ as the interface is approached from both sides. Then, it is straightforward to establish the following theorem for the jump discontinuity of a convolution integral:

$$\left[a(\mathbf{x},t)*b(\mathbf{x},t)\right]=\left[a(\mathbf{x},t)\right]*\left\langle b(\mathbf{x},t)\right\rangle+\left\langle a(\mathbf{x},t)\right\rangle*\left[b(\mathbf{x},t)\right].$$

Applying this to the interface summation term in equation (3.15) gives

$$\varepsilon_{ijk}\sum_{n=1}^{N}\int_{S_n}n_i(\mathbf{x})\left[e_j^A(\mathbf{x},t)*h_k^B(\mathbf{x},t)-e_j^B(\mathbf{x},t)*h_k^A(\mathbf{x},t)\right]dS(\mathbf{x})$$

$$=\varepsilon_{ijk}\sum_{n=1}^{N}\int_{S_n}n_i(\mathbf{x})\Big\{\left[e_j^A(\mathbf{x},t)\right]*\left\langle h_k^B(\mathbf{x},t)\right\rangle+\left\langle e_j^A(\mathbf{x},t)\right\rangle*\left[h_k^B(\mathbf{x},t)\right]$$
$$-\left[e_j^B(\mathbf{x},t)\right]*\left\langle h_k^A(\mathbf{x},t)\right\rangle-\left\langle e_j^B(\mathbf{x},t)\right\rangle*\left[h_k^A(\mathbf{x},t)\right]\Big\}\,dS(\mathbf{x})$$

$$=\sum_{n=1}^{N}\int_{S_n}\Big\{\varepsilon_{ijk}n_i(\mathbf{x})\left[e_j^A(\mathbf{x},t)\right]*\left\langle h_k^B(\mathbf{x},t)\right\rangle+\left\langle e_j^A(\mathbf{x},t)\right\rangle*\varepsilon_{ijk}n_i(\mathbf{x})\left[h_k^B(\mathbf{x},t)\right]$$
$$-\varepsilon_{ijk}n_i(\mathbf{x})\left[e_j^B(\mathbf{x},t)\right]*\left\langle h_k^A(\mathbf{x},t)\right\rangle-\left\langle e_j^B(\mathbf{x},t)\right\rangle*\varepsilon_{ijk}n_i(\mathbf{x})\left[h_k^A(\mathbf{x},t)\right]\Big\}\,dS(\mathbf{x})$$

$$=\sum_{n=1}^{N}\int_{S_n}\Big\{\varepsilon_{kij}n_i(\mathbf{x})\left[e_j^A(\mathbf{x},t)\right]*\left\langle h_k^B(\mathbf{x},t)\right\rangle-\left\langle e_j^A(\mathbf{x},t)\right\rangle*\varepsilon_{jik}n_i(\mathbf{x})\left[h_k^B(\mathbf{x},t)\right]$$
$$-\varepsilon_{kij}n_i(\mathbf{x})\left[e_j^B(\mathbf{x},t)\right]*\left\langle h_k^A(\mathbf{x},t)\right\rangle+\left\langle e_j^B(\mathbf{x},t)\right\rangle*\varepsilon_{jik}n_i(\mathbf{x})\left[h_k^A(\mathbf{x},t)\right]\Big\}\,dS(\mathbf{x})\,, \qquad (3.16)$$

where the last step arises from the previously-mentioned index symmetry properties of the permuting symbol.

Next, appeal to the interface conditions (2.4a and b) involving jump discontinuities in the electromagnetic field values:

$$\varepsilon_{kij}n_i(\mathbf{x})\left[e_j(\mathbf{x},t)\right]=0\,, \qquad \text{and} \qquad \varepsilon_{jik}n_i(\mathbf{x})\left[h_k(\mathbf{x},t)\right]=s_j(\mathbf{x},t)\,, \qquad (3.17\text{a,b})$$

where $s_i(\mathbf{x},t)$ is a surface current (SI unit: A/m) component on the interface S_n. In vector form, these interface conditions are written as the vector cross products

$$\mathbf{n}(\mathbf{x})\times\left[\mathbf{e}(\mathbf{x},t)\right]=\mathbf{0}, \qquad \text{and} \qquad \mathbf{n}(\mathbf{x})\times\left[\mathbf{h}(\mathbf{x},t)\right]=\mathbf{s}(\mathbf{x},t), \tag{3.17c,d}$$

which are equivalent to

$$\left[\mathbf{n}(\mathbf{x})\times\mathbf{e}(\mathbf{x},t)\right]=\mathbf{0}, \qquad \text{and} \qquad \left[\mathbf{n}(\mathbf{x})\times\mathbf{h}(\mathbf{x},t)\right]=\mathbf{s}(\mathbf{x},t). \tag{3.17e,f}$$

So, the interface conditions pertain to the *tangential* components of the electromagnetic field vectors, and not the *normal* components. de Hoop (1987, 1992) and de Hoop and de Hoop (2000) apparently discount the existence of surface currents $\mathbf{s}(\mathbf{x},t)$. Hence, their interface condition corresponding to equation (3.17b or d or f) here has $\mathbf{s}(\mathbf{x},t) = 0$. Inserting these results (3.17a and b) into (3.16) implies the summation term reduces to the form

$$\varepsilon_{ijk}\sum_{n=1}^{N}\int_{S_n} n_i(\mathbf{x})\left[e_j^A(\mathbf{x},t)*h_k^B(\mathbf{x},t)-e_j^B(\mathbf{x},t)*h_k^A(\mathbf{x},t)\right]dS(\mathbf{x})$$

$$=-\sum_{n=1}^{N}\int_{S_n}\left\{\left\langle e_i^A(\mathbf{x},t)\right\rangle * s_i^B(\mathbf{x},t)-\left\langle e_i^B(\mathbf{x},t)\right\rangle * s_i^A(\mathbf{x},t)\right\}dS(\mathbf{x}). \tag{3.18}$$

Hence, the *global reciprocity theorem of the time-convolution type* is finally written by combining equations (3.14), (3.15), and (3.18) as:

$$\int_V\left\{e_i^A(\mathbf{x},t)*j_i^{B-s}(\mathbf{x},t)-e_i^B(\mathbf{x},t)*j_i^{A-s}(\mathbf{x},t)\right\}dV(\mathbf{x})$$

$$-\int_V\left\{h_i^A(\mathbf{x},t)*k_i^{B-s}(\mathbf{x},t)-h_i^B(\mathbf{x},t)*k_i^{A-s}(\mathbf{x},t)\right\}dV(\mathbf{x})$$

$$+\int_V\left\{e_i^A(\mathbf{x},t)*l_i^{B-s}(\mathbf{x},t)-e_i^B(\mathbf{x},t)*l_i^{A-s}(\mathbf{x},t)\right\}dV(\mathbf{x})$$

$$+\int_V\frac{\partial}{\partial t}\left\{\left(\varphi_{ji}^A(\mathbf{x},t)-\varphi_{ij}^B(\mathbf{x},t)\right)*h_i^A(\mathbf{x},t)*h_j^B(\mathbf{x},t)\right\}dV(\mathbf{x})$$

$$-\int_V\frac{\partial}{\partial t}\left\{\left\{\left(\psi_{ji}^A(\mathbf{x},t)-\psi_{ij}^B(\mathbf{x},t)\right)+\left(\eta_{ji}^A(\mathbf{x},t)-\eta_{ij}^B(\mathbf{x},t)\right)*H(t)\right\}*e_i^A(\mathbf{x},t)*e_j^B(\mathbf{x},t)\right\}dV(\mathbf{x})$$

$$+\sum_{n=1}^{N}\int_{S_n}\left\{\left\langle e_i^A(\mathbf{x},t)\right\rangle * s_i^B(\mathbf{x},t)-\left\langle e_i^B(\mathbf{x},t)\right\rangle * s_i^A(\mathbf{x},t)\right\}dS(\mathbf{x})$$

$$+\varepsilon_{ijk}\int_S m_i(\mathbf{x})\left\{e_j^A(\mathbf{x},t)*h_k^B(\mathbf{x},t)-e_j^B(\mathbf{x},t)*h_k^A(\mathbf{x},t)\right\}dS(\mathbf{x})=0. \tag{3.19}$$

Recall that summation over repeated indices is implied. Each major term has physical dimension "energy", with SI unit J. This expression corresponds to equation (24) in de Hoop (1987) and equation (17) in de Hoop (1992), when i) displacement current body sources $\mathbf{l}^s(\mathbf{x},t)$ are neglected, ii) conductivity response functions $\eta_{ij}(\mathbf{x},t)$ are neglected, and iii) surface currents $\mathbf{s}(\mathbf{x},t)$ are neglected.

In many circumstances, the surface integral over the outer bounding surface S to volume V may be ignored. If S is removed to infinity, and the electromagnetic wavefields in the two states A and B satisfy "radiation conditions" (meaning, they decay sufficiently rapidly as distance from their source distributions approaches infinity), then this surface integral vanishes.

If the unconventional electromagnetic constitutive relations are used, then the left-hand-side of the global reciprocity theorem contains two additional volume integrals, corresponding to the second group of terms in the local interaction quantity (3.11) above. For the interesting case where media A and B are identical and satisfy adjoint conditions, the global reciprocity theorem (3.19) simplifies to (next page):

$$\int_V \left\{ e_i^A(\mathbf{x},t) * j_i^{B-s}(\mathbf{x},t) - e_i^B(\mathbf{x},t) * j_i^{A-s}(\mathbf{x},t) \right\} dV(\mathbf{x})$$

$$-\int_V \left\{ h_i^A(\mathbf{x},t) * k_i^{B-s}(\mathbf{x},t) - h_i^B(\mathbf{x},t) * k_i^{A-s}(\mathbf{x},t) \right\} dV(\mathbf{x})$$

$$+\int_V \left\{ e_i^A(\mathbf{x},t) * l_i^{B-s}(\mathbf{x},t) - e_i^B(\mathbf{x},t) * l_i^{A-s}(\mathbf{x},t) \right\} dV(\mathbf{x})$$

$$+\sum_{n=1}^{N} \int_{S_n} \left\{ \left\langle e_i^A(\mathbf{x},t) \right\rangle * s_i^B(\mathbf{x},t) - \left\langle e_i^B(\mathbf{x},t) \right\rangle * s_i^A(\mathbf{x},t) \right\} dS(\mathbf{x})$$

$$+\varepsilon_{ijk} \int_S m_i(\mathbf{x}) \left\{ e_j^A(\mathbf{x},t) * h_k^B(\mathbf{x},t) - e_j^B(\mathbf{x},t) * h_k^A(\mathbf{x},t) \right\} dS(\mathbf{x}) = 0 \,. \tag{3.20}$$

Only body and surface sources of the electromagnetic fields appear in the reciprocity theorem. This form will be used in the sequel.

3.3 Comparisons

As a check, we note the above global reciprocity expression is consistent with equation (8.11) in the more generalized reciprocity treatment of de Hoop and de Hoop (2000), under the specific assumptions:

1) Body source electric displacement currents $l_i(\mathbf{x},t)$ vanish, for both wavefields A and B.
2) All surface source currents $s_i(\mathbf{x},t)$ vanish.

With these assumptions, the global reciprocity theorem (3.20) becomes

$$\int_V \left\{ e_i^A(\mathbf{x},t) * j_i^{B-s}(\mathbf{x},t) - e_i^B(\mathbf{x},t) * j_i^{A-s}(\mathbf{x},t) \right\} dV(\mathbf{x})$$

$$- \int_V \left\{ h_i^A(\mathbf{x},t) * k_i^{B-s}(\mathbf{x},t) - h_i^B(\mathbf{x},t) * k_i^{A-s}(\mathbf{x},t) \right\} dV(\mathbf{x})$$

$$+ \varepsilon_{ijk} \int_S m_i(\mathbf{x}) \left\{ e_j^A(\mathbf{x},t) * h_k^B(\mathbf{x},t) - e_j^B(\mathbf{x},t) * h_k^A(\mathbf{x},t) \right\} dS(\mathbf{x}) = 0 \, , \tag{3.21}$$

which agrees with equation (8.11) in de Hoop and de Hoop (2000), after many notational conversions are made. Also, if magnetic current sources are neglected, and the bounding surface S is removed to infinity (with radiation conditions assumed), then the above reciprocity theorem (3.21) becomes

$$\int_V \left\{ e_i^A(\mathbf{x},t) * j_i^{B-s}(\mathbf{x},t) - e_i^B(\mathbf{x},t) * j_i^{A-s}(\mathbf{x},t) \right\} dV(\mathbf{x}) = 0 \, , \tag{3.22}$$

which agrees with equation (1) in Ru-Shao Cheo (1965). Interestingly, Ru-Shao Cheo (1965) obtains expression (3.21) here (with no magnetic current sources; his equation (11)), but then appears to argue that the surface integral over S is *always* zero, even if it is nearby the current source distributions.

de Hoop (1987) states that the reciprocity theorem developed by Welch (1961) is "of the time-convolution type". However, equation (6) in Welch (1961) does *not* contain convolution integrals. Suppose in our equation (3.21) above that the convolutions (which are functions of time t) are evaluated at the specific time $t = 0$. We obtain

$$\int_V \left\{ \int_{-\infty}^{+\infty} e_i^A(\mathbf{x},\tau) \bar{j}_i^{B-s}(\mathbf{x},\tau) \, d\tau - \int_{-\infty}^{+\infty} \bar{e}_i^B(\mathbf{x},\tau) j_i^{A-s}(\mathbf{x},\tau) \, d\tau \right\} dV(\mathbf{x})$$

$$- \int_V \left\{ \int_{-\infty}^{+\infty} h_i^A(\mathbf{x},\tau) \bar{k}_i^{B-s}(\mathbf{x},\tau) \, d\tau - \int_{-\infty}^{+\infty} \bar{h}_i^B(\mathbf{x},\tau) k_i^{A-s}(\mathbf{x},\tau) \, d\tau \right\} dV(\mathbf{x}) = 0 \, ,$$

where the outer bounding surface S is also removed to infinity with radiation conditions. Recall that the superposed bar notation implies time-reversal: $\bar{x}(t) = x(-t)$. All state B quantities (EM wavefields and body sources) are time-reversed. Although this expression resembles equation (6) in Welch (1960), there are two (perhaps significant?) differences:

1) It is not clear whether Welch's superposed tilde "~" notation is equivalent to time-reversal (our superposed bar "—" notation). To the contrary, Welch (1961) refers to "time-advanced" and "adjoint" solutions of Maxwell's equations.
2) Welch (1961) does not superpose a tilde on the body sources (electric and magnetic currents) of electromagnetic state B in his equation (6).

So, for the present, we do not agree with de Hoop's (1987) implication that Welch's (1961) reciprocity theorem identical to the time-convolution reciprocity theorem developed here. Moreover, Welch's theorem is derived under the restricting assumptions of isotropic EM constitutive relations.

Fourier transforming the above time-domain global reciprocity theorem (3.20) yields the equivalent frequency-domain expression:

$$\begin{aligned}
&\int_V \Big\{ E_i^A(\mathbf{x},\omega) J_i^{B-s}(\mathbf{x},\omega) - E_i^B(\mathbf{x},\omega) J_i^{A-s}(\mathbf{x},\omega) \Big\}\, dV(\mathbf{x}) \\
&- \int_V \Big\{ H_i^A(\mathbf{x},\omega) K_i^{B-s}(\mathbf{x},\omega) - H_i^B(\mathbf{x},\omega) K_i^{A-s}(\mathbf{x},\omega) \Big\}\, dV(\mathbf{x}) \\
&+ \int_V \Big\{ E_i^A(\mathbf{x},\omega) L_i^{B-s}(\mathbf{x},\omega) - E_i^B(\mathbf{x},\omega) L_i^{A-s}(\mathbf{x},\omega) \Big\}\, dV(\mathbf{x}) \\
&+ \sum_{n=1}^{N} \int_{S_n} \Big\{ \big\langle E_i^A(\mathbf{x},\omega) \big\rangle S_i^B(\mathbf{x},\omega) - \big\langle E_i^B(\mathbf{x},\omega) \big\rangle S_i^A(\mathbf{x},\omega) \Big\}\, dS(\mathbf{x}) \\
&+ \varepsilon_{ijk} \int_S m_i(\mathbf{x}) \Big\{ E_j^A(\mathbf{x},\omega) H_k^B(\mathbf{x},\omega) - E_j^B(\mathbf{x},\omega) H_k^A(\mathbf{x},\omega) \Big\}\, dS(\mathbf{x}) = 0\,,
\end{aligned} \tag{3.23}$$

where upper case symbols denote Fourier transform of lower case counterparts. The time-domain convolutions transform to frequency-domain multiplications. This form is appropriate for comparison with analogous frequency-domain reciprocity expressions stated (but not proved) in Crowley (1954). If the outer bounding surface S to volume V is removed to infinity, and the electromagnetic wavefields in the two states A and B satisfy "radiation conditions" (meaning, they decay sufficiently rapidly as distance from their source distributions approaches infinity), then the final term (the surface integral over S) in the above reciprocity expression vanishes. Moreover, neglecting displacement current body sources and surface current sources gives

$$\begin{aligned}
&\int_V \Big\{ E_i^A(\mathbf{x},\omega) J_i^{B-s}(\mathbf{x},\omega) - E_i^B(\mathbf{x},\omega) J_i^{A-s}(\mathbf{x},\omega) \Big\}\, dV(\mathbf{x}) \\
&- \int_V \Big\{ H_i^A(\mathbf{x},\omega) K_i^{B-s}(\mathbf{x},\omega) - H_i^B(\mathbf{x},\omega) K_i^{A-s}(\mathbf{x},\omega) \Big\}\, dV(\mathbf{x}) = 0\,.
\end{aligned}$$

This agrees with equation (1) given in Crowley (1954), although that expression is stated under several rather restricting conditions (i.e., media are isotropic, wavefields A and B are monochromatic with the same frequency, and conductive media with non-zero conductivity σ are ignored). The above expression also agrees with equation (2) in Rumsey (1954), who cites Schelkunoff (1943, page 477) for a proof of “the reciprocity theorem”. Curiously, the above expression does *not* agree with equation (57) in Bojarski (1983); the signs of the magnetic current sources are reversed. However, it is possible that Bojarski’s magnetic current sources (never defined in his paper) are merely the negatives of those used here.

With no body sources of electromagnetic fields (and assuming the radiation conditions apply), the global (Fourier transformed) reciprocity theorem (3.21) above reduces to

$$\sum_{n=1}^{N}\int_{S_n}\left\{\left\langle E_i^A(\mathbf{x},\omega)\right\rangle S_i^B(\mathbf{x},\omega)-\left\langle E_i^B(\mathbf{x},\omega)\right\rangle S_i^A(\mathbf{x},\omega)\right\}\,dS(\mathbf{x})=0\,, \tag{3.24}$$

containing only surface currents. For comparison with analogous results in Crowley (1954), consider the case where there are only two internal interfaces ($N = 2$). Then

$$\int_{S_1}\left\{\left(E_i^A(\mathbf{x}^+,\omega)+E_i^A(\mathbf{x}^-,\omega)\right)S_i^B(\mathbf{x},\omega)-\left(E_i^B(\mathbf{x}^+,\omega)+E_i^B(\mathbf{x}^-,\omega)\right)S_i^A(\mathbf{x},\omega)\right\}\,dS(\mathbf{x})$$

$$+\int_{S_2}\left\{\left(E_i^A(\mathbf{x}^+,\omega)+E_i^A(\mathbf{x}^-,\omega)\right)S_i^B(\mathbf{x},\omega)-\left(E_i^B(\mathbf{x}^+,\omega)+E_i^B(\mathbf{x}^-,\omega)\right)S_i^A(\mathbf{x},\omega)\right\}\,dS(\mathbf{x})=0\,,$$

where the averaged electric field values on each interface are inserted. Next, assume that the surface current sources for electromagnetic wavefields A and B are confined to the separate interfaces S_1 and S_2, respectively. Then the reciprocity expression simplifies to

$$\int_{S_2}\left\{\left(E_i^A(\mathbf{x}^+,\omega)+E_i^A(\mathbf{x}^-,\omega)\right)S_i^B(\mathbf{x},\omega)\right\}\,dS(\mathbf{x})=\int_{S_1}\left\{\left(E_i^B(\mathbf{x}^+,\omega)+E_i^B(\mathbf{x}^-,\omega)\right)S_i^A(\mathbf{x},\omega)\right\}\,dS(\mathbf{x})\,.$$

This form appears identical to the sum of equations (2) and (2a) in Crowley (1954) (and surface *magnetic* currents are neglected in those expressions). Crowley (1954) takes both surfaces S_1 and S_2 to be closed, although this does not seem to be required in the present analysis.

The combination of body current sources (electric and magnetic, but no displacement currents) with surface electric current sources yields the (Fourier transformed) reciprocity expression

$$\int_{V}\left\{E_i^A(\mathbf{x},\omega)J_i^{B-s}(\mathbf{x},\omega)-E_i^B(\mathbf{x},\omega)J_i^{A-s}(\mathbf{x},\omega)\right\}\,dV(\mathbf{x})$$

$$-\int_{V}\left\{H_i^A(\mathbf{x},\omega)K_i^{B-s}(\mathbf{x},\omega)-H_i^B(\mathbf{x},\omega)K_i^{A-s}(\mathbf{x},\omega)\right\}\,dV(\mathbf{x})$$

$$+\sum_{n=1}^{N}\int_{S_n}\left\{\left\langle E_i^A(\mathbf{x},\omega)\right\rangle S_i^B(\mathbf{x},\omega)-\left\langle E_i^B(\mathbf{x},\omega)\right\rangle S_i^A(\mathbf{x},\omega)\right\}\,dS(\mathbf{x})=0\,, \tag{3.25}$$

(again assuming radiation conditions hold at infinite distance). If the A field is sourced solely by surface currents flowing on a single interface S_1 (i.e., $\mathbf{j}^{A\text{-}s}(\mathbf{x},t) = \mathbf{k}^{A\text{-}s}(\mathbf{x},t) = \mathbf{0}$) and the B field is sourced by solely by body currents (i.e., $\mathbf{s}^{B\text{-}s}(\mathbf{x},t) = \mathbf{0}$), then this becomes

$$\int_V \left\{ E_i^A(\mathbf{x},\omega) J_i^{B-s}(\mathbf{x},\omega) - H_i^A(\mathbf{x},\omega) K_i^{B-s}(\mathbf{x},\omega) \right\} dV(\mathbf{x}) = \int_{S_1} \left\{ \left\langle E_i^B(\mathbf{x},\omega) \right\rangle S_i^A(\mathbf{x},\omega) \right\} dS(\mathbf{x}),$$

which appears identical to equation (3) in Crowley (1954) (when surface *magnetic* currents on S_1 are ignored there). Crowley (1954) does not state how the discontinuous electric field $\mathbf{E}^{\mathbf{B}}(\mathbf{x},\omega)$ is defined on interface S_1, although it may be reasonably inferred that the average of the two limiting values on each side is taken.

These successful comparisons with the results of Crowley (1954) give us confidence that the surface current terms, apparently ignored by Ru-Shao Cheo (1965), de Hoop (1987, 1992), and de Hoop and de Hoop (2000), are indeed legitimate.

Finally, Fourier transforming the local reciprocity theorem (3.13) (appropriate for identical media in states A and B), and ignoring displacement current body sources, gives (in direct notation)

$$\begin{aligned}
-\operatorname{div}\left\{ \mathbf{E}^A(\mathbf{x},\omega) \times \mathbf{H}^B(\mathbf{x},\omega) - \mathbf{E}^B(\mathbf{x},\omega) \times \mathbf{H}^A(\mathbf{x},\omega) \right\} = \\
+\left\{ \mathbf{E}^A(\mathbf{x},\omega) \cdot \mathbf{J}^{B-s}(\mathbf{x},\omega) - \mathbf{E}^B(\mathbf{x},\omega) \cdot \mathbf{J}^{A-s}(\mathbf{x},\omega) \right\} \\
-\left\{ \mathbf{H}^A(\mathbf{x},\omega) \cdot \mathbf{K}^{B-s}(\mathbf{x},\omega) - \mathbf{H}^B(\mathbf{x},\omega) \cdot \mathbf{J}^{A-s}(\mathbf{x},\omega) \right\}.
\end{aligned}$$

This agrees with the first (un-numbered) equation in Welch (1960), who attributes the original deduction to Lorentz (1895). However, there is a subtle difference in interpretation. Welch (1960) considers the electric and magnetic vectors in the above expression to be *real-valued* fields with simple harmonic time dependence of frequency ω. The present development indicates that **E** and **H** (in both states A and B) are *complex-valued Fourier transforms* of real-valued, time-domain electromagnetic wavefields. Welch (1960) would assign SI unit (V-A)/m^3 = N/(m^2-s) to the above expression, whereas we would assign SI unit (V-A)/(m^3-Hz^2) = (N-s)/m^2. Both interpretations are valid.

4.0 POINT SOURCE RECIPROCITY

The local and global reciprocity theorems developed in the previous section readily accommodate electromagnetic wavefield source terms (either body sources or surface sources) with arbitrary spatial support. However, an important and common situation pertains to a *point source* with support restricted to a single isolated location in space. A point source at position $\mathbf{x}_s$ is proportional to the spatial Dirac delta function $\delta(\mathbf{x}-\mathbf{x}_s)$. For a body source, the Dirac function is three-dimensional (with SI unit m^{-3}), whereas for a surface source, it is two-dimensional (with SI unit m^{-2}). Reciprocity statements for various point source combinations are developed in this section. The following analysis bears some similarity to that contained in section 5 of de Hoop (1992), although our notation is far simpler, and a greater diversity of electromagnetic source types is considered.

We consider the two media supporting electromagnetic wave propagation in states A and B to be identical, and additionally to satisfy the adjoint conditions (3.10a,b,c) and (3.12c). Then, the operative global reciprocity theorem in volume V is equation (3.20):

$$\int_V \left\{ e_i^A(\mathbf{x},t) * j_i^{B-s}(\mathbf{x},t) - e_i^B(\mathbf{x},t) * j_i^{A-s}(\mathbf{x},t) \right\} dV(\mathbf{x})$$

$$-\int_V \left\{ h_i^A(\mathbf{x},t) * k_i^{B-s}(\mathbf{x},t) - h_i^B(\mathbf{x},t) * k_i^{A-s}(\mathbf{x},t) \right\} dV(\mathbf{x})$$

$$+\int_V \left\{ e_i^A(\mathbf{x},t) * l_i^{B-s}(\mathbf{x},t) - e_i^B(\mathbf{x},t) * l_i^{A-s}(\mathbf{x},t) \right\} dV(\mathbf{x})$$

$$+\sum_{n=1}^{N} \int_{S_n} \left\{ \left\langle e_i^A(\mathbf{x},t) \right\rangle * s_i^B(\mathbf{x},t) - \left\langle e_i^B(\mathbf{x},t) \right\rangle * s_i^A(\mathbf{x},t) \right\} dS(\mathbf{x}) = 0 \,, \tag{4.1}$$

where wavefield radiation conditions are also assumed to eliminate the integral on the outer bounding surface *S*.

Similar to the strategy pursued by Aldridge and Symons (2001) with seismic energy sources, we now construct a table of all possible pairs of point electromagnetic sources that can activate the two wavefield states A and B. Equation (4.1) indicates that there are four types of sources. Hence, there are (5 × 4)/2 = 10 distinct source pairs, which we order as follows:

	State A	State B
1)	body electric current	body electric current
2)	body magnetic current	body magnetic current
3)	body displacement current	body displacement current
4)	surface current	surface current
5)	body electric current	body displacement current

6) body electric current	body magnetic current
7) body displacement current	body magnetic current
8) surface current	body electric current
9) surface current	body displacement current
10) surface current	body magnetic current

The first four table entries correspond to identical sources types activating wavefield states A and B. Entries 5 through 7 correspond to different types of body sources, and the remaining entries 8 through 10 involve surface source – body source pairs. de Hoop (1992) considers only two body source types: electric current and magnetic current. However, he arrives at a total of four (= 2^2) source pairs instead of 3 (= (3 × 2)/2) *distinct* source pairings. His selection corresponds to entries #1, #2, and #6 in the above table, with the additional {A – body magnetic current; B – body electric current}. This last pairing is physically the same as #6, and only differs in notation.

4.1 Two Similar Point Sources

The first four entries in the above table involve pairs of point sources of the same type. Items #1 through #3 are body sources and item #4 involves surface sources. Consider first the case of two point current density sources. A point body source of electric (conduction) current located at position $\mathbf{x}_s$ may be represented by a current density vector of the form

$$\mathbf{j}_s(\mathbf{x},t) = J\mathbf{d}w(t)\delta(\mathbf{x}-\mathbf{x}_s). \tag{4.2}$$

Here J is a current amplitude scalar with SI unit A-m. Vector $\mathbf{d}$ is a dimensionless unit vector characterizing the orientation of the current source, and $w(t)$ is a dimensionless source waveform. It is convenient to normalize the maximum absolute amplitude of this waveform to unity: $\max\limits_t |w(t)| = 1$. Then, the magnitude of the source is represented solely by scalar J. Recall that the three-dimensional spatial Dirac delta function $\delta(\mathbf{x})$ has SI unit m^{-3}, so that current density $\mathbf{j}_s(\mathbf{x},t)$ has the proper SI unit A/m^2.

We distinguish the two point current density vectors that activate electromagnetic wavefields A and B within volume V via superscripts as follows:

$$\mathbf{j}_s^A(\mathbf{x},t) = J^A\mathbf{d}^A w^A(t)\delta(\mathbf{x}-\mathbf{x}_s^A), \qquad \mathbf{j}_s^B(\mathbf{x},t) = J^B\mathbf{d}^B w^B(t)\delta(\mathbf{x}-\mathbf{x}_s^B). \tag{4.3a,b}$$

All other electromagnetic sources (either body sources or surface sources) are assumed to vanish. Then, substituting expressions (4.3a and b) into the global reciprocity theorem (4.1) and (trivially) performing the volume integration over $\mathbf{x}$ yields

$$J^A w^A(t) * \mathbf{d}^A \cdot \mathbf{e}^B(\mathbf{x}_s^A,t) = J^B w^B(t) * \mathbf{d}^B \cdot \mathbf{e}^A(\mathbf{x}_s^B,t)\,. \tag{4.4a}$$

This is an explicit, time-domain relationship between the two electric vectors generated by point current density sources situated at the two different locations $\mathbf{x}_s^A$ and $\mathbf{x}_s^B$. Interestingly, the SI unit of each side is A-V-s = N-m = J, or the SI unit of energy. Equation (4.4a) here is analogous to equation (26) in de Hoop (1992), although he considers the source waveforms to be temporal Dirac delta functions.

If the amplitudes and waveforms of the two point sources are identical (i.e., $J^A = J^B = J$ and $w^A(t) = w^B(t) = w(t)$), then they cancel from each side of expression (4.4a), giving the simpler form

$$\mathbf{d}^A \cdot \mathbf{e}^B(\mathbf{x}_s^A, t) = \mathbf{d}^B \cdot \mathbf{e}^A(\mathbf{x}_s^B, t)\,, \tag{4.4b}$$

(with SI unit V/m). In words, equation (4.4b) indicates that the electric vector observed at position $\mathbf{x}_s^A$ due to a point current source at position $\mathbf{x}_s^B$ is *identical* to the electric vector observed at position $\mathbf{x}_s^B$ due to a point current source at position $\mathbf{x}_s^A$, as long as the electric vectors are measured along the respective source directions $\mathbf{d}^A$ and $\mathbf{d}^B$. This is the common (and perhaps intuitive) notion of reciprocity; the time signal is invariant with respect to interchanging the locations of point source and point receiver. Figure 4.1 depicts this situation graphically.

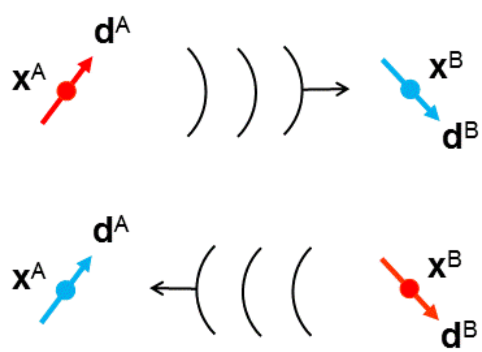

Figure 4.1. Illustration of signal invariance with point body current density sources (**red** vectors) and point electric field receivers (**blue** vectors). In the top panel, a point current density source at $\mathbf{x}^A$ and oriented in direction $\mathbf{d}^A$ generates an electric field component recorded by a receiver at $\mathbf{x}^B$ along direction $\mathbf{d}^B$. This signal is identical to that generated by the point source and point receiver configuration depicted in the bottom panel. Curved black lines are intended to convey a sense of propagating EM wavefronts.

The analysis proceeds in exactly the same manner for two point magnetic current sources (case #2 in above table) and two point displacement current sources (case #3). Thus, a point magnetic current source has the form

$$\mathbf{k}_s(\mathbf{x}, t) = K\mathbf{d}w(t)\delta(\mathbf{x} - \mathbf{x}_s), \tag{4.5}$$

where amplitude scalar K has SI unit V-m. Then, assuming all other electromagnetic wavefield sources vanish, inserting this form into the global reciprocity theorem (4.1) immediately leads to

$$K^A w^A(t) * \mathbf{d}^A \cdot \mathbf{h}^B(\mathbf{x}_s^A, t) = K^B w^B(t) * \mathbf{d}^B \cdot \mathbf{h}^A(\mathbf{x}_s^B, t)\,. \tag{4.6a}$$

This is analogous to equation (29) in de Hoop (1992). Identical source amplitudes and waveforms in the two states A and B imply the simpler relation

$$\mathbf{d}^A \cdot \mathbf{h}^B(\mathbf{x}_s^A, t) = \mathbf{d}^B \cdot \mathbf{h}^A(\mathbf{x}_s^B, t) \,. \tag{4.6b}$$

The interpretation of equation (4.6b) is the same as illustrated in figure 4.1 for current density sources. Similarly, a point displacement current source is represented by

$$\mathbf{l}_s(\mathbf{x}, t) = L\mathbf{d}w(t)\delta(\mathbf{x} - \mathbf{x}_s), \tag{4.7}$$

where amplitude scalar L has SI unit A-m. Inserting this into global reciprocity theorem (4.1) yields

$$L^A w^A(t) * \mathbf{d}^A \cdot \mathbf{e}^B(\mathbf{x}_s^A, t) = L^B w^B(t) * \mathbf{d}^B \cdot \mathbf{e}^A(\mathbf{x}_s^B, t) \,, \tag{4.8}$$

which simplifies to the previous expression (4.4b) when identical source amplitudes and waveforms are assumed. Similar to expression (4.4a), the reciprocity relations (4.6a) and (4.8) have physical dimension "energy", with SI unit V-A-s = J.

In order to avoid future confusion, it is appropriate at this time to discuss the dimensionless (and normalized) source waveform $w(t)$ assumed for the magnetic current (4.5) and displacement current (4.7) point source vectors. Recall that magnetic current and displacement current sources are defined by time derivatives in equations (3.6a,b) as

$$\mathbf{k}_s(\mathbf{x}, t) \equiv \frac{\partial \mathbf{b}_s(\mathbf{x}, t)}{\partial t}, \qquad \mathbf{l}_s(\mathbf{x}, t) \equiv \frac{\partial \mathbf{d}_s(\mathbf{x}, t)}{\partial t}, \tag{3.6a,b}$$

respectively. Here $\mathbf{b}_s(\mathbf{x},t)$ and $\mathbf{d}_s(\mathbf{x},t)$ are source magnetic and electric *flux density* vectors, each appearing on the right side of the relevant electromagnetic constitutive relations (2.2a,b). In many circumstances, it is appropriate to prescribe a point body source of flux density rather than of current. For example, point sources of magnetic and electric flux density are readily defined as

$$\mathbf{b}_s(\mathbf{x}, t) = B\mathbf{d}w_b(t)\delta(\mathbf{x} - \mathbf{x}_s), \qquad \mathbf{d}_s(\mathbf{x}, t) = D\mathbf{d}w_d(t)\delta(\mathbf{x} - \mathbf{x}_s), \tag{4.9a,b}$$

where amplitude scalars B and D have SI units V-m-s and A-m-s, respectively. Note that the source waveforms (dimensionless and normalized) are subscripted to denote the associated source type. It is easy to deduce that the equivalent *magnetic current* source parameters (for use in (4.5)) are

$$K = B\left(\max_t |w_b'(t)|\right), \qquad w(t) = \frac{w_b'(t)}{\max_t |w_b'(t)|}, \tag{4.10a,b}$$

and the equivalent *electric displacement current* source parameters (for use in (4.7)) are

$$L = D\left(\max_t |w_d'(t)|\right), \qquad w(t) = \frac{w_d'(t)}{\max_t |w_d'(t)|}, \tag{4.11a,b}$$

where primes denote differentiation. However, the product $Kw(t)$ in reciprocity relation (4.6a) and $Lw(t)$ in relation (4.8) do not depend on the maximum absolute values. Thus, reciprocity relations involving magnetic and electric flux density sources are

$$B^A \frac{\partial w_b^A(t)}{\partial t} * \mathbf{d}^A \cdot \mathbf{h}^B(\mathbf{x}_s^A, t) = B^B \frac{\partial w_b^B(t)}{\partial t} * \mathbf{d}^B \cdot \mathbf{h}^A(\mathbf{x}_s^B, t), \tag{4.12a}$$

and

$$D^A \frac{\partial w_d^A(t)}{\partial t} * \mathbf{d}^A \cdot \mathbf{e}^B(\mathbf{x}_s^A, t) = D^B \frac{\partial w_d^B(t)}{\partial t} * \mathbf{d}^B \cdot \mathbf{e}^A(\mathbf{x}_s^B, t), \tag{4.12b}$$

respectively. The reciprocity relations have time-derivatives of the (dimensionless and normalized) source waveforms, as intuitively expected. Both expressions have physical dimension "energy", with SI unit J. For identical source amplitudes and waveforms in the two states A and B, they reduce to the familiar simplified forms.

Finally, case #4 in the above table, for two point *surface* current sources, is slightly different. A point surface current source is represented by the vector

$$\mathbf{s}(\mathbf{x},t) = S\mathbf{d}w(t)\delta(\mathbf{x} - \mathbf{x}_s), \tag{4.13}$$

where scalar S has SI unit A-m. Position vectors $\mathbf{x}$ and $\mathbf{x}_s$ are restricted to reside on a material parameter discontinuity surface S_n within volume V. Hence, the Dirac delta function in (4.13) is two-dimensional, implying the surface current density vector has SI unit A/m (as opposed to a body current density vector, with SI unit A/m^2). Also, orientation vector $\mathbf{d}$ is restricted to be tangent to the surface S_n at position $\mathbf{x}_s$.

Substituting form (4.13) into the global reciprocity theorem (4.1) (and assuming all other source types vanish) yields the reciprocity relation

$$S^A w^A(t) * \mathbf{d}^A \cdot \left\langle \mathbf{e}^B(\mathbf{x}_s^A, t) \right\rangle = S^B w^B(t) * \mathbf{d}^B \cdot \left\langle \mathbf{e}^A(\mathbf{x}_s^B, t) \right\rangle, \tag{4.14a}$$

again with dimension energy. Identical amplitudes and waveforms in states A and B imply the simpler form

$$\mathbf{d}^A \cdot \left\langle \mathbf{e}^B(\mathbf{x}_s^A, t) \right\rangle = \mathbf{d}^B \cdot \left\langle \mathbf{e}^A(\mathbf{x}_s^B, t) \right\rangle. \tag{4.14b}$$

Recall that the paired braces $\langle \ \rangle$ refer to an average value as the interface is approached from the two sides. Thus, the average value of an electric vector is

$$\left\langle \mathbf{e}(\mathbf{x},t) \right\rangle = \frac{\mathbf{e}(\mathbf{x}^+, t) + \mathbf{e}(\mathbf{x}^-, t)}{2},$$

where the positions labeled + and **–** are referenced to the unit normal $\mathbf{n}(\mathbf{x})$ to interface S_n. The physical interpretation of equation (4.14b) is illustrated in figure 4.2, which is a slight generalization of figure 4.1 to include interfaces (i.e., the dashed lines).

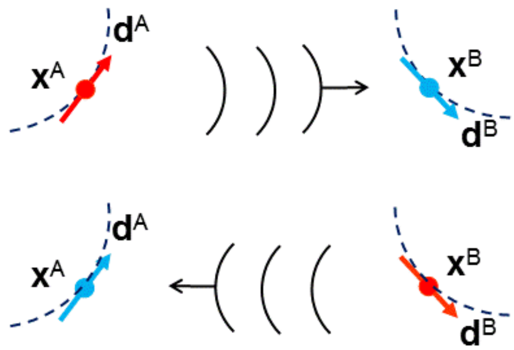

Figure 4.2. Illustration of signal invariance with point surface current density sources (**red** vectors) and point electric field receivers (**blue** vectors). In the top panel, a point surface current density source at $\mathbf{x}^A$ and oriented in direction $\mathbf{d}^A$ generates an electric field component recorded by a receiver at $\mathbf{x}^B$ along direction $\mathbf{d}^B$. This signal is identical to that generated by the point source and point receiver configuration depicted in the bottom panel. Dashed lines depict interfaces (discontinuity surfaces of material parameters). Surface current orientation vectors $\mathbf{d}^A$ and $\mathbf{d}^B$ are tangent to these interfaces.

4.2 Two Dissimilar Point Sources

Consider now the three entries #5, #6, and #7 in the above table, which correspond to dissimilar point body sources activating the two electromagnetic wavefield states A and B. In case #5, state A is sourced by an electric (conduction) current and state B by an electric (displacement) current. The point current density vectors are

$$\mathbf{j}_s^A(\mathbf{x},t) = J^A\mathbf{d}^A w^A(t)\delta\left(\mathbf{x}-\mathbf{x}_s^A\right), \qquad \mathbf{l}_s^B(\mathbf{x},t) = L^B\mathbf{d}^B w^B(t)\delta\left(\mathbf{x}-\mathbf{x}_s^B\right), \tag{4.15a,b}$$

where all symbols are previously defined. Substituting these into the global reciprocity theorem (4.1) (and assuming all other source types vanish) gives

$$J^A w^A(t) * \mathbf{d}^A \cdot \mathbf{e}^B(\mathbf{x}_s^A,t) = L^B w^B(t) * \mathbf{d}^B \cdot \mathbf{e}^A(\mathbf{x}_s^B,t)\,, \tag{4.16a}$$

which is virtually identical to reciprocity relation (4.4a); amplitude scalar L^B replaces J^B on the right hand side. If source amplitudes and waveforms in the two states are the same, then the simplification

$$\mathbf{d}^A \cdot \mathbf{e}^B(\mathbf{x}_s^A,t) = \mathbf{d}^B \cdot \mathbf{e}^A(\mathbf{x}_s^B,t)\,, \tag{4.16b}$$

immediately follows.

Case #6 corresponds to state A sourced by an electric (conduction) current and state B by a magnetic current. The point source vectors are

$$\mathbf{j}_s^A(\mathbf{x},t) = J^A\mathbf{d}^A w^A(t)\delta\left(\mathbf{x}-\mathbf{x}_s^A\right), \qquad \mathbf{k}_s^B(\mathbf{x},t) = K^B\mathbf{d}^B w^B(t)\delta\left(\mathbf{x}-\mathbf{x}_s^B\right). \tag{4.17a,b}$$

The global reciprocity theorem (4.1) implies

$$J^A w^A(t) * \mathbf{d}^A \cdot \mathbf{e}^B(\mathbf{x}_s^A, t) = -K^B w^B(t) * \mathbf{d}^B \cdot \mathbf{h}^A(\mathbf{x}_s^B, t) . \tag{4.18a}$$

Note the negative sign on the right-hand-side! This negative sign can also be found in de Hoop's (1992) development corresponding to our case #5 (i.e., his equation (30)).

Both sides of (4.18a) have physical dimension "energy", with SI unit J. In this case of different source types, it is perhaps unrealistic to assume identical source waveforms. Nevertheless, assuming $w^A(t) = w^B(t) = w(t)$ and convolving with the inverse wavelet $w^{-1}(t)$, we obtain the simpler form

$$J^A \mathbf{d}^A \cdot \mathbf{e}^B(\mathbf{x}_s^A, t) = -K^B \mathbf{d}^B \cdot \mathbf{h}^A(\mathbf{x}_s^B, t) , \tag{4.18b}$$

with SI unit J/s = W (i.e., unit of energy per time, or power).

Finally case #7, corresponding to state A sourced by an electric (displacement) current and state B sourced by a magnetic current, is virtually identical to the preceding case #6. Merely replace the amplitude scalar J^A in reciprocity relations (4.18a and b) with L^A (which has the same SI unit A-m). The negative sign persists; this occurs whenever an electric current source (either conduction or displacement) and a magnetic current source are paired.

Figure 4.3 illustrates signal invariance when point source and point receiver are swapped for this more complicated case of dissimilar electromagnetic wavefield sources. When account is taken of the negative sign in (4.18b), the two EM time signals are identical!

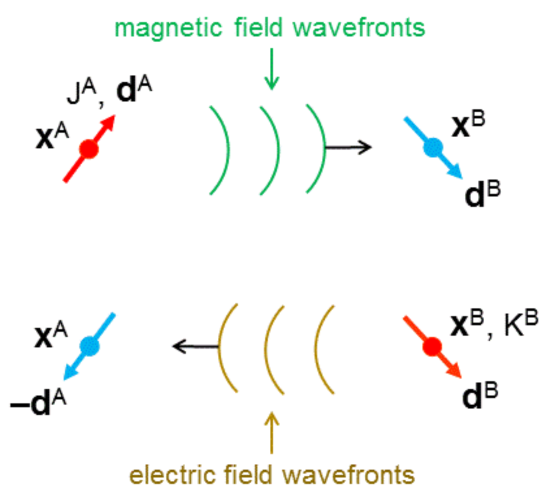

Figure 4.3. Illustration of electromagnetic signal invariance with dissimilar point sources (**red** vectors) and dissimilar point receivers (**blue** vectors). In the top panel, a point electric current density source at $\mathbf{x}^A$ and oriented in direction $\mathbf{d}^A$ generates a magnetic component recorded by a receiver at $\mathbf{x}^B$ along direction $\mathbf{d}^B$. Propagating magnetic field wavefronts are depicted in **green**. In the bottom panel, a point magnetic current density source at $\mathbf{x}^B$ and oriented in direction $\mathbf{d}^B$ generates and electric component recorded by a receiver back at $\mathbf{x}^A$ along the opposite original direction $-\mathbf{d}^A$. Propagating electric field wavefronts are depicted in **brown**. Assuming the source magnitudes J^A and K^B are both positive, then the two electromagnetic time signals are identical.

The last three cases #8, #9, and #10 in the above table involve pairing a surface electric current source in state A with a body source (either electric or magnetic) in state B. A point surface current source is represented by the vector

$$\mathbf{s}(\mathbf{x},t) = S\mathbf{d}w(t)\delta(\mathbf{x}-\mathbf{x}_s), \tag{4.19}$$

where scalar S has SI unit A-m. Recall that position vectors $\mathbf{x}$ and $\mathbf{x}_s$ are restricted to reside on a material parameter discontinuity surface S_n contained within volume V, implying that the Dirac delta function is two-dimensional, rather than three-dimensional. Straightforward application of global reciprocity theorem (4.1) yields the three reciprocity relations:

***Case #8*: *State A = surface electric current*; *State B = body conduction current*:**

$$S^A w^A(t) * \mathbf{d}^A \cdot \left\langle \mathbf{e}^B(\mathbf{x}_s^A,t) \right\rangle = J^B w^B(t) * \mathbf{d}^B \cdot \mathbf{e}^A(\mathbf{x}_s^B,t)\,. \tag{4.20}$$

***Case #9*: *State A = surface electric current*; *State B = body conduction current*:**

$$S^A w^A(t) * \mathbf{d}^A \cdot \left\langle \mathbf{e}^B(\mathbf{x}_s^A,t) \right\rangle = L^B w^B(t) * \mathbf{d}^B \cdot \mathbf{e}^A(\mathbf{x}_s^B,t)\,. \tag{4.21}$$

***Case #10*: *State A = surface electric current*; *State B = body magnetic current*:**

$$S^A w^A(t) * \mathbf{d}^A \cdot \left\langle \mathbf{e}^B(\mathbf{x}_s^A,t) \right\rangle = -K^B w^B(t) * \mathbf{d}^B \cdot \mathbf{h}^A(\mathbf{x}_s^B,t)\,. \tag{4.22}$$

Expressions (4.20) and (4.21) are virtually identical. Note the negative sign in equation (4.22), which again arises when an electric current source and a magnetic current source are paired. All three expressions have physical dimension "energy", with SI unit J. The graphical illustration of equation (4.22) would be the same as in figure 4.3, except the surface electric current point source $\mathbf{s}(\mathbf{x}^{\mathbf{A}},t)$ is restricted to be tangent to an interface (as in figure 4.2).

4.3 Uniform Wholespace

The point source – point receiver reciprocity relations developed in the previous section may be explicitly verified for media with sufficiently simple material properties. In this section, we consider a homogeneous and isotropic wholespace characterized by the three electromagnetic response functions

$$\varphi_{ij}(\mathbf{x},t)=\mu\delta_{ij}\delta(t)\,,\qquad \psi_{ij}(\mathbf{x},t)=\varepsilon\delta_{ij}\delta(t)\,,\qquad \eta_{ij}(\mathbf{x},t)=\sigma\delta_{ij}\delta(t)\,,\tag{4.23a,b,c}$$

where δ_{ij} is the Kronecker delta symbol and $\delta(t)$ is the temporal Dirac delta function. Then, the convolutional constitutive relations (2.2a,b,c) reduce to the familiar forms

$$b_i(\mathbf{x},t)=\mu h_i(\mathbf{x},t)+b_i^s(\mathbf{x},t)\,,\qquad d_i(\mathbf{x},t)=\varepsilon e_j(\mathbf{x},t)+d_i^s(\mathbf{x},t)\,,\qquad j_i(\mathbf{x},t)=\sigma e_j(\mathbf{x},t)+j_i^s(\mathbf{x},t)\,,\tag{4.24a,b,c}$$

appropriate for a homogeneous and isotropic medium. Constants μ, ε, and σ are the electric permittivity, magnetic permeability, and current conductivity, respectively. Since the response functions (4.23) are proportional to the Dirac delta function, the medium is considered to be “instantaneously reacting” in the language of de Hoop (1987, 1992).

A homogeneous wholespace has no internal interfaces S_n (i.e., discontinuity surfaces in medium properties) and does not have an outer bounding surface S. Hence, the time-domain global reciprocity theorem (3.20) simplifies to

$$\begin{aligned}
&\int_V \left\{ e_i^A(\mathbf{x},t) * j_i^{B-s}(\mathbf{x},t) - e_i^B(\mathbf{x},t) * j_i^{A-s}(\mathbf{x},t) \right\} dV(\mathbf{x}) \\
-&\int_V \left\{ h_i^A(\mathbf{x},t) * k_i^{B-s}(\mathbf{x},t) - h_i^B(\mathbf{x},t) * k_i^{A-s}(\mathbf{x},t) \right\} dV(\mathbf{x}) \\
+&\int_V \left\{ e_i^A(\mathbf{x},t) * l_i^{B-s}(\mathbf{x},t) - e_i^B(\mathbf{x},t) * l_i^{A-s}(\mathbf{x},t) \right\} dV(\mathbf{x}) = 0\,.
\end{aligned}\tag{4.25}$$

Fourier transforming yields the frequency-domain counterpart

$$\begin{aligned}
&\int_V \left\{ E_i^A(\mathbf{x},\omega) J_i^{B-s}(\mathbf{x},\omega) - E_i^B(\mathbf{x},\omega) J_i^{A-s}(\mathbf{x},\omega) \right\} dV(\mathbf{x}) \\
-&\int_V \left\{ H_i^A(\mathbf{x},\omega) K_i^{B-s}(\mathbf{x},\omega) - H_i^B(\mathbf{x},\omega) K_i^{A-s}(\mathbf{x},\omega) \right\} dV(\mathbf{x}) \\
+&\int_V \left\{ E_i^A(\mathbf{x},\omega) L_i^{B-s}(\mathbf{x},\omega) - E_i^B(\mathbf{x},\omega) L_i^{A-s}(\mathbf{x},\omega) \right\} dV(\mathbf{x})\,,
\end{aligned}\tag{4.26a}$$

in which convolutions transform to multiplications. Clearly, in direct (vector) notation, this is written as

$$\int_V \left\{ \mathbf{E}^A(\mathbf{x},\omega)\cdot\mathbf{J}^{B-s}(\mathbf{x},\omega) - \mathbf{E}^B(\mathbf{x},\omega)\cdot\mathbf{J}^{A-s}(\mathbf{x},\omega) \right\} dV(\mathbf{x})$$

$$-\int_V \left\{ \mathbf{H}^A(\mathbf{x},\omega)\cdot\mathbf{K}^{B-s}(\mathbf{x},\omega) - \mathbf{H}^B(\mathbf{x},\omega)\cdot\mathbf{K}^{A-s}(\mathbf{x},\omega) \right\} dV(\mathbf{x})$$

$$+\int_V \left\{ \mathbf{E}^A(\mathbf{x},\omega)\cdot\mathbf{L}^{B-s}(\mathbf{x},\omega) - \mathbf{E}^B(\mathbf{x},\omega)\cdot\mathbf{L}^{A-s}(\mathbf{x},\omega) \right\} dV(\mathbf{x}) = 0\,, \tag{4.26b}$$

which will subsequently be useful.

4.3.1 Exact Frequency-Domain Formulae

Aldridge (2013) develops frequency-domain formulae for the (Fourier-transformed) electric vector $\mathbf{E}(\mathbf{x},\omega)$ and magnetic vector $\mathbf{H}(\mathbf{x},\omega)$ generated by various types of point sources situated in a homogeneous and isotropic wholespace. These formulae are considered "exact" in the sense that no frequency limitations (like low- or high-frequencies) are adopted in the derivations. In this section, we demonstrate that these exact frequency-domain formulae are indeed reciprocal, for a variety of point current source and point magnetic source combinations.

First, some mathematical definitions are required. Let a point source be located at position $\mathbf{x}_s$ and a point receiver at position $\mathbf{x}_r$. The vector pointing *from* source *to* receiver is $\mathbf{r} = \mathbf{x}_r - \mathbf{x}_s$. A unit vector pointing in this same direction is $\mathbf{e}_r = \mathbf{r}/r$, where r is the scalar distance between source and receiver $r = \|\mathbf{r}\| = \sqrt{(x_r - x_s)^2 + (y_r - y_s)^2 + (z_r - z_s)^2}$.

The frequency-domain expressions make repeated use of the *complex wavenumber* $k(\omega)$. The squared complex wavenumber is given by

$$k(\omega)^2 = \frac{\omega^2 + i\omega_t\omega}{c_\infty^2}\,, \tag{4.27}$$

where $\omega_t = \sigma/\varepsilon$ is the transition angular frequency and $c_\infty = 1/\sqrt{\varepsilon\mu}$ is the infinite-frequency phase speed. The transition frequency roughly separates the EM diffusion regime (for $|\omega| < \omega_t$) from the EM propagation regime (for $|\omega| > \omega_t$). It can be demonstrated (Loseth et al., 2006; Aldridge, 2013) that the complex wavenumber can be written as

$$k(\omega) = \frac{1}{c_\infty}\left[\frac{\omega}{s(\omega/\omega_t)} + i\,\frac{\omega_t s(\omega/\omega_t)}{2} \right], \tag{4.28a}$$

where the dimensionless function $s(x)$ is defined as

$$s(x) \equiv \sqrt{2|x|}\,\left[\sqrt{1+x^2} - |x|\right]^{+1/2}. \tag{4.28b}$$

$s(x)$ is a positive and even function that approaches zero and unity as $x \to 0$ and $x \to \pm\infty$, respectively.

We now verify various time-domain reciprocity relations involving two point electromagnetic sources by Fourier transforming the relations to the frequency domain.

***Point electric conduction current (or electric dipole) sources*:**

A point source of electric (conduction) current located at $\mathbf{x}_s$ is represented as

$$\mathbf{j}_s(\mathbf{x},t) = J\mathbf{d}w(t)\delta(\mathbf{x}-\mathbf{x}_s), \tag{4.29}$$

where J is an amplitude scalar (SI unit: A-m), $\mathbf{d}$ is a unit orientation vector, and $w(t)$ is a dimensionless waveform (normalized to unit maximum absolute amplitude). This is often referred to as a point "electric dipole" source. The Fourier-transformed electric vector (SI unit: (V/m)/Hz) generated by this source at a point receiver position $\mathbf{x}_r$ is given by Aldridge (2013) as

$$\mathbf{E}(\mathbf{x}_r,\omega;\mathbf{x}_s) = \left(\frac{J\mu}{4\pi r}\right)[(-i\omega)W(\omega)]\exp[+ik(\omega)r] \times \\ \left\{[(\mathbf{d}\cdot\mathbf{e}_r)\mathbf{e}_r - \mathbf{d}] - [3(\mathbf{d}\cdot\mathbf{e}_r)\mathbf{e}_r - \mathbf{d}]\left(\frac{1}{ik(\omega)r} - \frac{1}{(ik(\omega)r)^2}\right)\right\}, \tag{4.30}$$

where $W(\omega)$ is the transform of $w(t)$. With this expression, it is straightforward to verify that the frequency-domain reciprocity relation

$$J^A W^A(\omega)\,\mathbf{d}^A\cdot\mathbf{E}(\mathbf{x}_s^A,\omega;\mathbf{x}_s^B) = J^B W^B(\omega)\,\mathbf{d}^B\cdot\mathbf{E}(\mathbf{x}_s^B,\omega;\mathbf{x}_s^A), \tag{4.31}$$

is identically satisfied for two point electric current sources located at the separate positions $\mathbf{x}_s^A$ and $\mathbf{x}_s^B$. This reciprocity relation is just the forward Fourier transform of time-domain relation (4.4a), with slightly different notation for identifying the electric fields. The trivial proof relies on the obvious fact that unit vectors satisfy $\mathbf{e}_r^B = -\mathbf{e}_r^A$ (i.e., they point in opposing directions) and the source-receiver distance r is invariant between the two states A and B.

***Point magnetic induction (or magnetic dipole) sources*:**

A point source of magnetic induction (often referred to as a point "magnetic dipole") is represented by the magnetic induction vector

$$\mathbf{b}_s(\mathbf{x},t) = B\mathbf{d}w_b(t)\delta(\mathbf{x}-\mathbf{x}_s), \tag{4.32}$$

where amplitude scalar B has SI unit T-m^3 = V-m-s. The source waveform $w_b(t)$ is subscripted with "b" in order to distinguish it from a magnetic current waveform (as in the prior section (4.1)). The Fourier-transformed magnetic vector (SI unit: (A/m)/Hz) generated by this source at a point receiver position $\mathbf{x}_r$ is given by Aldridge (2013) as

$$\mathbf{H}(\mathbf{x}_r,\omega;\mathbf{x}_s) = \left(\frac{B\sigma}{4\pi r}\right)\left[\left(1 - i\frac{\omega}{\omega_t}\right)(-i\omega)W_b(\omega)\right]\exp\left[+ik(\omega)r\right] \times$$

$$\left\{\left[(\mathbf{d}\cdot\mathbf{e}_r)\mathbf{e}_r - \mathbf{d}\right] - \left[3(\mathbf{d}\cdot\mathbf{e}_r)\mathbf{e}_r - \mathbf{d}\right]\left(\frac{1}{ik(\omega)r} - \frac{1}{(ik(\omega)r)^2}\right)\right\}, \tag{4.33}$$

where $W_b(\omega)$ is the Fourier transform of $w_b(t)$. Clearly, the structure of equation (4.33) is quite similar to that of formula (4.30) for the electric vector generated by a point electric dipole. Hence, we anticipate that source-receiver reciprocity will hold in this situation as well. The equivalent time-domain reciprocity relation for two point magnetic dipole sources is equation (4.12a), repeated here:

$$B^A \frac{\partial w_b^A(t)}{\partial t} * \mathbf{d}^A \cdot \mathbf{h}(\mathbf{x}_s^A, t; \mathbf{x}_s^B) = B^B \frac{\partial w_b^B(t)}{\partial t} * \mathbf{d}^B \cdot \mathbf{h}(\mathbf{x}_s^B, t; \mathbf{x}_s^A) . \tag{4.12a}$$

This has Fourier transform

$$B^A W_b^A(\omega)\, \mathbf{d}^A \cdot \mathbf{H}(\mathbf{x}_s^A, \omega; \mathbf{x}_s^B) = B^B W_b^B(\omega)\, \mathbf{d}^B \cdot \mathbf{H}(\mathbf{x}_s^B, \omega; \mathbf{x}_s^A) , \tag{4.34}$$

in which the common term ($-i\omega$) cancels from each side. Once again, it is trivial to demonstrate that two the magnetic vectors of the form (4.33) satisfy this reciprocity relation identically.

***Point electric dipole and point magnetic dipole sources*:**

Finally, we examine the more complicated situation of two dissimilar point sources in a homogeneous and isotropic wholespace. The current density vector for a point electric dipole and the magnetic induction vector for a point magnetic dipole are

$$\mathbf{j}_s^A(\mathbf{x},t) = J^A \mathbf{d}^A w^A(t)\delta\left(\mathbf{x} - \mathbf{x}_s^A\right), \qquad \mathbf{b}_s^B(\mathbf{x},t) = B^B \mathbf{d}^B w_b^B(t)\delta\left(\mathbf{x} - \mathbf{x}_s^B\right), \tag{4.35a,b}$$

respectively. However, the equivalent magnetic current source, from equations (3.6a) and (4.10a,b) above, is

$$\mathbf{k}_s^B(\mathbf{x},t) = K^B \mathbf{d}^B w^B(t)\delta\left(\mathbf{x} - \mathbf{x}_s^B\right), \qquad \text{with} \qquad K^B w^B(t) = B^B \frac{\partial w_b^B(t)}{\partial t} . \tag{4.35c,d}$$

The *magnetic* field generated by a point *electric* dipole source is given by Aldridge (2013) as

$$\mathbf{H}(\mathbf{x}_r,\omega;\mathbf{x}_s) = \left(\frac{JW(\omega)}{4\pi r^2}\right)\exp\left[+ik(\omega)r\right]\left(ik(\omega)r - 1\right)(\mathbf{e}_r \times \mathbf{d}) . \tag{4.36}$$

Similarly, the *electric* field generated by a point *magnetic* dipole source is also given by Aldridge (2013) as

$$\mathbf{E}(\mathbf{x}_r,\omega;\mathbf{x}_s) = \left(\frac{B}{4\pi r^2}\right)\left[(-i\omega)W(\omega)\right]\exp\left[+ik(\omega)r\right]\left(ik(\omega)r - 1\right)(\mathbf{d} \times \mathbf{e}_r) . \tag{4.37}$$

Note the opposite ordering of the two vector cross products, which is crucial for obtaining the proper reciprocity result! The reciprocity relation pertinent for these two particular point source types is obtained from the frequency-domain global theorem (4.26b) as

$$J^A W^A(\omega)\mathbf{d}^A \cdot \mathbf{E}(\mathbf{x}_s^A, \omega; \mathbf{x}_s^B) = -B^B(-i\omega)W_b^B(\omega)\mathbf{d}^B \cdot \mathbf{H}(\mathbf{x}_s^B, \omega; \mathbf{x}_s^A)\,. \tag{4.38}$$

Note the additional power of frequency ($-i\omega$) on the right hand side, which is also crucial. Substituting in the electric and magnetic field vectors, and cancelling common terms leads to

$$\mathbf{d}^A \cdot \left(\mathbf{d}^B \times \mathbf{e}_r^B\right) = -\mathbf{d}^B \cdot \left(\mathbf{e}_r^A \times \mathbf{d}^A\right).$$

However, recalling that $\mathbf{e}_r^A = -\mathbf{e}_r^B$ and utilizing the symmetry properties of the triple scalar product (i.e., Gradshteyn and Ryzhik, 1995, page 1114), the right side is seen to be identical to the left side! Point source – point receiver reciprocity holds in this situation of dissimilar source types.

In conclusion, closed-form mathematical formulae for electromagnetic responses generated by point sources situated in a homogeneous and isotropic wholespace explicitly satisfy reciprocity relations derived from the global reciprocity theorem.

4.3.2 Approximate Time-Domain Formulae

As mentioned previously, the electromagnetic responses derived in Aldridge (2013) and stated in the previous section do not adopt any approximating mathematical assumptions like low or high frequencies. Within the context of EM wave propagation within a homogenous and isotropic wholespace, the formulae are mathematically exact. Hence, it is no surprise that these formulae are fully consistent with reciprocity relations that are also rigorously developed from fundamental physical and mathematical principles.

In general, the frequency-domain formulae in Aldridge (2013) are not amenable to inverse Fourier transformation to obtain the analogous time-domain responses. Many authors, most recently perhaps Loseth et al. (2006), have noted this as well. However, if low- or high-frequency approximations are introduced, then the formulae *can* be inverted to the time-domain. It is interesting to note that these *approximate* time-domain electromagnetic response formulae *also* satisfy reciprocity (exactly). We demonstrate this situation in this section.

4.3.2.1 Low Frequency Responses

For frequencies ω well less than the transition frequency $\omega_t = \sigma/\varepsilon$, Aldridge (2013) and others demonstrate that the inverse Fourier transforms of the frequency-domain electromagnetic field expressions can be performed analytically. The approach involves replacing the exact form (4.28a,b) of the complex wavenumber $k(\omega)$ by the low-frequency approximation

$$k(\omega) = \frac{\omega_t}{c_\infty}\sqrt{\frac{|\omega|}{\omega_t}}\left[\frac{\mathrm{sgn}(\omega) + i\left(1 - \delta_0(\omega)\right)}{\sqrt{2}}\right],$$

where $\mathrm{sgn}(\omega)$ is the sign function (= $\omega/|\omega|$ for $\omega \neq 0$, zero otherwise) and $\delta_0(\omega)$ is the null function (= 0 for $\omega \neq 0$, one otherwise) (Bracewell, 1965).

***Point electric conduction current (or electric dipole) sources*:**

In the low-frequency approximation, Aldridge (2013) and others demonstrate that the inverse Fourier transform of the electric vector expression (4.30) can be performed analytically. The result is a convolutional expression for the time-domain electric vector given by

$$\mathbf{e}(\mathbf{x}_r,t;\mathbf{x}_s) = w(t) * \mathbf{g}_e(\mathbf{x}_r,t;\mathbf{x}_s), \tag{4.39}$$

where $w(t)$ is a dimensionless current source waveform, and $\mathbf{g}_e(\mathbf{x}_r,t;\mathbf{x}_s)$ is a causal impulse response function with SI unit (V/m)/s. This impulse response is given by

$$\mathbf{g}_e(\mathbf{x}_r,t;\mathbf{x}_s) = \left(\frac{J\mu}{4\pi^{3/2}rt^2}\right)\left\{\mathbf{d}\big(\kappa(t)r\big) + \big[(\mathbf{d}\cdot\mathbf{e}_r)\mathbf{e}_r - \mathbf{d}\big]\big(\kappa(t)r\big)^3\right\}\exp\left[-\big(\kappa(t)r\big)^2\right]H(t), \tag{4.40a}$$

where function $\kappa(t)$ (with SI unit 1/m) is defined as

$$\kappa(t) \equiv \frac{1}{2c_\infty}\sqrt{\frac{\omega_t}{|t|}} = \sqrt{\frac{\sigma\mu}{4|t|}}. \tag{4.40b}$$

Note that $\kappa(t)$ does not depend on the electric permittivity ε. It is a positive and even function of time that decays to zero as $t \to \pm\infty$. Also note that $\kappa(0) \to +\infty$. This formulation clearly indicates that, in the low-frequency approximation, the electric field is independent of the electric permittivity ε. Note that the impulse response has the symmetry $\mathbf{g}_e(\mathbf{x}_r,t;\mathbf{x}_s) = \mathbf{g}_e(\mathbf{x}_s,t;\mathbf{x}_r)$.

Next, considering two point electric current sources of the form (4.29), it is straightforward to demonstrate that the time-domain reciprocity relation (4.4a) holds exactly:

$$J^A w^A(t) * \mathbf{d}^A \cdot \mathbf{e}(\mathbf{x}_s^A,t;\mathbf{x}_s^B) = J^B w^B(t) * \mathbf{d}^B \cdot \mathbf{e}(\mathbf{x}_s^B,t;\mathbf{x}_s^A). \tag{4.4a}$$

Remarkably, point source – point receiver reciprocity holds even for the approximate (low-frequency) time-domain responses for electric dipole sources!

***Point magnetic induction (or magnetic dipole) sources*:**

Next, consider a point magnetic induction (or magnetic dipole) source of the form (4.32) above. Aldridge (2013) and others demonstrate that, in the low frequency approximation, the time-domain magnetic field vector is given by the convolution

$$\mathbf{h}(\mathbf{x}_r,t;\mathbf{x}_s) = w(t) * \left(\frac{B\sigma}{J\mu}\mathbf{g}_e(\mathbf{x}_r,t;\mathbf{x}_s)\right). \tag{4.41}$$

where the impulse response is defined above in (4.40a,b). Using this form for the magnetic vector, it is straightforward to verify that the time-domain reciprocal relation (4.12a) holds exactly:

$$B^A \frac{\partial w_b^A(t)}{\partial t} * \mathbf{d}^A \cdot \mathbf{h}(\mathbf{x}_s^A,t;\mathbf{x}_s^B) = B^B \frac{\partial w_b^B(t)}{\partial t} * \mathbf{d}^B \cdot \mathbf{h}(\mathbf{x}_s^B,t;\mathbf{x}_s^A). \tag{4.12a}$$

Reciprocity holds with the low-frequency approximations for the magnetic dipole source responses.

***Point electric dipole and point magnetic dipole sources*:**

Finally, we examine the remaining (and more complicated) case of mixed point source types. In the low-frequency approximation, the time-domain *magnetic* vector generated by a point *electric* dipole source is

$$\mathbf{h}(\mathbf{x}_r,t;\mathbf{x}_s) = w(t) * \mathbf{g}_h(\mathbf{x}_r,t;\mathbf{x}_s)\,, \tag{4.42}$$

where the impulse response (with SI unit (A/m)/s) is defined by

$$\mathbf{g}_h(\mathbf{x}_r,t;\mathbf{x}_s) = \left(\frac{J}{2\pi^{3/2} r^2 |t|}\right)(\kappa(t)r)^3 \exp\left[-(\kappa(t)r)^2\right] H(t)\ (\mathbf{d}\times\mathbf{e}_r)\,, \tag{4.43}$$

(Aldridge, 2013). Note that this has symmetry $\mathbf{g}_h(\mathbf{x}_r,t;\mathbf{x}_s) = -\mathbf{g}_h(\mathbf{x}_s,t;\mathbf{x}_r)$. Similarly, the time-domain *electric* vector generated by a point *magnetic* dipole source is

$$\mathbf{e}(\mathbf{x}_r,t;\mathbf{x}_s) = \frac{\partial w_b(t)}{\partial t} * \left(-\frac{B}{J}\mathbf{g}_h(\mathbf{x}_r,t;\mathbf{x}_s)\right), \tag{4.44}$$

(Aldridge, 2013). The time-domain reciprocity relation applying to this situation is equation (4.18a), repeated here with a magnetic dipole waveform $w_b(t)$ rather than a magnetic current waveform $w(t)$:

$$J^A w^A(t) * \mathbf{d}^A \cdot \mathbf{e}(\mathbf{x}_s^A,t;\mathbf{x}_s^B) = -B^B \frac{\partial w_b^B(t)}{\partial t} * \mathbf{d}^B \cdot \mathbf{h}(\mathbf{x}_s^B,t;\mathbf{x}_s^A)\,. \tag{4.18a}$$

Recall the negative sign. Substituting in the forms (4.42) and (4.44) for the magnetic and electric vectors yields

$$\mathbf{d}^A \cdot \left(\mathbf{d}^B \times \mathbf{e}_r^B\right) = \mathbf{d}^B \cdot \left(\mathbf{d}^A \times \mathbf{e}_r^A\right),$$

which is an identity, via the triple scalar product rule. Hence, point source – point receiver reciprocity holds for the approximate low-frequency electromagnetic responses for the electric dipole/magnetic dipole source pairing.

4.3.2.2 High Frequency Responses

For frequency ω well above the transition frequency ω_t, the complex wavenumber may be approximated as

$$k(\omega) \approx \frac{\omega}{c_\infty} + i\alpha_\infty\,,$$

where $c_\infty = 1/\sqrt{\varepsilon\mu}$ is the infinite-frequency phase speed, and $\alpha_\infty \equiv \omega_t/2c_\infty = (\sigma/2)\sqrt{\mu/\varepsilon}$ is the infinite-frequency attenuation factor. Note that this attenuation factor vanishes for σ = 0 S/m. Time-domain expressions for the electric and magnetic vectors in this approximation are given in Aldridge (2013) as

Electric vector generated by a point electric dipole:

$$\mathbf{e}(\mathbf{x}_r,t;\mathbf{x}_s)=\left(\frac{J\mu}{4\pi r}\right)\exp\left(-\alpha_\infty r\right)\left\{\left[(\mathbf{d}\cdot\mathbf{e}_r)\mathbf{e}_r-\mathbf{d}\right]w'\left(t-\frac{r}{c_\infty}\right)\right.$$
$$+\left[3(\mathbf{d}\cdot\mathbf{e}_r)\mathbf{e}_r-\mathbf{d}\right]\left[\frac{c_\infty}{r}\left(\exp\left[-\omega_t t/2\right]H(t)\right)*w'\left(t-\frac{r}{c_\infty}\right)\right.$$
$$\left.\left.+\frac{c_\infty^2}{r^2}\left(\exp\left[-\omega_t t/2\right]R(t)\right)*w'\left(t-\frac{r}{c_\infty}\right)\right]\right\}. \qquad (4.45a)$$

Magnetic vector generated by a point electric dipole:

$$\mathbf{h}(\mathbf{x}_r,t;\mathbf{x}_s)=\left(\frac{J}{4\pi c_\infty r}\right)\exp\left(-\alpha_\infty r\right)\left\{w'\left(t-\frac{r}{c_\infty}\right)+\left(\alpha_\infty c_\infty+\frac{c_\infty}{r}\right)w\left(t-\frac{r}{c_\infty}\right)\right\}(\mathbf{d}\times\mathbf{e}_r). \qquad (4.45b)$$

Electric vector generated by a point magnetic dipole:

$$\mathbf{e}(\mathbf{x}_r,t;\mathbf{x}_s)=\left(\frac{B}{4\pi c_\infty r}\right)\exp\left(-\alpha_\infty r\right)\left\{w_b''\left(t-\frac{r}{c_\infty}\right)+\left(\alpha_\infty c_\infty+\frac{c_\infty}{r}\right)w_b'\left(t-\frac{r}{c_\infty}\right)\right\}(\mathbf{e}_r\times\mathbf{d}). \qquad (4.46a)$$

Magnetic vector generated by a point magnetic dipole:

$$\mathbf{h}(\mathbf{x}_r,t;\mathbf{x}_s)=\left(\frac{B\varepsilon}{4\pi r}\right)\exp\left(-\alpha_\infty r\right)\left\{\left[(\mathbf{d}\cdot\mathbf{e}_r)\mathbf{e}_r-\mathbf{d}\right]w_b''\left(t-\frac{r}{c_\infty}\right)\right.$$
$$+\left[3(\mathbf{d}\cdot\mathbf{e}_r)\mathbf{e}_r-\mathbf{d}\right]\left[\frac{c_\infty}{r}w'\left(t-\frac{r}{c_\infty}\right)+\frac{c_\infty^2}{r^2}w\left(t-\frac{r}{c_\infty}\right)\right.$$
$$-\frac{c_\infty\omega_t}{2r}\left(\exp\left(-\omega_t t/2\right)H(t)\right)*\left(w'\left(t-\frac{r}{c_\infty}\right)+\frac{2c_\infty}{r}w\left(t-\frac{r}{c_\infty}\right)\right)$$
$$\left.\left.+\frac{c_\infty^2\omega_t^2}{4r^2}\left(\exp\left(-\omega_t t/2\right)R(t)\right)*w\left(t-\frac{r}{c_\infty}\right)\right]\right\}. \qquad (4.46b)$$

$H(t)$ is the Heaviside unit step function, and $R(t) = tH(t)$ is the ramp function of unit slope. Each response is proportional to a range-dependent exponential decay factor $\exp(-\alpha_\infty r)$, which becomes unity for a vacuum with vanishing conductivity σ. For a causal source waveform (i.e., $w(t) = 0$ for $t < 0$) all electromagnetic responses are causal (at the receiver location $\mathbf{x}_r$) with respect to the arrival time r/c_∞.

Recalling that the transition frequency is $\omega_t = \sigma/\varepsilon$, it is straightforward to obtain the vacuum (or free space) forms of (4.45) and (4.46) as $\sigma \to 0$. Since the current conductivity σ does not appear in the derivations in Welch (1960), it appears that his time-domain reciprocity theorems pertain only to these free space electromagnetic response formulae.

Without engaging in the derivational details, we merely state that the three time-domain point source – point receiver reciprocal relations (4.4a), (4.12a), and (4.18a) are exactly satisfied with these high-frequency approximate responses. This can be easily verified.

Hence, the remarkable conclusion of this section is that various time-domain electromagnetic responses, obtained either as low-frequency or high-frequency *approximations* to the *exact* frequency-domain response formulae, also satisfy the point source – point receiver reciprocity relations.

5.0 EXAMPLES

There are two important practical applications of reciprocity theory in the area of numerical simulation of geophysical fields:

1) In situations where there are many more source locations than receiver locations (e.g., an ocean bottom seismic recording survey with towed air-gun sources) reciprocity may be used to dramatically reduce the computational modeling burden by swapping source and receiver positions. Symons and Aldridge (2000) employed this strategy in the numerical modeling a of dual-borehole seismic survey.

2) Adherence to reciprocity constitutes a validation test for a numerical simulation algorithm. Algorithm correctness is often assessed by comparing numerically-calculated responses with so-called "analytic" responses. However, such responses are available only for sufficiently simple earth models (e.g., a homogeneous medium, or a one-dimensional layered medium) or recording geometries. Reciprocity affords the opportunity to test an algorithm under much more general conditions (i.e., three-dimensional heterogeneity, anisotropy, interfaces, etc.). In particular, reciprocity is useful for verifying the proper implementation of sources and receivers in a numerical algorithm. Aldridge and Symons (2001) develop reciprocity rules for various source and receiver types used in seismic modeling algorithms. Non-reciprocity indicates a code implementation error, although sometimes the mathematical theory of reciprocity is (mistakenly!) challenged (e.g., Muerdter and Kelly, 2001; Muerdter, D., Kelly, M., and Ratcliff, D., 2001).

In this section, we demonstrate adherence to reciprocity with two electromagnetic forward modeling algorithms. Algorithm EMHOLE (Aldridge, 2013) solves the frequency-domain expressions (4.30) and (4.33) for the electric and magnetic vectors generated by a point source situated in a homogeneous and isotropic wholespace. It has the advantage of extreme rapidity of calculation. Algorithm FDEM (Aldridge, 2014, *in progress*) is an explicit, time-domain, finite-difference solution of the coupled partial differential equations governing electromagnetic wave propagation in a 3D heterogeneous and isotropic medium. Proper spatial and temporal gridding is required for accurate response calculations, and algorithm execution time is significantly larger. Algorithmic issues are not discussed here.

Point source/point receiver reciprocity is illustrated in the following sequence of electromagnetic trace plots, where various details of the calculations are given in the figure captions.

5.1 Algorithm EMHOLE

All numerical simulations with algorithm EMHOLE utilize the homogeneous wholespace medium parameters: relative electric permittivity κ_ε = 10.0, relative magnetic permeability κ_μ = 1.0, and relative current conductivity κ_σ = 0.5 (relative to 1.0 S/m).

Figure 5.1 illustrates the first reciprocity test with algorithm EMHOLE. The situation involves point current density sources recorded by point electric field receivers, for which the appropriate reciprocity relation is equation (4.4a), repeated here:

$$J^A w^A(t) * \mathbf{d}^A \cdot \mathbf{e}^B(\mathbf{x}_s^A, t) = J^B w^B(t) * \mathbf{d}^B \cdot \mathbf{e}^A(\mathbf{x}_s^B, t) \,. \tag{4.4a again}$$

However, source waveforms of states A and B in this example are taken as identical (i.e., $w^A(t) = w^B(t) = w(t)$) and hence may be deconvolved from the above expression, yielding the simpler form

$$J^A \mathbf{d}^A \cdot \mathbf{e}^B(\mathbf{x}_s^A, t) = J^B \mathbf{d}^B \cdot \mathbf{e}^A(\mathbf{x}_s^B, t) \,.$$

The common source waveform is an alternating polarity square pulse sequence, consisting of 0.5 s on (positive) time, 0.5 s off (zero) time, 0.5 s on (negative) time, and 0.5 s off (zero) time. Trace plots are 2 s long, corresponding to one period of source performance. Each trace has physical dimension corresponding to a scaled electric vector $J\mathbf{e}$, or SI unit (A-m) × (V/m) = (A-V) = J/s = W (energy per time, or power).

Clearly, the blue (state B) trace overplots the red (state A) trace in figure 5.1, indicating that point source/point receiver reciprocity is satisfied. The electric vector responses exhibit an initial exponential growth toward a saturation value (near ~500 ms) followed by an exponential decay toward zero (after ~ 1000 ms). Interestingly, the magnitude of the negative pulse (near ~1500 ms) *exceeds* the magnitude of the positive pulse (near ~500 ms). This is because the decaying response overshoots the zero level (near ~1000 ms). After a few source periods (not displayed here) the responses assume a symmetrical form.

Figure 5.2 depicts point source/point receiver reciprocity for the case of magnetic induction sources and magnetic field receivers, for which the appropriate reciprocity relation is equation (4.12a), repeated here:

$$B^A \frac{\partial w_b^A(t)}{\partial t} * \mathbf{d}^A \cdot \mathbf{h}^B(\mathbf{x}_s^A, t) = B^B \frac{\partial w_b^B(t)}{\partial t} * \mathbf{d}^B \cdot \mathbf{h}^A(\mathbf{x}_s^B, t). \qquad \text{(4.12a again)}$$

With identical source waveforms in states A and B, this reduces to

$$B^A \mathbf{d}^A \cdot \mathbf{h}^B(\mathbf{x}_s^A, t) = B^B \mathbf{d}^B \cdot \mathbf{h}^A(\mathbf{x}_s^B, t).$$

The common source waveform is a 1st-derivative Gaussian pulse with characteristic frequency 2.0 s, and delayed (from time t = 0) by 0.5 s. Reciprocity is clearly satisfied for this source-receiver configuration. Each trace has physical dimension corresponding to scaled magnetic vector $B\mathbf{h}$, or SI unit (V-m-s) × (A/m) = V-A-s = J (energy).

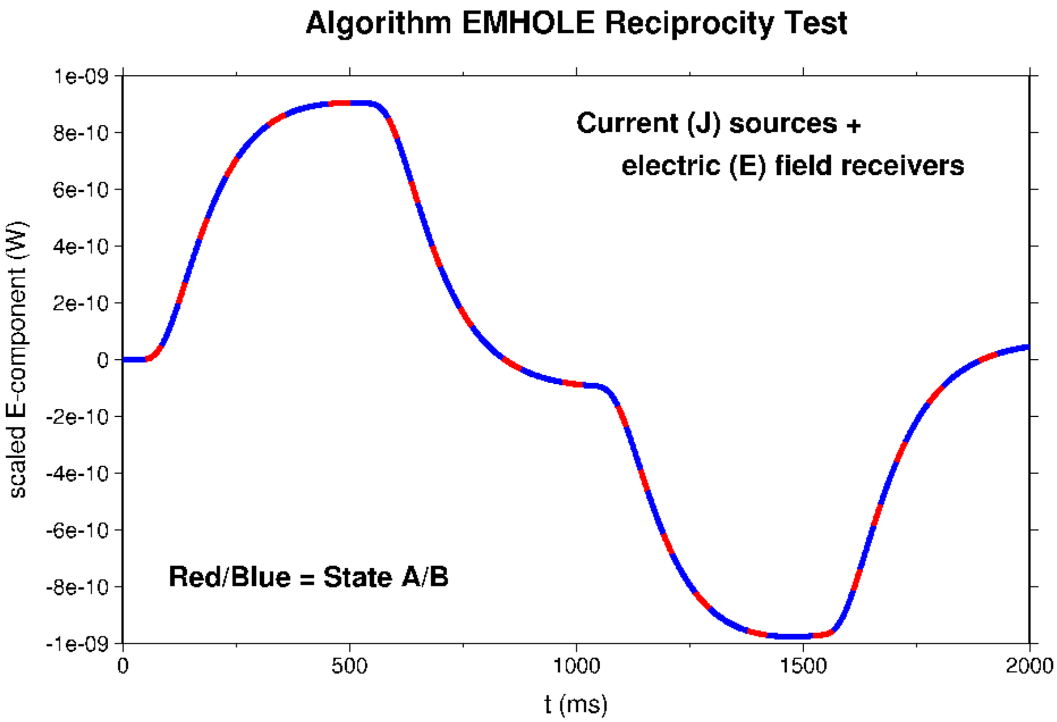

Figure 5.1. Validation of point source/point receiver reciprocity for Green function electromagnetic modeling algorithm EMHOLE. **Red** and **blue** traces correspond to states A and B, respectively. **Blue** trace overplots **red** trace, confirming that reciprocity holds for this case of *current density sources* and *electric field receivers* in both states. This test validates reciprocity relation (4.4a) for the case where source waveforms of states A and B are identical. The situation is depicted in figure 4.1.

State A specifications:
Point current density source located at (x_s, y_s, z_s) = (–200, 742, 1500) m; source amplitude J = 6.7 A-m; orientation angles $(\theta, \varphi) = (60^o, 30^o)$.
Point electric vector sensor located at (x_r, y_r, z_r) = (572, 347, –124) m; receiver amplification = 4.2; orientation angles $(\theta, \varphi) = (120^o, 215^o)$.

State B specifications:
Point current density source located at (x_s, y_s, z_s) = (572, 347, –124) m; source amplitude J = 4.2 A-m; orientation angles $(\theta, \varphi) = (120^o, 215^o)$.
Point electric vector sensor located at (x_r, y_r, z_r) = (–200, 742, 1500) m; receiver amplification = 6.7; orientation angles $(\theta, \varphi) = (60^o, 30^o)$.

Earth model parameters:
Relative electric permittivity κ_ε = 10.0; relative magnetic permeability κ_μ = 1.0; relative current conductivity κ_σ = 0.5 (relative to 1.0 S/m).

Source waveform:
Alternating polarity sequence of square pulses; duration = 0.5 s; pulse lag = 1.0 s; unit magnitude.

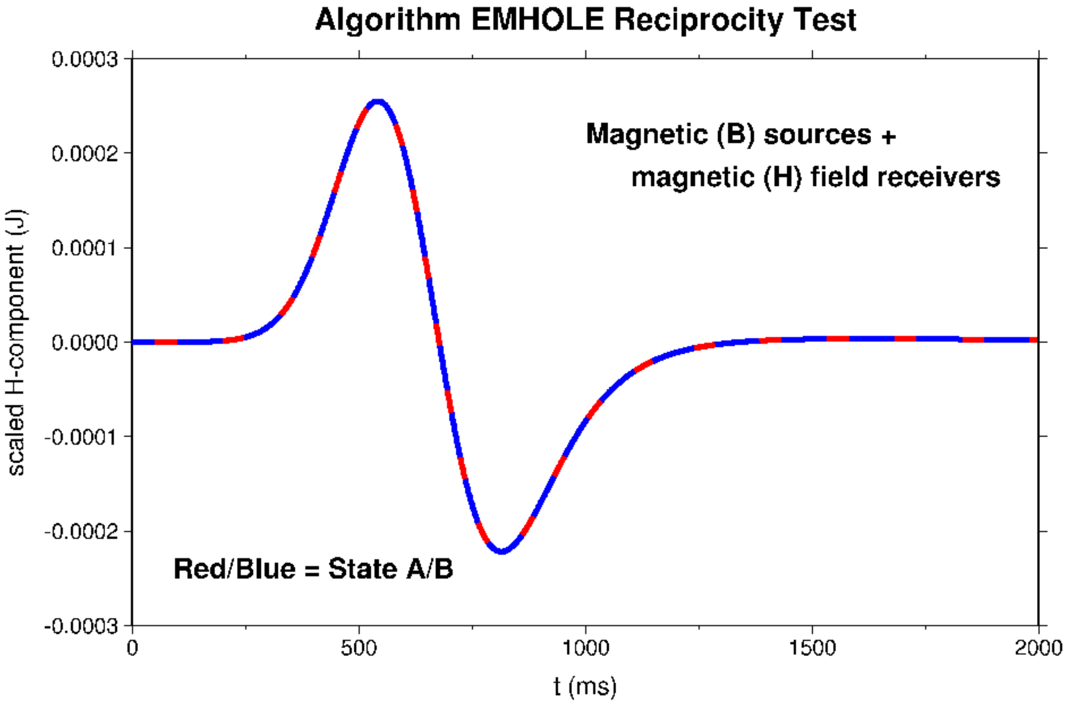

Figure 5.2. Validation of point source/point receiver reciprocity for Green function electromagnetic modeling algorithm EMHOLE. **Red** and **blue** traces correspond to states A and B, respectively. **Blue** trace overplots **red** trace, confirming that reciprocity holds for this case of *magnetic induction sources* and *magnetic field receivers* in both states. This test validates reciprocity relation (4.12a) for the case where source waveforms of states A and B are identical.

***State A specifications*:**
Point magnetic induction source located at $(x_s, y_s, z_s) = (-200, 742, 1500)$ m; source amplitude $B = 6.7$ V-m-s; orientation angles $(\theta, \varphi) = (60^o, 30^o)$.
Point magnetic vector sensor located at $(x_r, y_r, z_r) = (572, 347, -124)$ m; receiver amplification = 4.2; orientation angles $(\theta, \varphi) = (120^o, 215^o)$.

***State B specifications*:**
Point magnetic induction source located at $(x_s, y_s, z_s) = (572, 347, -124)$ m; source amplitude $B = 4.2$ V-m-s; orientation angles $(\theta, \varphi) = (120^o, 215^o)$.
Point magnetic vector sensor located at $(x_r, y_r, z_r) = (-200, 742, 1500)$ m; receiver amplification = 6.7; orientation angles $(\theta, \varphi) = (60^o, 30^o)$.

***Earth model parameters*:**
Relative electric permittivity $\kappa_\varepsilon = 10.0$; relative magnetic permeability $\kappa_\mu = 1.0$; relative current conductivity $\kappa_\sigma = 0.5$ (relative to 1.0 S/m).

***Source waveform*:**
First derivative Gaussian pulse with characteristic frequency 2.0 Hz; time delay = 0.5 s; unit magnitude.

A more complicated reciprocity situation involving *dissimilar* point source types is depicted in figure 5.3. The appropriate reciprocity relation is equation (4.18a), repeated in modified form here as:

$$J^A w^A(t) * \mathbf{d}^A \cdot \mathbf{e}^B(\mathbf{x}_s^A, t) = -B^B \frac{\partial w_b^B(t)}{\partial t} * \mathbf{d}^B \cdot \mathbf{h}^A(\mathbf{x}_s^B, t) . \qquad \text{(modified from 4.18a)}$$

Wavelet $w_b^B(t)$ is the source *magnetic induction* waveform for state B; its time derivative is the source *magnetic current* waveform. Figure 4.3 is a graphic depiction of this reciprocal situation. Note that the receiver in state B is oriented in the opposite direction to the source in state A.

State A (blue trace in figure 5.3) is activated by a point current density source with the alternating polarity square pulse sequence (same as in figure 5.1). The receiver is a point magnetic field $\mathbf{h}(\mathbf{x},t)$ receiver. State B (red trace in figure 5.3) is activated by a point magnetic induction source with a trapezoidal waveform (i.e., linear ramp on to unity at 500 ms, constant from 500 ms to 1000 ms, and linear ramp off to zero at 1500 ms) whose time-derivative is proportional to the square pulse sequence of state A. [Proportionality constant = $2/T_{\text{base}}$ where T_{base} has dimension time.] Hence, this common waveform may be deconvolved from each side of the above expression, yielding

$$J^A \mathbf{d}^A \cdot \mathbf{e}^B(\mathbf{x}_s^A, t) = -B^B \left(\frac{2}{T_{\text{base}}} \right) \mathbf{d}^B \cdot \mathbf{h}^A(\mathbf{x}_s^B, t) .$$

[The trapezoidal waveform is achieved in algorithm EMHOLE by summing two triangular pulses of base width $T_{\text{base}} = 1.0$ s, and lagged by 0.5 s.] The receiver in state B is a point electric field $\mathbf{e}(\mathbf{x},t)$ receiver. As shown in figure 5.3, the two responses (which have SI unit W) overplot, indicating reciprocity is exactly satisfied. Note that source amplitudes and receiver sensitivities are handled with care!

Suppose the source waveforms used to activate electromagnetic wavefield states A and B differ. Figure 5.4 indicates that the electric vector responses generated by two point current density sources are *not* equal. The appropriate reciprocity relation for differing current source waveforms is not

$$J^A \mathbf{d}^A \cdot \mathbf{e}^B(\mathbf{x}_s^A, t) = J^B \mathbf{d}^B \cdot \mathbf{e}^A(\mathbf{x}_s^B, t) ,$$

(which is what is plotted in figure 5.4), but rather

$$J^A w^A(t) * \mathbf{d}^A \cdot \mathbf{e}^B(\mathbf{x}_s^A, t) = J^B w^B(t) * \mathbf{d}^B \cdot \mathbf{e}^A(\mathbf{x}_s^B, t) . \qquad \text{(4.4a again)}$$

Convolving the state B electric response with $w^A(t)$ and the state A electric response with $w^B(t)$ yields the trace plots of figure 5.5. Reciprocity is clearly satisfied with these convolved responses (which have the common SI unit J). This constitutes a strong reciprocity validation test for algorithm EMHOLE.

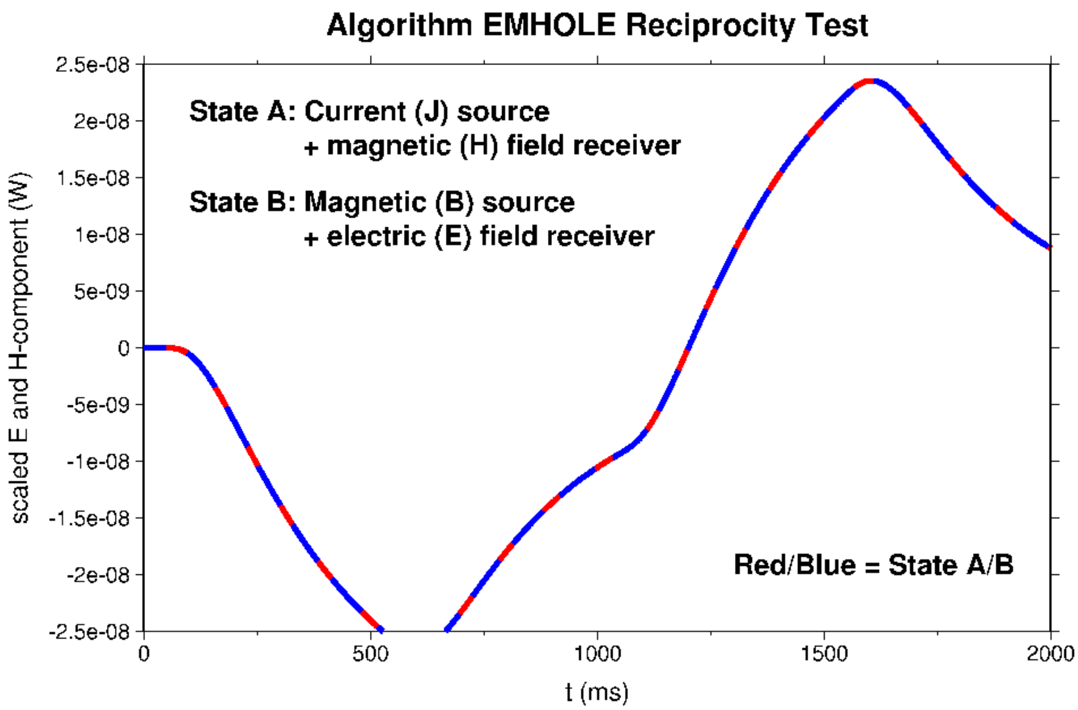

Figure 5.3. Validation of point source/point receiver reciprocity for Green function electromagnetic modeling algorithm EMHOLE. **Red** and **blue** traces correspond to states A and B, respectively. **Blue** trace overplots **red** trace, confirming that reciprocity holds for this case of *dissimilar source types* in the two states. This test validates reciprocity relation (4.18a), and is graphically depicted in figure 4.3.

***State A specifications*:**
Point current density source located at $(x_s, y_s, z_s) = (-200, 742, 1500)$ m; source amplitude $J = 6.7$ A-m; orientation angles $(\theta, \varphi) = (60^o, 30^o)$.
Point magnetic vector sensor located at $(x_r, y_r, z_r) = (572, 347, -124)$ m; receiver amplification = 4.2; orientation angles $(\theta, \varphi) = (120^o, 215^o)$.

***State B specifications*:**
Point magnetic induction source located at $(x_s, y_s, z_s) = (572, 347, -124)$ m; source amplitude $B = 2.1$ V-m-s; orientation angles $(\theta, \varphi) = (120^o, 215^o)$.
Point electric vector sensor located at $(x_r, y_r, z_r) = (-200, 742, 1500)$ m; receiver amplification = 6.7; orientation angles $(\theta, \varphi) = (120^o, 210^o)$.

***Earth model parameters*:**
Relative electric permittivity $\kappa_\varepsilon = 10.0$; relative magnetic permeability $\kappa_\mu = 1.0$; relative current conductivity $\kappa_\sigma = 0.5$ (relative to 1.0 S/m).

***Source waveforms*:**
State A: Alternating polarity sequence of square pulses; duration = 0.5 s; pulse lag = 1.0 s; unit magnitude.
State B: Sum of two triangular pulses; duration = 1.0 s; pulse lag = 0.5 s, yielding a trapezoidal pulse of duration 1.5 s and unit magnitude.

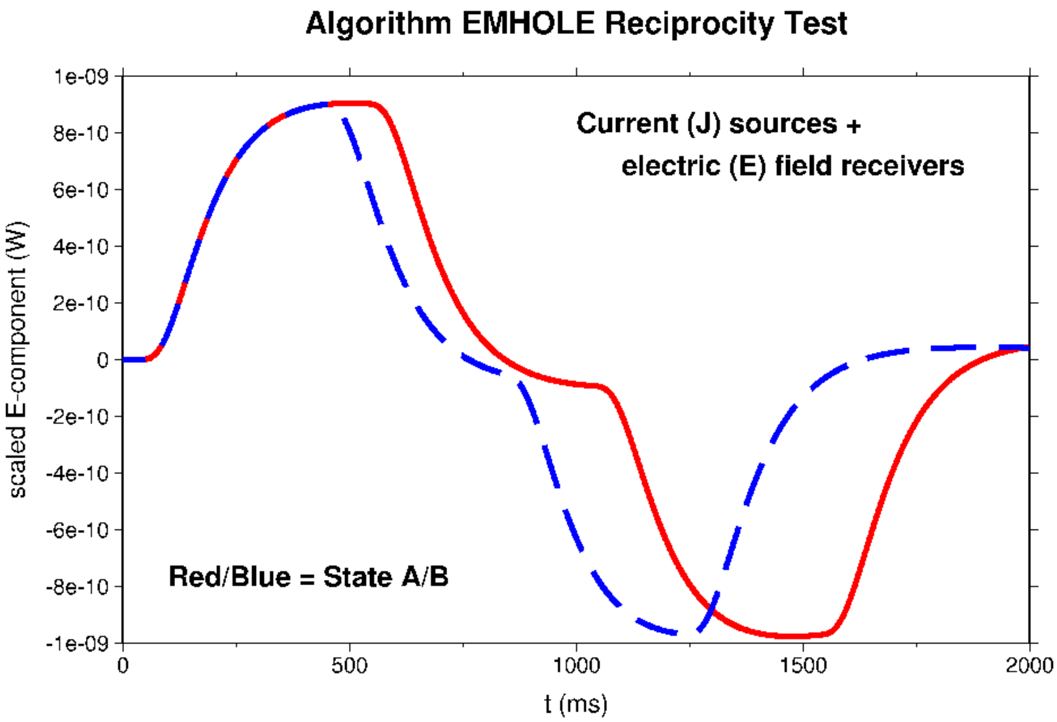

Figure 5.4. Test of point source/point receiver reciprocity for Green function electromagnetic modeling algorithm EMHOLE. **Red** and **blue** traces correspond to states A and B, respectively. **Blue** trace *does not* overplot **red** trace, indicating that reciprocity *does not* hold for this case of current density sources and electric field receivers, when the two source waveforms *differ*.

***State A specifications*:**
Point current density source located at $(x_s, y_s, z_s) = (-200, 742, 1500)$ m; source amplitude $J = 6.7$ A-m; orientation angles $(\theta, \varphi) = (60^o, 30^o)$.
Point electric vector sensor located at $(x_r, y_r, z_r) = (572, 347, -124)$ m; receiver amplification = 4.2; orientation angles $(\theta, \varphi) = (120^o, 215^o)$.

***State B specifications*:**
Point current density source located at $(x_s, y_s, z_s) = (572, 347, -124)$ m; source amplitude $J = 4.2$ A-m; orientation angles $(\theta, \varphi) = (120^o, 215^o)$.
Point electric vector sensor located at $(x_r, y_r, z_r) = (-200, 742, 1500)$ m; receiver amplification = 6.7; orientation angles $(\theta, \varphi) = (60^o, 30^o)$.

***Earth model parameters*:**
Relative electric permittivity $\kappa_\varepsilon = 10.0$; relative magnetic permeability $\kappa_\mu = 1.0$; relative current conductivity $\kappa_\sigma = 0.5$ (relative to 1.0 S/m).

***Source waveform*:**
State A: Sequence of two opposite polarity square pulses; duration = 0.5 s; pulse lag = 1.0 s; unit magnitude.
State B: Sequence of two opposite polarity square pulses; duration = 0.4 s; pulse lag = 0.8 s; unit magnitude.

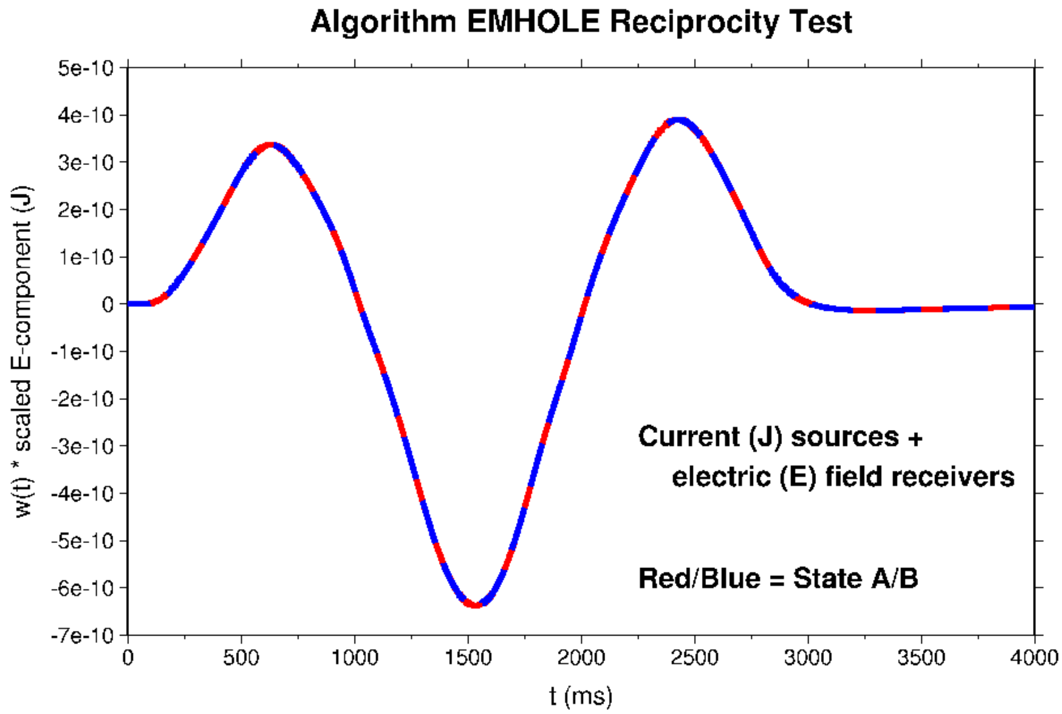

Figure 5.5. Validation of point source/point receiver reciprocity for Green function electromagnetic modeling algorithm EMHOLE. **Red** and **blue** traces correspond to states A and B, respectively. **Blue** trace overplots **red** trace, confirming that reciprocity holds for this case of current density sources and electric field receivers in both states. This test validates reciprocity relation (4.4a) for the case where source waveforms of states A and B *differ*. Note doubled time scale compared to figure 5.4.

***State A specifications*:**
Point current density source located at $(x_s, y_s, z_s) = (-200, 742, 1500)$ m; source amplitude $J = 6.7$ A-m; orientation angles $(\theta, \varphi) = (60^o, 30^o)$.
Point electric vector sensor located at $(x_r, y_r, z_r) = (572, 347, -124)$ m; receiver amplification = 4.2; orientation angles $(\theta, \varphi) = (120^o, 215^o)$.

***State B specifications*:**
Point current density source located at $(x_s, y_s, z_s) = (572, 347, -124)$ m; source amplitude $J = 4.2$ A-m; orientation angles $(\theta, \varphi) = (120^o, 215^o)$.
Point electric vector sensor located at $(x_r, y_r, z_r) = (-200, 742, 1500)$ m; receiver amplification = 6.7; orientation angles $(\theta, \varphi) = (60^o, 30^o)$.

***Earth model parameters*:**
Relative electric permittivity $\kappa_\varepsilon = 10.0$; relative magnetic permeability $\kappa_\mu = 1.0$; relative current conductivity $\kappa_\sigma = 0.5$ (relative to 1.0 S/m).

***Source waveform*:**
State A: Sequence of two opposite polarity square pulses; duration = 0.5 s; pulse lag = 1.0 s; unit magnitude.
State B: Sequence of two opposite polarity square pulses; duration = 0.4 s; pulse lag = 0.8 s; unit magnitude.

5.2 Algorithm FDEM

It is not surprising that electromagnetic responses calculated by algorithm EMHOLE satisfy reciprocity, as illustrated in the previous section 5.1. Algorithm EMHOLE is a direct numerical implementation of frequency-domain expressions (4.30) and (4.33) for the electric and magnetic vectors due to a point body source in a wholespace. These *mathematical* expressions have already been shown to theoretically satisfy reciprocity in section (4.3.1).

It is somewhat more challenging to demonstrate that a "purely numerical" electromagnetic simulation algorithm obeys reciprocity. Algorithm FDEM (Aldridge, 2014, *in progress*) uses an explicit, time-domain, finite-difference (FD) numerical technique to solve the coupled, first-order, linear, inhomogeneous partial differential system

$$\varepsilon(\mathbf{x})\frac{\partial\,\mathbf{e}(\mathbf{x},t)}{\partial t}+\sigma(\mathbf{x})\mathbf{e}(\mathbf{x},t)-\mathbf{curl}\,\mathbf{h}(\mathbf{x},t)=-\mathbf{j}_s(\mathbf{x},t)-\frac{\partial\mathbf{d}_s(\mathbf{x},t)}{\partial t}\,, \tag{5.1a}$$

$$\mu(\mathbf{x})\frac{\partial\,\mathbf{h}(\mathbf{x},t)}{\partial t}+\mathbf{curl}\,\mathbf{e}(\mathbf{x},t)=-\frac{\partial\,\mathbf{b}_s(\mathbf{x},t)}{\partial t}\,. \tag{5.1b}$$

We refer to equations (5.1a and b) as the "EH system", after the two unknown dependent variables $\mathbf{e}(\mathbf{x},t)$ and $\mathbf{h}(\mathbf{x},t)$ contained therein. Inhomogeneous terms on the right-hand-sides are electromagnetic body sources. The EH system is easily derived by combining isotropic constitutive relations with the Faraday law (2.1a) and the Ampere-Maxwell law (2.1b). The medium is characterized by the three parameters electric permittivity $\varepsilon(\mathbf{x})$, magnetic permeability $\mu(\mathbf{x})$, and current conductivity $\sigma(\mathbf{x})$, which appear as spatially-variable coefficients in the EH system.

FDEM solution methodology consists of approximating all temporal and spatial derivatives in the EH system (5.1) with centered finite-differences, and then explicitly solving for the discretized EM wavefield on a 3D grid at future times in terms of present and past values. Accurate responses are obtained provided spatial and temporal gridding intervals are sufficiently fine. A detailed discussion of algorithm FDEM is found in Aldridge (2014, *in progress*).

The reciprocity tests displayed in the following figures use a one-dimensional (1D) layered conductivity model consisting of the five homogeneous layers:

layer #	conductivity (S/m)	thickness (m)
1	1.00	600
2	0.10	1000
3	0.50	800
4	0.01	200
5	2.00	400

Light convolutional smoothing is applied across interfaces between layers. For all layers, the magnetic permeability is $\mu(\mathbf{x}) = \mu_0$ and the electric permittivity is $\varepsilon(\mathbf{x}) = 300{,}000\ \varepsilon_0$. [The reason for the artificially-large permittivity is to maintain numerical stability in algorithm FDEM. Calculated responses are not adversely impacted, provided the source waveform has sufficiently low frequency content.] This 1D earth model is then discretized on a 301 × 301 × 301 3D rectangular grid, with grid interval Δh = 10 m in all three coordinate directions. Hence, there are slightly more than 27 million gridpoints in the earth model.

Figure 5.6 displays electric vector component traces for state A (**red**) and state B (**blue**), both generated by point current density sources. Trace duration is 100 ms, sampled at $\Delta t = 1 \times 10^{-5}$ s (or 10 μs) for a total of 10,001 samples. The current source waveform is an alternating polarity sequence of square pulses, characterized by 25 ms on (positive), 25 ms off (zero), 25 ms on (negative), and 25 ms off (zero). Red and blue traces overplot, confirming that algorithm FDEM satisfies reciprocity for this case of point current density sources and point electric field receivers. Interestingly, the calculated responses exhibit a noticeable time lag relative to the source waveform (which turns on at t = 0 ms), perhaps due to the particular conductivity structure used. The responses have not decayed to zero at the simulation end time t = 100 ms.

The *orders* of the temporal and spatial FD operators used for the calculations in figure 5.6 are $M = 2$ and $N = 2$, respectively. Algorithm FDEM is restricted to a 2nd-order FD operator in time (i.e., $M = 2$), but the spatial FD operator order may range from $N = 2$ to $N = 10$. For comparison, figure 5.7 depicts the same electromagnetic traces calculated with a higher spatial FD operator order $N = 4$. A smaller FD timestep of $\Delta t = 8 \times 10^{-6}$ s (or 8 μs) is required to maintain numerical stability, leading to 12,651 samples per trace. Once again, reciprocity is clearly satisfied. There appears to be no visible difference in the O(2,2) and O(2,4) responses calculated by algorithm FDEM.

Finally, figure 5.8 illustrates electromagnetic reciprocity for the case of dissimilar source types, each activated by a different source waveform. The appropriate reciprocity relation is

$$J^A w^A(t) * \mathbf{d}^A \cdot \mathbf{e}^B(\mathbf{x}_s^A, t) = -B^B \frac{\partial w_b^B(t)}{\partial t} * \mathbf{d}^B \cdot \mathbf{h}^A(\mathbf{x}_s^B, t) . \qquad \text{(modified from 4.18a)}$$

State A (**red** trace) is the magnetic field response generated by a point current density source, and convolved with the (time-differentiated) state B source waveform $\partial w_b^B(t) / \partial t$. State B (**blue** trace) is the electric field response generated by a point magnetic induction source, and convolved with the state A source waveform $w^A(t)$. Traces have the common SI unit J (energy). Source waveforms are alternating polarity square pulse sequences, but with different pulse widths. So, the derivative of the state B waveform is a set of four impulses (i.e., Dirac delta functions), with magnitudes $1/\Delta t = 10^5$ s^{-1}. Within the simulation window 0 ms to 100 ms, the two convolved responses agree, indicating reciprocity is satisfied.

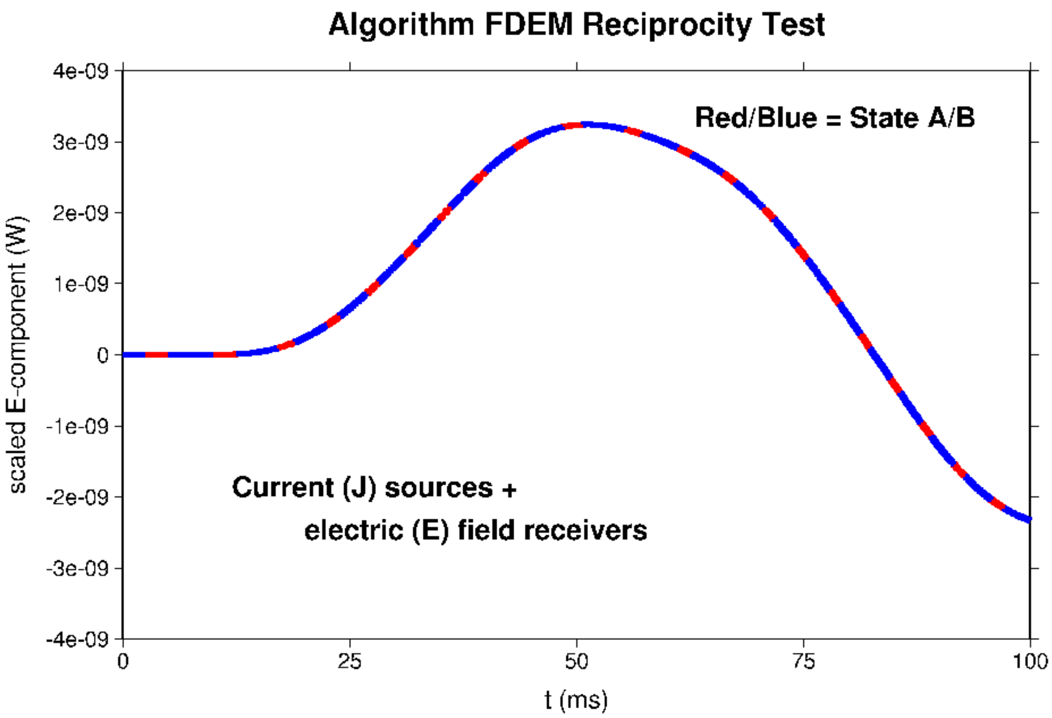

Figure 5.6. Validation of point source/point receiver reciprocity for finite-difference electromagnetic modeling algorithm FDEM. **Red** and **blue** traces correspond to states A and B, respectively. **Blue** trace overplots **red** trace, confirming that reciprocity holds for this case of *current density sources* and *electric field receivers* in both states. This test validates reciprocity relation (4.4a) for the case where source waveforms of states A and B are identical. The situation is depicted in figure 4.1. Temporal/spatial FD operator orders are O(M,N) = O(2,2).

***State A specifications*:**
Point current density source located at $(x_s, y_s, z_s) = (-200, 742, 640)$ m; source amplitude $J = 6.7$ A-m; orientation angles $(\theta, \varphi) = (60^o, 30^o)$.
Point electric vector sensor located at $(x_r, y_r, z_r) = (572, 347, -124)$ m; receiver amplification = 4.2; orientation angles $(\theta, \varphi) = (120^o, 215^o)$.

***State B specifications*:**
Point current density source located at $(x_s, y_s, z_s) = (572, 347, -124)$ m; source amplitude $J = 4.2$ A-m; orientation angles $(\theta, \varphi) = (120^o, 215^o)$.
Point electric vector sensor located at $(x_r, y_r, z_r) = (-200, 742, 640)$ m; receiver amplification = 6.7; orientation angles $(\theta, \varphi) = (60^o, 30^o)$.

***Earth model parameters*:**
1D layered conductivity model.

***Source waveform*:**
Alternating polarity sequence of square pulses (25 ms positive, 25 ms zero , 25 ms negative, 25 ms zero); unit magnitude.

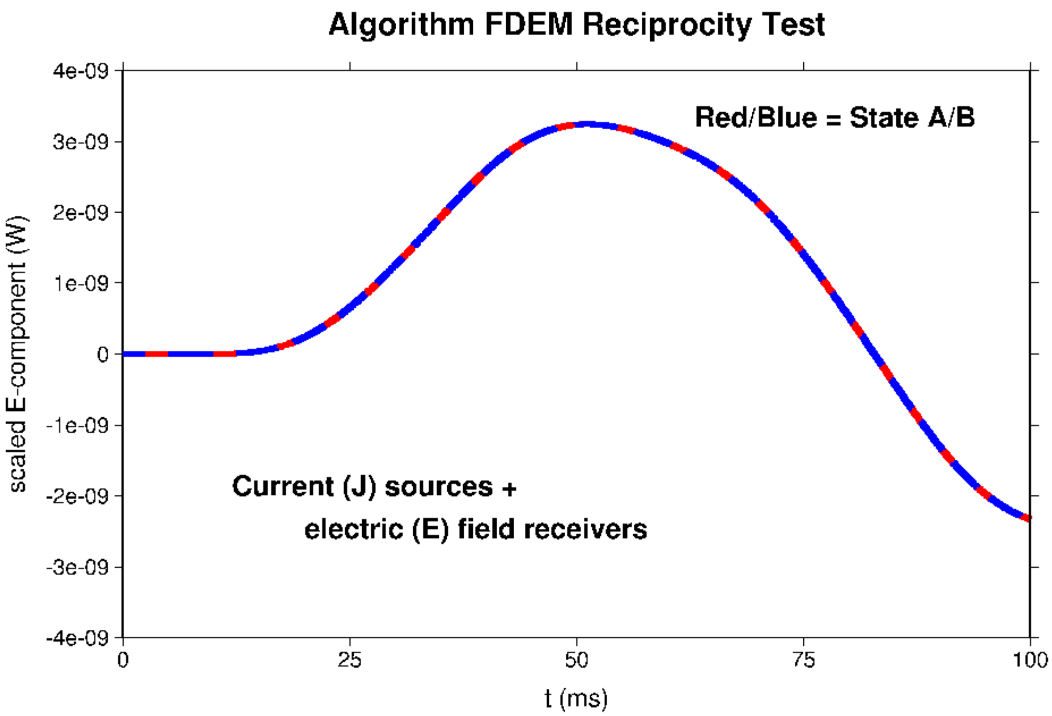

Figure 5.7. Same as figure 5.6, except the temporal/spatial FD operator orders in algorithm FDEM are O(M,N) = O(2,4). These EM responses are virtually identical to those displayed in figure 5.6.

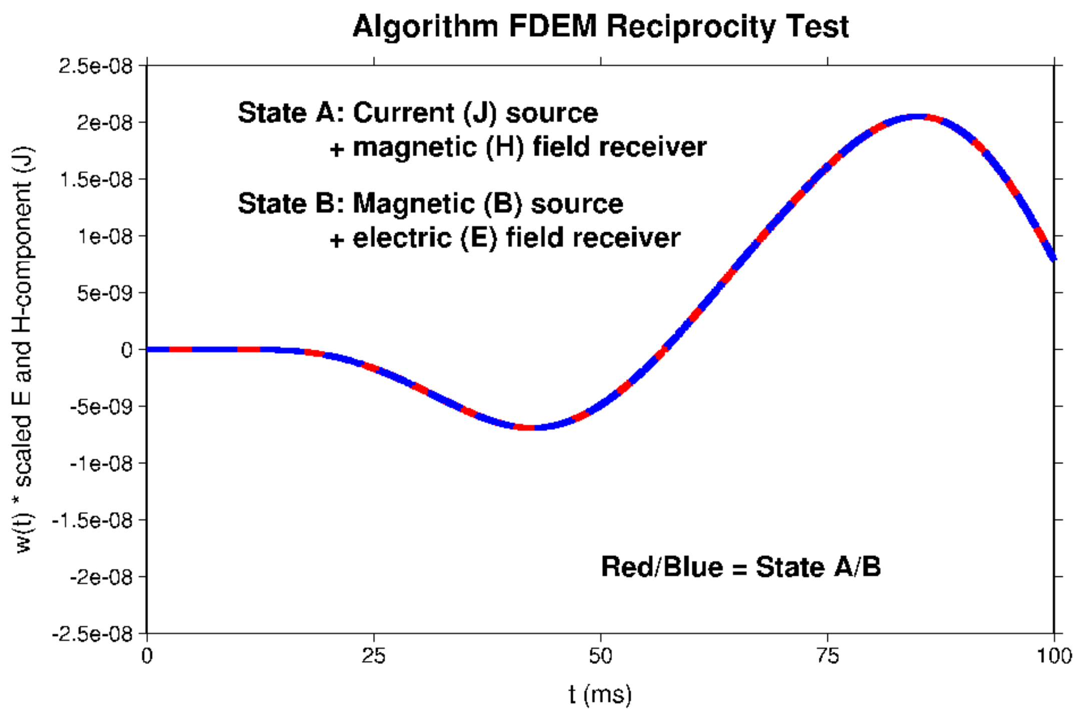

Figure 5.8. Validation of point source/point receiver reciprocity for finite-difference electromagnetic modeling algorithm FDEM. **Red** and **blue** traces correspond to states A and B, respectively. **Blue** trace overplots **red** trace, confirming that reciprocity holds for this case of *dissimilar source types* activated by *different waveforms* in the two states.

***State A specifications*:**
Point current density source located at $(x_s, y_s, z_s) = (-200, 742, 1500)$ m; source amplitude $J = 6.7$ A-m; orientation angles $(\theta, \varphi) = (60^o, 30^o)$.
Point magnetic vector sensor located at $(x_r, y_r, z_r) = (572, 347, -124)$ m; receiver amplification = 4.2; orientation angles $(\theta, \varphi) = (120^o, 215^o)$.

***State B specifications*:**
Point magnetic induction source located at $(x_s, y_s, z_s) = (572, 347, -124)$ m; source amplitude $B = 2.1$ V-m-s; orientation angles $(\theta, \varphi) = (120^o, 215^o)$.
Point electric vector sensor located at $(x_r, y_r, z_r) = (-200, 742, 1500)$ m; receiver amplification = 6.7; orientation angles $(\theta, \varphi) = (120^o, 210^o)$.

***Earth model parameters*:**
Relative electric permittivity κ_ε = 10.0; relative magnetic permeability κ_μ = 1.0; relative current conductivity κ_σ = 0.5 (relative to 1.0 S/m).

***Source waveforms*:**
State A: Alternating polarity sequence of square pulses; duration = 0.5 s; pulse lag = 1.0 s; unit magnitude.
State B: Alternating polarity sequence of square pulses; duration = 0.4 s; pulse lag = 0.8 s; unit magnitude.

6.0 THEORETICAL USES OF RECIPROCITY

The previous chapter emphasizes two important practical uses of reciprocity in numerical modeling work. However, reciprocity also has strong theoretical utility. In fact, de Hoop (1992) states that "field reciprocity theorems can be considered to be the most basic relations that exist in the theory of classical fields and waves". In this chapter, we utilize the time-convolution reciprocity theorem (3.20) to derive two mathematical objects of fundamental importance in the analysis of electromagnetic wavefields:

1) A "Green's function" represents the electromagnetic field generated by a unit magnitude, impulsive, point body source oriented in a particular coordinate direction. Integration of the Green's function over a three-dimensional volume then yields the physical electromagnetic field generated by a body source distribution with arbitrary location, magnitude, waveform, and orientation. The volume integral formula is often referred to as a "representation theorem" for the electromagnetic wavefield.

2) "Frechet derivatives" are, broadly speaking, quantitative measures of the sensitivity of electromagnetic data to changes in various parameters. In particular, Fréchet derivatives are often calculated with respect to the parameters characterizing a medium supporting an electromagnetic wavefield. Fréchet derivatives are basic tools required for solving the full waveform electromagnetic inverse problem.

In the derivations that follow, the two media supporting electromagnetic wave propagation in states A and B are taken to be identical, and additionally satisfy the adjoint conditions (3.10a,b,c) and (3.12c). Then, the operative global reciprocity theorem in volume V is equation (3.20):

$$\int_V \left\{ e_i^A(\mathbf{x},t) * j_i^{B-s}(\mathbf{x},t) - e_i^B(\mathbf{x},t) * j_i^{A-s}(\mathbf{x},t) \right\} dV(\mathbf{x})$$

$$-\int_V \left\{ h_i^A(\mathbf{x},t) * k_i^{B-s}(\mathbf{x},t) - h_i^B(\mathbf{x},t) * k_i^{A-s}(\mathbf{x},t) \right\} dV(\mathbf{x})$$

$$+\int_V \left\{ e_i^A(\mathbf{x},t) * l_i^{B-s}(\mathbf{x},t) - e_i^B(\mathbf{x},t) * l_i^{A-s}(\mathbf{x},t) \right\} dV(\mathbf{x})$$

$$+\sum_{n=1}^{N} \int_{S_n} \left\{ \left\langle e_i^A(\mathbf{x},t) \right\rangle * s_i^B(\mathbf{x},t) - \left\langle e_i^B(\mathbf{x},t) \right\rangle * s_i^A(\mathbf{x},t) \right\} dS(\mathbf{x}) = 0 \,, \tag{6.1}$$

where wavefield radiation conditions are also assumed to eliminate the integral on the outer bounding surface S.

6.1 Representation Theorem

We consider the simple case where the electromagnetic wavefields of states A and B are generated via body sources of conduction current. Then, the time-convolution reciprocity theorem (6.1) reduces to

$$\int_V \left\{ e_i^A(\mathbf{x},t) * j_i^{B-s}(\mathbf{x},t) - e_i^B(\mathbf{x},t) * j_i^{A-s}(\mathbf{x},t) \right\} dV(\mathbf{x}) = 0 \,. \tag{6.2}$$

That is, there are no active body magnetic currents, body displacement currents, or surface currents. Next, assume that the electromagnetic field of state B is generated by a *point* current density source located at position $\mathbf{x}_s^B$:

$$j_i^{B-s}(\mathbf{x},t) = J^B w^B(t) d_i^B \delta\left(\mathbf{x} - \mathbf{x}_s^B\right). \tag{6.3}$$

Here J^B is a magnitude scalar (SI unit: A-m), $w^B(t)$ is a dimensionless current waveform (and normalized to unit maximum absolute amplitude), and $\mathbf{d}^B$ is a unit orientation vector. Substituting this point source into the reciprocity relation (6.2) and performing the volume integral over the Dirac delta function yields

$$d_i^B e_i^A(\mathbf{x}_s^B, t) = \int_V \left[\frac{w^B(t)^{-1} * e_i^B(\mathbf{x},t)}{J^B} \right] * j_i^{A-s}(\mathbf{x},t)\, dV(\mathbf{x}). \tag{6.4}$$

In this expression, we have convolved each side with the inverse function $w^B(t)^{-1}$ of the source current waveform, which satisfies $w^B(t) * w^B(t)^{-1} = w^B(t)^{-1} * w^B(t) = \delta(t)$. This inverse wavelet has SI unit $1/s^2$. The quantity in square brackets on the right-hand-side (with SI unit (V/m)/(A-m-s) is clearly a vector (not a tensor or a "dyadic"). However, consider separately the three cases where the orientation vector $\mathbf{d}^B$ of the point current density source activating EM wavefield B is aligned with the Cartesian coordinate axes:

$$\mathbf{d}^B = (1,0,0) = \mathbf{e}_1, \qquad \mathbf{d}^B = (0,1,0) = \mathbf{e}_2, \qquad \mathbf{d}^B = (0,0,1) = \mathbf{e}_3.$$

Boldface vectors $\mathbf{e}_i$ (i = 1,2,3) are an orthonormal basis triad for the rectangular coordinate system. Then, expression (6.4) above may be re-written in the more generalized form

$$e_i^A(\mathbf{x}_s^B, t) = \int_V \left[\frac{w^B(t)^{-1} * e_j(\mathbf{x}', t; \mathbf{x}_s^B, \mathbf{e}_i)}{J^B} \right] * j_j^{A-s}(\mathbf{x}',t)\, dV(\mathbf{x}'). \tag{6.5}$$

The dummy volume integration variable is changed from $\mathbf{x}$ to $\mathbf{x}'$, and the dummy summation index in the integrand is changed from i to j. The earlier notation for electric vector component $e_i^B(\mathbf{x},t)$ is changed to $e_j(\mathbf{x},t;\mathbf{x}_s^B,\mathbf{e}_i)$, where the two arguments after the semi-colon stand for the position and orientation of the point source activating EM wavefield B. Now, it is merely a matter of re-labeling mathematical symbols to write equation (6.5) in the completely equivalent form

$$e_i(\mathbf{x},t) = \int_V g_{ij}(\mathbf{x}',t;\mathbf{x}) * j_j^s(\mathbf{x}',t)\, dV(\mathbf{x}'), \tag{6.6}$$

where the second-rank tensor function $g_{ij}(\mathbf{x},t;\mathbf{x}_s)$ is defined by

$$g_{ij}(\mathbf{x},t;\mathbf{x}_s) \equiv \frac{w^B(t)^{-1} * e_j(\mathbf{x},t;\mathbf{x}_s,\mathbf{e}_i)}{J^B}. \tag{6.7}$$

Note that the identifying superscript "A" referring to EM state A is dropped from equation (6.6). Tensor component g_{ij} in (6.7) is interpreted as the j^{th} component of the electric vector at position $\mathbf{x}$ and time t, generated by a *unit magnitude and impulsive* point current density source at position $\mathbf{x}_s$, and oriented in the i^{th} coordinate direction. This tensor, referred to by de Hoop (1992) and others as a "Green's function", has SI unit (V/m)/(A-m-s). Normally, a point current density source with SI unit A/m^2 is specified by the vector components $j_i^s(\mathbf{x},t) = Jd_i w(t)\delta(\mathbf{x}-\mathbf{x}_s)$ as in equation (6.3) above. The Green's function (6.7) is thought of as the j^{th} electric vector component generated by a different "point current density source" described by

$$\frac{w(t)^{-1} * j_i^s(\mathbf{x},t)}{J} = d_i\delta(t)\delta(\mathbf{x}-\mathbf{x}_s),$$

with unit *dimensionless* magnitude and *dimensioned* time variation given by the temporal Dirac delta function $\delta(t)$. Moreover, the unit magnitude source orientation vector $\mathbf{d}$ is restricted to one of the three coordinate directions $\mathbf{e}_i$ (i = 1,2,3)

Equation (6.6) is a *representation theorem* giving an electric field as a three-dimensional volume integral over the source current density distribution in volume V supporting the electromagnetic wavefield. Fourier transforming yields the frequency-domain equivalent

$$E_i(\mathbf{x},\omega) = \int_V G_{ij}(\mathbf{x}',\omega;\mathbf{x})\, J_j^s(\mathbf{x}',\omega)\, dV(\mathbf{x}')\,, \tag{6.8}$$

where upper case symbols denote Fourier transforms of lower case counterparts. The Fourier-transformed Green's function G_{ij} has SI unit (V/m)/(A-m-s)/Hz = (V/m)/(A-m).

Unfortunately, the mathematical form of the Green's function is usually not known except for rather simple media. For example, the (frequency-domain) electric vector generated by a point current density source situated in a homogeneous and isotropic wholespace is given by the previous equation (4.30) as

$$\mathbf{E}(\mathbf{x},\omega;\mathbf{x}_s) = \left(\frac{J\mu}{4\pi r}\right)[(-i\omega)W(\omega)]\exp[+ik(\omega)r] \times$$
$$\left\{[(\mathbf{d}\cdot\mathbf{e}_r)\mathbf{e}_r - \mathbf{d}] - [3(\mathbf{d}\cdot\mathbf{e}_r)\mathbf{e}_r - \mathbf{d}]\left(\frac{1}{ik(\omega)r} - \frac{1}{(ik(\omega)r)^2}\right)\right\}. \tag{4.30 again}$$

r is the distance from the source at $\mathbf{x}_s$ to a field point at $\mathbf{x}$, in the direction of unit vector $\mathbf{e}_r$. Hence, $\mathbf{e}_r = (x_i - x_i^s)\mathbf{e}_i / r$. Divide by $JW(\omega)$ and dot with coordinate vector $\mathbf{e}_j$ to obtain

$$\frac{E_j(\mathbf{x},\omega;\mathbf{x}_s,\mathbf{d})}{JW(\omega)} = \left(\frac{(-i\omega)\mu}{4\pi r}\right)\exp[+ik(\omega)r] \times$$
$$\left\{\left[\frac{(x_j - x_j^s)(x_i - x_i^s)}{r^2} - \delta_{ji}\right] - \left[\frac{3(x_j - x_j^s)(x_i - x_i^s)}{r^2} - \delta_{ji}\right]\left(\frac{1}{ik(\omega)r} - \frac{1}{(ik(\omega)r)^2}\right)\right\} d_i\,. \tag{6.9}$$

The left-hand-side is the j^{th} component of the (Fourier-transformed) electric vector at field point $\mathbf{x}$, due to a point source at $\mathbf{x}_s$ and pointing in direction $\mathbf{d}$, and normalized by the source magnitude and spectrum.

Hence, the right-hand-side *must* be the (Fourier-transformed) Green function product $G_{ji}(\mathbf{x},\omega;\mathbf{x}_s)d_i$, with proper SI unit (V/m)/(A-m). Recall that $k(\omega)$ is the complex wavenumber and δ_{ji} is the Kronecker delta symbol.

Inspection of expression (6.9) indicates that the Green's tensor for a homogeneous and isotropic wholespace has the symmetries

$$G_{ji}(\mathbf{x},\omega;\mathbf{x}_s) = G_{ij}(\mathbf{x},\omega;\mathbf{x}_s) = G_{ij}(\mathbf{x}_s,\omega;\mathbf{x}). \tag{6.10}$$

In fact, the extreme left/right equality in (6.10) is a general property of the electromagnetic Green's tensor (i.e., for heterogeneous, anisotropic, non-instantaneously reacting media, etc.) as demonstrated by de Hoop (1992). Hence, the frequency-domain representation theorem (6.8) is expressed alternately as

$$E_i(\mathbf{x},\omega) = \int_V G_{ji}(\mathbf{x},\omega;\mathbf{x}')\, J_j^s(\mathbf{x}',\omega)\, dV(\mathbf{x}'). \tag{6.11}$$

Repeating the physical interpretation above, G_{ji} is the i^{th} component of the electric vector due to a point current density source pointing in the j^{th} coordinate direction. Now the volume integration appears over the point source location coordinate $\mathbf{x}'$, which seems more logical. The analogous time-domain representation theorem is

$$e_i(\mathbf{x},t) = \int_V g_{ji}(\mathbf{x},t;\mathbf{x}') * j_j^s(\mathbf{x}',t)\, dV(\mathbf{x}'). \tag{6.12}$$

6.2 Fréchet Derivatives

Broadly speaking, Fréchet derivatives are quantitative measures of the sensitivity of synthetic (or predicted, or modeled) data with respect to various parameters. Commonly, Fréchet derivatives are calculated with respect to medium parameters, as with the permittivity, permeability, and conductivity characterizing an isotropic electromagnetic medium. However, they may also be calculated with respect to the body sources, boundary conditions, and initial conditions that generate an EM wavefield. Fréchet derivatives are analogous to, but not identical to, the partial derivatives of a multi-variable function. In the following analysis, we develop expressions for Fréchet derivatives of electromagnetic data with respect to the three isotropic medium parameters, and the current density body source.

6.2.1 EH Partial Differential System

The initial step is to develop the system of partial differential equations governing electromagnetic phenomena. In vector notation, the Faraday law (2.1a) and the Ampere-Maxwell law (2.1b) of Maxwell's electromagnetic equations are written as

$$\frac{\partial\,\mathbf{b}(\mathbf{x},t)}{\partial t} + \mathbf{curl}\;\mathbf{e}(\mathbf{x},t) = \mathbf{0}\,, \qquad \text{(Faraday law)}$$

and

$$\frac{\partial\,\mathbf{d}(\mathbf{x},t)}{\partial t} - \mathbf{curl}\;\mathbf{h}(\mathbf{x},t) + \mathbf{j}(\mathbf{x},t) = \mathbf{0}\,, \qquad \text{(Ampere-Maxwell law)}$$

respectively. The five dependent variables in these equations are

$\mathbf{b}(\mathbf{x},t)$**:** magnetic flux density, SI unit: $\dfrac{\mathrm{V\text{-}s}}{\mathrm{m}^2}$,

$\mathbf{d}(\mathbf{x},t)$**:** electric flux density, SI unit: $\dfrac{\mathrm{A\text{-}s}}{\mathrm{m}^2}$,

$\mathbf{e}(\mathbf{x},t)$**:** electric field strength, SI unit: $\dfrac{\mathrm{V}}{\mathrm{m}}$,

$\mathbf{h}(\mathbf{x},t)$**:** magnetic field strength, SI unit: $\dfrac{\mathrm{A}}{\mathrm{m}}$,

$\mathbf{j}(\mathbf{x},t)$**:** conduction current density, SI unit: $\dfrac{\mathrm{A}}{\mathrm{m}^2}$.

Next, introduce three electromagnetic constitutive relations appropriate for linear, time-independent, local, instantaneously reacting, and isotropic media as:

$$\mathbf{d}(\mathbf{x},t) = \varepsilon(\mathbf{x})\,\mathbf{e}(\mathbf{x},t) + \mathbf{d}_s(\mathbf{x},t)\,,$$

$$\mathbf{j}(\mathbf{x},t) = \sigma(\mathbf{x})\,\mathbf{e}(\mathbf{x},t) + \mathbf{j}_s(\mathbf{x},t)\,,$$

$$\mathbf{b}(\mathbf{x},t) = \mu(\mathbf{x})\,\mathbf{h}(\mathbf{x},t) + \mathbf{b}_s(\mathbf{x},t)\,.$$

The medium is characterized by the three scalar parameters

$\varepsilon(\mathbf{x})$: electric permittivity, SI unit: $\dfrac{\text{A/m}}{\text{V/s}} = \dfrac{\text{F}}{\text{m}}$,

$\mu(\mathbf{x})$: magnetic permeability, SI unit: $\dfrac{\text{V/m}}{\text{A/s}} = \dfrac{\text{H}}{\text{m}}$,

$\sigma(\mathbf{x})$: current conductivity, SI unit: $\dfrac{\text{A/m}}{\text{V}} = \dfrac{\text{S}}{\text{m}}$.

Although we include all body sources (symbols with subscript "s") in the constitutive relations, we will only calculate the Fréchet derivative for the current density body source $\mathbf{j}_\mathbf{s}(\mathbf{x},t)$. The body source terms for electric and magnetic flux vectors will be needed for reference.

Combining the three constitutive equations with the two Maxwell equations yields the coupled first-order system of inhomogeneous partial differential equations:

$$\varepsilon(\mathbf{x})\frac{\partial\,\mathbf{e}(\mathbf{x},t)}{\partial t} + \sigma(\mathbf{x})\mathbf{e}(\mathbf{x},t) - \mathbf{curl}\,\mathbf{h}(\mathbf{x},t) = -\mathbf{j}_s(\mathbf{x},t) - \frac{\partial \mathbf{d}_s(\mathbf{x},t)}{\partial t}, \tag{6.13a}$$

$$\mu(\mathbf{x})\frac{\partial\,\mathbf{h}(\mathbf{x},t)}{\partial t} + \mathbf{curl}\,\mathbf{e}(\mathbf{x},t) = -\frac{\partial \mathbf{b}_s(\mathbf{x},t)}{\partial t}. \tag{6.13b}$$

We refer to these equations as the "EH system", after the two dependent variables contained therein. Right-hand-side terms are body sources of electromagnetic wavefields.

6.2.2 First Born Approximation

The next step in the development is to consider the effects of small variations in medium parameters and body sources. Although there are various ways to pursue the analysis, an approach based on the First Born Approximation is particularly straightforward.

Consider a three-dimensional volume V bounded by a surface S (which may be infinitely far away). Within this volume the electric and magnetic vectors satisfy the EH partial differential system (6.13a,b) above. On surface S the electric and magnetic vectors satisfy boundary conditions. Here we adopt the simple point of view that the boundary conditions are prescribed functions of the electric and magnetic vectors:

$$\mathbf{e}(\mathbf{x},t) = \mathbf{v}(\mathbf{x},t) \quad \text{for } \mathbf{x} \in S_v, \qquad \mathbf{h}(\mathbf{x},t) = \mathbf{w}(\mathbf{x},t) \quad \text{for } \mathbf{x} \in S_w, \tag{6.13c,d}$$

where $S_v \cup S_w = S$. Clearly, more complicated BCs can be contemplated. Finally, at time t_0, the initial conditions

$$\mathbf{e}(\mathbf{x},t_0) = \mathbf{e}_0(\mathbf{x}), \qquad \mathbf{h}(\mathbf{x},t_0) = \mathbf{h}_0(\mathbf{x}), \tag{6.13e,f}$$

are taken to hold throughout V and on S.

Now consider a medium occupying the same volume V, but characterized by the *slightly different* electromagnetic parameters

$$\varepsilon(\mathbf{x}) + \delta\varepsilon(\mathbf{x}), \qquad \mu(\mathbf{x}) + \delta\mu(\mathbf{x}), \qquad \sigma(\mathbf{x}) + \delta\sigma(\mathbf{x}).$$

The two space-dependent perturbations $\delta\varepsilon(\mathbf{x})$ and $\delta\mu(\mathbf{x})$ are considered small compared to $\varepsilon(\mathbf{x})$ and $\mu(\mathbf{x})$, respectively. However, in the case where the conductivity $\sigma(\mathbf{x})$ vanishes (i.e., the medium at $\mathbf{x}$ is vacuum), then perturbation $\delta\sigma(\mathbf{x})$ is obviously not small compared to $\sigma(\mathbf{x})$. We take the boundary and initial conditions applied to the perturbed medium to be identical to those applied to the unperturbed medium. Moreover, the perturbed medium is subject to the same flux density body sources $\mathbf{d}_s(\mathbf{x},t)$ and $\mathbf{b}_s(\mathbf{x},t)$, as well as a perturbed current density body source given by the sum

$$\mathbf{j}_s(\mathbf{x},t) + \boldsymbol{\delta}\mathbf{j}_s(\mathbf{x},t).$$

This EM wavefield source generates electric and magnetic field vectors given by

$$\mathbf{e}(\mathbf{x},t) + \boldsymbol{\delta}\mathbf{e}(\mathbf{x},t), \qquad \mathbf{h}(\mathbf{x},t) + \boldsymbol{\delta}\mathbf{h}(\mathbf{x},t),$$

respectively, where $\boldsymbol{\delta}\mathbf{e}(\mathbf{x},t)$ and $\boldsymbol{\delta}\mathbf{h}(\mathbf{x},t)$ represent corresponding small variations. These new electric and magnetic fields must satisfy the coupled EH PDE system

$$\left[\varepsilon(\mathbf{x}) + \delta\varepsilon(\mathbf{x})\right]\left[\frac{\partial\,\mathbf{e}(\mathbf{x},t)}{\partial t} + \frac{\partial\,\boldsymbol{\delta}\mathbf{e}(\mathbf{x},t)}{\partial t}\right] + \left[\sigma(\mathbf{x}) + \delta\sigma(\mathbf{x})\right]\left[\mathbf{e}(\mathbf{x},t) + \boldsymbol{\delta}\mathbf{e}(\mathbf{x},t)\right]$$

$$- \mathbf{curl}\;\mathbf{h}(\mathbf{x},t) - \mathbf{curl}\;\boldsymbol{\delta}\mathbf{h}(\mathbf{x},t) = -\mathbf{j}_s(\mathbf{x},t) - \boldsymbol{\delta}\mathbf{j}_s(\mathbf{x},t) - \frac{\partial\,\mathbf{d}_s(\mathbf{x},t)}{\partial t}, \tag{6.14a}$$

$$\left[\mu(\mathbf{x}) + \delta\mu(\mathbf{x})\right]\left[\frac{\partial\,\mathbf{h}(\mathbf{x},t)}{\partial t} + \frac{\partial\,\boldsymbol{\delta}\mathbf{h}(\mathbf{x},t)}{\partial t}\right] + \mathbf{curl}\;\mathbf{e}(\mathbf{x},t) + \mathbf{curl}\;\boldsymbol{\delta}\mathbf{e}(\mathbf{x},t) = -\frac{\partial\,\mathbf{b}_s(\mathbf{x},t)}{\partial t}, \tag{6.14b}$$

for $\mathbf{x}$ within V. On the bounding surface S, the electric and magnetic conditions

$$\mathbf{e}(\mathbf{x},t) + \boldsymbol{\delta}\mathbf{e}(\mathbf{x},t) = \mathbf{v}(\mathbf{x},t) \quad \text{for } \mathbf{x} \in S_v, \tag{6.14c}$$

$$\mathbf{h}(\mathbf{x},t) + \boldsymbol{\delta}\mathbf{h}(\mathbf{x},t) = \mathbf{w}(\mathbf{x},t) \quad \text{for } \mathbf{x} \in S_w, \tag{6.14d}$$

hold. Finally, at time t_0, the total EM wavefield satisfies the initial conditions

$$\mathbf{e}(\mathbf{x},t_0) + \boldsymbol{\delta}\mathbf{e}(\mathbf{x},t_0) = \mathbf{e}_0(\mathbf{x}), \tag{6.14e}$$

$$\mathbf{h}(\mathbf{x},t_0) + \boldsymbol{\delta}\mathbf{h}(\mathbf{x},t_0) = \mathbf{h}_0(\mathbf{x}), \tag{6.14f}$$

throughout V and on S.

Subtracting equations (6.13a,b) from (6.14a,b) gives

$$\varepsilon(\mathbf{x})\frac{\partial\,\boldsymbol{\delta}\mathbf{e}(\mathbf{x},t)}{\partial t}+\sigma(\mathbf{x})\boldsymbol{\delta}\mathbf{e}(\mathbf{x},t)-\mathbf{curl}\,\boldsymbol{\delta}\mathbf{h}(\mathbf{x},t)+\left[\delta\varepsilon(\mathbf{x})\frac{\partial\,\boldsymbol{\delta}\mathbf{e}(\mathbf{x},t)}{\partial t}+\delta\sigma(\mathbf{x})\boldsymbol{\delta}\mathbf{e}(\mathbf{x},t)\right]$$
$$=-\boldsymbol{\delta}\mathbf{j}_s(\mathbf{x},t)-\delta\sigma(\mathbf{x})\,\mathbf{e}(\mathbf{x},t)-\delta\varepsilon(\mathbf{x})\frac{\partial\mathbf{e}(\mathbf{x},t)}{\partial t},$$

$$\mu(\mathbf{x})\frac{\partial\,\boldsymbol{\delta}\mathbf{h}(\mathbf{x},t)}{\partial t}+\mathbf{curl}\,\boldsymbol{\delta}\mathbf{e}(\mathbf{x},t)+\left[\delta\mu(\mathbf{x})\frac{\partial\,\boldsymbol{\delta}\mathbf{h}(\mathbf{x},t)}{\partial t}\right]=-\delta\mu(\mathbf{x})\frac{\partial\,\mathbf{h}(\mathbf{x},t)}{\partial t}.$$

This system is simplified by neglecting the terms (in square brackets) that are products in small quantities. Thus, the PDEs governing the perturbation in the electromagnetic wavefield become

$$\varepsilon(\mathbf{x})\frac{\partial\,\boldsymbol{\delta}\mathbf{e}(\mathbf{x},t)}{\partial t}+\sigma(\mathbf{x})\boldsymbol{\delta}\mathbf{e}(\mathbf{x},t)-\mathbf{curl}\,\boldsymbol{\delta}\mathbf{h}(\mathbf{x},t)\approx-\boldsymbol{\delta}\mathbf{j}_s(\mathbf{x},t)-\delta\sigma(\mathbf{x})\,\mathbf{e}(\mathbf{x},t)-\delta\varepsilon(\mathbf{x})\frac{\partial\mathbf{e}(\mathbf{x},t)}{\partial t},\tag{6.15a}$$

$$\mu(\mathbf{x})\frac{\partial\,\boldsymbol{\delta}\mathbf{h}(\mathbf{x},t)}{\partial t}+\mathbf{curl}\,\boldsymbol{\delta}\mathbf{e}(\mathbf{x},t)\approx-\delta\mu(\mathbf{x})\frac{\partial\,\mathbf{h}(\mathbf{x},t)}{\partial t},\tag{6.15b}$$

where approximate equality symbols indicate that the expressions are appropriate for small material parameter perturbations. Additionally, subtracting equations (6.13c-f)) from (6.14c-f) implies that the perturbation wavefield satisfies (exactly!) the homogeneous boundary and initial conditions

$$\boldsymbol{\delta}\mathbf{e}(\mathbf{x},t)=\mathbf{0}\ \text{ for }\ \mathbf{x}\in S_v,\qquad\qquad \boldsymbol{\delta}\mathbf{h}(\mathbf{x},t)=\mathbf{0}\ \text{ for }\ \mathbf{x}\in S_w,\tag{6.15c,d}$$

$$\boldsymbol{\delta}\mathbf{e}(\mathbf{x},t_0)=\mathbf{0},\qquad\qquad \boldsymbol{\delta}\mathbf{h}(\mathbf{x},t_0)=\mathbf{0}.\tag{6.15e,f}$$

The solution of equations (6.15) constitutes the *First Born Approximation* for the perturbation wavefield. These expressions have a straightforward interpretation. The perturbation wavefield [$\boldsymbol{\delta}\mathbf{e}(\mathbf{x},t)$ and $\boldsymbol{\delta}\mathbf{h}(\mathbf{x},t)$] propagates within the original (i.e., unperturbed) medium characterized by parameters $\varepsilon(\mathbf{x})$, $\mu(\mathbf{x})$, and $\sigma(\mathbf{x})$. The wavefield is generated by the current density body source $\boldsymbol{\delta}\mathbf{j}_s(\mathbf{x},t)$ as well as *effective body sources* that depend on the material parameter perturbations:

$$\boldsymbol{\delta}\mathbf{j}_{eff}(\mathbf{x},t)\equiv\delta\sigma(\mathbf{x})\mathbf{e}(\mathbf{x},t),\tag{6.16a}$$

$$\boldsymbol{\delta}\mathbf{d}_{eff}(\mathbf{x},t)\equiv\delta\varepsilon(\mathbf{x})\mathbf{e}(\mathbf{x},t),\tag{6.16b}$$

$$\boldsymbol{\delta}\mathbf{b}_{eff}(\mathbf{x},t)\equiv\delta\mu(\mathbf{x})\mathbf{h}(\mathbf{x},t).\tag{6.16c}$$

Clearly, these effective body sources vanish at positions $\mathbf{x}$ in V where the material property perturbations equal zero. The sources are also directly proportional to the primary (or "incident" or "reference") electromagnetic field vectors $\mathbf{e}(\mathbf{x},t)$ and $\mathbf{h}(\mathbf{x},t)$. Now it becomes clear why body sources of the electric and magnetic flux densities were included in the original EH system (6.13a,b). Comparing with equations (6.16b,c) indicates that small perturbations in electric permittivity $\delta\varepsilon(\mathbf{x})$ and magnetic permeability $\delta\mu(\mathbf{x})$ constitute *effective* electric and magnetic flux density sources for the perturbation field.

The first Born approximation may be advantageously utilized in electromagnetic modeling in the following manner: First, given a "background" model represented by the three EM parameters $\sigma(\mathbf{x}), \varepsilon(\mathbf{x}), \text{and } \mu(\mathbf{x}),$ the partial differential system (6.13a,b) is solved for the electric vector $\mathbf{e}(\mathbf{x},t)$ and magnetic vector $\mathbf{h}(\mathbf{x},t)$ [subject to the boundary conditions (6.13c,d) and initial conditions (6.13e,f)]. This electromagnetic field is referred to as the "primary" or "incident" EM wavefield. During this modeling run $\mathbf{e}(\mathbf{x},t)$ and $\mathbf{h}(\mathbf{x},t)$ are stored at all positions $\mathbf{x}$ in the model where perturbations to the three medium parameters $\delta\sigma(\mathbf{x}), \delta\varepsilon(\mathbf{x}), \text{and } \delta\mu(\mathbf{x})$ are subsequently inserted into the background model. Next, partial differential system (6.15a,b) is solved for perturbations $\boldsymbol{\delta}\mathbf{e}(\mathbf{x},t)$ and $\boldsymbol{\delta}\mathbf{h}(\mathbf{x},t)$ to the primary electric and magnetic vectors. PDE system (6.15a,b) is mathematically identical to the original system (6.13a,b); only the right-hand-side terms representing body sources of EM waves change. Hence, the *same* numerical algorithm may be used for solution. According to equations (6.16a,b,c), the effective body sources (conduction current, displacement current, magnetic induction) for this second modeling run are localized in the EM earth model at precisely those positions where corresponding perturbations to medium parameters $\delta\sigma(\mathbf{x}), \delta\varepsilon(\mathbf{x}), \text{and } \delta\mu(\mathbf{x})$ are inserted. The sources are directly proportional to the strength of the perturbations, as well as the primary field vectors. The total electromagnetic response at any receiver position $\mathbf{x}_r$ should be well-approximated by the sums $\mathbf{e}(\mathbf{x}_r,t) + \boldsymbol{\delta}\mathbf{e}(\mathbf{x}_r,t)$ and $\mathbf{h}(\mathbf{x}_r,t) + \boldsymbol{\delta}\mathbf{h}(\mathbf{x}_r,t)$.

6.2.3. Time-Varying Sensitivity Equation

The reciprocity theorem (6.1) can now used to obtain a volume integral expression containing the unknown perturbations in the material parameters $\delta\varepsilon(\mathbf{x})$, $\delta\mu(\mathbf{x})$ and $\delta\sigma(\mathbf{x})$ and the current density source $\boldsymbol{\delta}\mathbf{j}_s(\mathbf{x},t)$. For conceptual, mathematical, and notational simplicity, the derivation pertains to the case where full waveform electromagnetic data are generated by a single point source (at position $\mathbf{x}^S$) and recorded by a single point receiver (at position $\mathbf{x}^R$). A simple graphic depiction of the situation is given in figure 6.1. Generalization to multiple point sources and/or point receivers is straightforward, but is not treated in this report.

In order to utilize the reciprocity theorem, two wavefields (A and B) satisfying the EH system (6.13a,b) in V and on S must be identified. The previous section indicates that a perturbation wavefield induced by small variations in material properties and current density source satisfies this system. Hence, the electric and magnetic vector components of wavefield A are taken to be

$$e_i^A(\mathbf{x},t) = \delta e_i(\mathbf{x},t), \qquad h_i^A(\mathbf{x},t) = \delta h_i(\mathbf{x},t), \tag{6.17a,b}$$

respectively. The first Born approximation indicates that the sources of wavefield A are spatially distributed, and are a combination of body conduction current density, body electric flux density, and body magnetic flux density given by

1) Body conduction current density $j_i^{A-s}(\mathbf{x},t) = \delta j_i^s(\mathbf{x},t) + \delta\sigma(\mathbf{x})e_i(\mathbf{x},t;\mathbf{x}^S)$, (6.18a)
2) Body electric flux density $d_i^{A-s}(\mathbf{x},t) = \delta\varepsilon(\mathbf{x})e_i(\mathbf{x},t;\mathbf{x}^S)$, (6.18b)
3) Body magnetic flux density $b_i^{A-s}(\mathbf{x},t) = \delta\mu(\mathbf{x})h_i(\mathbf{x},t;\mathbf{x}^S)$. (6.18c)

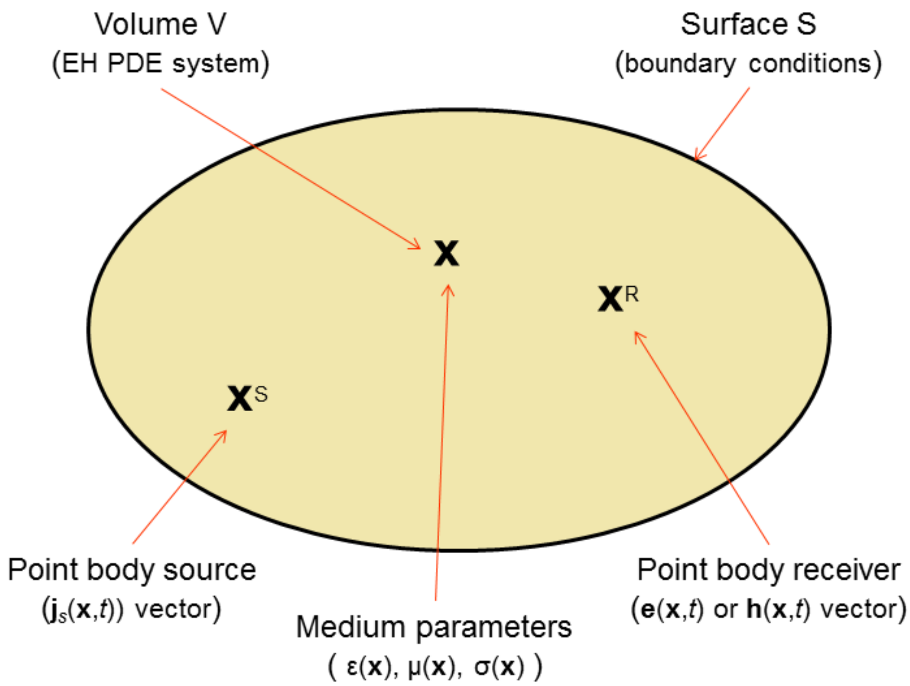

Figure 6.1. Setup of the Fréchet derivative problem. An isotropic electromagnetic medium occupies volume V bounded by surface S. The medium is characterized by the three scalar material properties electric permittivity $\varepsilon(\mathbf{x})$, magnetic permeability $\mu(\mathbf{x})$, and current conductivity $\sigma(\mathbf{x})$. All parameters are functions of position $\mathbf{x}$ within V and on S, and are independent of time t. A point current density source is located at $\mathbf{x}^S$ and a point receiver (of either the electric or magnetic field) is located at $\mathbf{x}^R$. Within V, the electromagnetic wavefield satisfies the EH system of coupled first-order partial differential equations. On S, boundary conditions are imposed.

The electric and magnetic fields in equations (6.18) originate at the point source located at $\mathbf{x}^S$. Recall that these fields propagate within the reference medium in volume V with material properties $\varepsilon(\mathbf{x})$, $\mu(\mathbf{x})$, and $\sigma(\mathbf{x})$.

Next, electromagnetic wavefield B is assumed to arise from a point source located at the receiver position $\mathbf{x}^R$. Thus, the electric and magnetic vector components are designated

$$e_i^B(\mathbf{x},t) = e_i(\mathbf{x},t;\mathbf{x}^R), \qquad h_i^B(\mathbf{x},t) = h_i(\mathbf{x},t;\mathbf{x}^R). \tag{6.19a,b}$$

This "receiver activated" wavefield is assumed to be generated by a point conduction current density source given by

$$j_i^{B-s}(\mathbf{x},t) = J^R d_i^R w^R(t)\delta\left(\mathbf{x}-\mathbf{x}^R\right), \tag{6.20}$$

where J^R is a source magnitude scalar (SI unit: A-m), d_i^R are the components of a (dimensionless) unit magnitude orientation vector, and $w^R(t)$ is a (dimensionless) current waveform (normalized to unit maximum absolute amplitude). This is sole body source type for wavefield B. In particular, we assume electric flux density source $d_i^{B-s}(\mathbf{x},t) = 0$ and magnetic flux density source $b_i^{B-s}(\mathbf{x},t) = 0$.

All of the mathematical elements required for utilizing the reciprocity theorem are now in place. Equations (6.17) through (6.20) are substituted into the reciprocity convolution integral relation (6.1), yielding

$$\int_V \delta\sigma(\mathbf{x})\left[e_i\left(\mathbf{x},t;\mathbf{x}^S\right)*e_i\left(\mathbf{x},t;\mathbf{x}^R\right)\right]dV(\mathbf{x})+\int_V \delta\varepsilon(\mathbf{x})\left[\frac{\partial e_i(\mathbf{x},t;\mathbf{x}^S)}{\partial t}*e_i\left(\mathbf{x},t;\mathbf{x}^R\right)\right]dV(\mathbf{x})$$

$$-\int_V \delta\mu(\mathbf{x})\left[\frac{\partial h_i(\mathbf{x},t;\mathbf{x}^S)}{\partial t}*h_i\left(\mathbf{x},t;\mathbf{x}^R\right)\right]dV(\mathbf{x})+\int_V \delta j_i^s(\mathbf{x},t)*e_i\left(\mathbf{x},t;\mathbf{x}^R\right)dV(\mathbf{x}).$$

$$=J^R w^R(t)*d_i^R\delta e_i(\mathbf{x}^R,t). \qquad (6.21)$$

This is an integral relation between the *unknown* parameter perturbations $\delta\sigma(\mathbf{x})$, $\delta\mu(\mathbf{x})$, $\delta\varepsilon(\mathbf{x})$, and $\boldsymbol{\delta}\mathbf{j}_s(\mathbf{x},t)$ (on the left) and the *known* electric vector residual recordings $\delta e_i(\mathbf{x}^R,t)$ (on the right). Various analytical and/or numerical techniques can be utilized to solve the expression for the perturbations.

The residuals are taken to be $\delta e_i(\mathbf{x}^R,t) = e_i^{obs}(\mathbf{x}^R,t;\mathbf{x}^S) - e_i^{prd}(\mathbf{x}^R,t;\mathbf{x}^S)$, or the difference between observed and predicted electric vector component data associated with source position $\mathbf{x}^S$ and receiver position $\mathbf{x}^R$.

Let $w^R(t)^{-1}$ denote the inverse of the source activation waveform used at the receiver position $\mathbf{x}^R$, satisfying $w^R(t)^{-1}*w^R(t)=w^R(t)*w^R(t)^{-1}=\delta(t)$ where $\delta(t)$ is the temporal Dirac delta function. Since $w^R(t)$ is dimensionless, its inverse $w^R(t)^{-1}$ has dimension "inverse-time-squared", or SI unit s^{-2}. Convolve (6.21) with $w^R(t)^{-1}$ and divide by the source amplitude scalar J^R to obtain the final result:

$$\int_V \delta\sigma(\mathbf{x})\left[e_i\left(\mathbf{x},t;\mathbf{x}^S\right)*\hat{e}_i\left(\mathbf{x},t;\mathbf{x}^R\right)\right]dV(\mathbf{x})+\int_V \delta\varepsilon(\mathbf{x})\left[\frac{\partial e_i(\mathbf{x},t;\mathbf{x}^S)}{\partial t}*\hat{e}_i\left(\mathbf{x},t;\mathbf{x}^R\right)\right]dV(\mathbf{x})$$

$$+\int_V \delta\mu(\mathbf{x})\left[-\frac{\partial h_i(\mathbf{x},t;\mathbf{x}^S)}{\partial t}*\hat{h}_i\left(\mathbf{x},t;\mathbf{x}^R\right)\right]dV(\mathbf{x})+\int_V \delta j_i^s(\mathbf{x},t)*\hat{e}_i\left(\mathbf{x},t;\mathbf{x}^R\right)dV(\mathbf{x})$$

$$=d_i^R\delta e_i(\mathbf{x}^R,t). \qquad (6.22)$$

New dependent variables, denoted by a superposed "hat" symbol, are defined as follows:

$$\hat{e}_i\left(\mathbf{x},t;\mathbf{x}^R\right)\equiv\frac{e_i(\mathbf{x},t;\mathbf{x}^R)*w^R(t)^{-1}}{J^R}, \qquad \hat{h}_i\left(\mathbf{x},t;\mathbf{x}^R\right)\equiv\frac{h_i(\mathbf{x},t;\mathbf{x}^R)*w^R(t)^{-1}}{J^R}, \qquad (6.23a,b)$$

$\hat{e}_i$ and $\hat{h}_i$ are electric and magnetic field strengths, respectively, due to a point current density body source placed at the receiver location $\mathbf{x}^R$, and activated by a unit magnitude Dirac delta function temporal waveform $\delta(t)$. Hence, they are properly referred to as *impulse response wavefields*. These variables have the SI units:

$$\hat{e}_i(\mathbf{x},t;\mathbf{x}^R) \;\rightarrow\; \frac{\text{V/m}}{\text{A-m-s}}, \qquad\qquad \hat{h}_i(\mathbf{x},t;\mathbf{x}^R) \;\rightarrow\; \frac{\text{A/m}}{\text{A-m-s}},$$

Thus, the impulse response wavefields have dimensions "physical quantity per current density impulse". Mathematical justification for designating these as impulse response wavefields is provided by the EH partial differential system (6.13a,b).

The *time-varying sensitivity equation* (6.22) is the basic tool used for derivation of Fréchet derivatives. Note that each term has the dimension of "electric field" and SI unit V/m.

6.2.4 Fréchet Derivatives of Electric Field Data

Examination of the kernels of the three volume integrals involving the material parameter perturbations in the time-varying sensitivity equation (6.22) suggests the following definitions of Fréchet derivatives:

$$\mathcal{F}_{\sigma}^{e_d}(\mathbf{x},t;\mathbf{x}^S,\mathbf{x}^R) \equiv e_i(\mathbf{x},t;\mathbf{x}^S) * \hat{e}_i(\mathbf{x},t;\mathbf{x}^R), \qquad \text{SI unit: } \frac{\text{V/m}}{\left(\dfrac{\text{A/m}}{\text{V}}\right)\text{-m}^3}, \tag{6.24a}$$

$$\mathcal{F}_{\varepsilon}^{e_d}(\mathbf{x},t;\mathbf{x}^S,\mathbf{x}^R) \equiv \frac{\partial e_i(\mathbf{x},t;\mathbf{x}^S)}{\partial t} * \hat{e}_i(\mathbf{x},t;\mathbf{x}^R), \qquad \text{SI unit: } \frac{\text{V/m}}{\left(\dfrac{\text{A/m}}{\text{V/s}}\right)\text{-m}^3}, \tag{6.24b}$$

$$\mathcal{F}_{\mu}^{e_d}(\mathbf{x},t;\mathbf{x}^S,\mathbf{x}^R) \equiv -\frac{\partial h_i(\mathbf{x},t;\mathbf{x}^S)}{\partial t} * \hat{h}_i(\mathbf{x},t;\mathbf{x}^R), \qquad \text{SI unit: } \frac{\text{V/m}}{\left(\dfrac{\text{V/m}}{\text{A/s}}\right)\text{-m}^3}. \tag{6.24c}$$

The "script F" notation on the left-hand-side denotes a Fréchet derivative, with superscript indicating the quantity to be differentiated, and subscript indicating the differentiation variable. Each Fréchet derivative is a function of position $\mathbf{x}$ and time t, and is parameterized by the locations of the point source $\mathbf{x}^S$ (a current density vector) and the point receiver $\mathbf{x}^R$ (sensitive to the electric vector). Hence, a Fréchet derivative is a rather complicated mathematical entity! The SI units indicate that it may be thought of as a "volume density of partial derivative" of electric field with respect to a medium parameter. Basically, it presents a four-dimensional map (in $\mathbf{x}$ and t) regarding where the electric vector component (sourced at $\mathbf{x}^S$ and recorded at $\mathbf{x}^R$ in direction $\mathbf{d}^R$) is sensitive to the pertaining medium parameter.

The right-hand-side of each definition indicates that the Fréchet derivative is a scalar (i.e., summed over all components i = 1,2,3) and is formed from a convolution of the physical EM wavefield sourced at $\mathbf{x}^S$ and the impulse response EM wavefield sourced at $\mathbf{x}^R$. Finally, recall that these Frechet derivatives correspond to the component of the electric field measured along the receiver direction $\mathbf{d}^R$ (which provides yet another free variable in the problem!).

A clearer understanding of the Fréchet derivative of the electric field with respect to the source current density function is obtained by writing out the relevant convolution integral in the time-varying sensitivity equation (6.22):

$$\int_V \delta j_i^s(\mathbf{x},t)*\hat{e}_i(\mathbf{x},t;\mathbf{x}^R)\,dV(\mathbf{x}) = \int_V \int_{-\infty}^{+\infty} \delta j_i^s(\mathbf{x},\tau)\hat{e}_i(\mathbf{x},t-\tau;\mathbf{x}^R)\,d\tau\, dV(\mathbf{x})\,.$$

Examination of the kernel function in this space-time integral indicates that the Fréchet derivative may be defined as

$$\mathcal{F}^{e_d}_{j^s-i}(\mathbf{x},t;\mathbf{x}^R,\tau) \equiv \hat{e}_i(\mathbf{x},\tau-t;\mathbf{x}^R), \qquad \text{SI unit: } \frac{\text{V/m}}{\left(\frac{\text{A}}{\text{m}^2}\right)\text{-}\ \text{m}^3\text{-}\ \text{s}}\,. \tag{6.24d}$$

This Fréchet derivative is a *vector* (as the rather complicated notation on the left-hand-side attempts to convey). At position **x** within volume V, the i^{th} component of the receiver-side impulse response wavefield is reversed in time. This function is a measure of the sensitivity of the recorded electric vector (d^{th} component) to the i^{th} component of the source current density function at the same **x** (and as a function of time t). Whew! The role of the time shift parameter τ is not yet fully understood; perhaps it can be set equal to zero. Also, in contrast to the situation with Fréchet derivatives with respect to medium parameters, the derivative (6.24d) does not depend on the original point source position $\mathbf{x}^S$. Dimensional analysis indicates that this Fréchet derivative may be thought of as a "partial derivative per unit volume per unit time". This additional complicating feature does not arise with Fréchet derivatives with respect to the three medium parameters, because they are not functions of time t.

6.2.5 Magnetic Field Recording

The aforementioned Fréchet derivatives pertain to *electric* vector recording (along direction $\mathbf{d}^R$) at receiver station $\mathbf{x}^R$. How does the situation change if the *magnetic* vector $\mathbf{h}(\mathbf{x}^R,t)$ is observed? In this case, replace the point current density source (6.20) located at $\mathbf{x}^R$ with the point magnetic flux density source:

$$b_i^{B-s}(\mathbf{x},t) = B^R d_i^R w^R(t)\delta(\mathbf{x}-\mathbf{x}^R), \tag{6.25}$$

where amplitude scalar B^R has SI unit T-m^3 = V-m-s. All other mathematical quantities used in the reciprocity theorem (6.1) remain the same. Then, analogous to equation (6.21), we obtain the volume integral expression

$$\int_V \delta\sigma(\mathbf{x})\left[e_i(\mathbf{x},t;\mathbf{x}^S)*e_i(\mathbf{x},t;\mathbf{x}^R)\right]dV(\mathbf{x}) + \int_V \delta\varepsilon(\mathbf{x})\left[\frac{\partial e_i(\mathbf{x},t;\mathbf{x}^S)}{\partial t}*e_i(\mathbf{x},t;\mathbf{x}^R)\right]dV(\mathbf{x})$$

$$+\int_V \delta\mu(\mathbf{x})\left[-\frac{\partial h_i(\mathbf{x},t;\mathbf{x}^S)}{\partial t}*h_i(\mathbf{x},t;\mathbf{x}^R)\right]dV(\mathbf{x}) + \int_V \delta j_i^s(\mathbf{x},t)*e_i(\mathbf{x},t;\mathbf{x}^R)\,dV(\mathbf{x})\,.$$

$$= -B^R\,\frac{\partial w^R(t)}{\partial t}*d_i^R\delta h_i(\mathbf{x}^R,t)\,. \tag{6.26}$$

Note that the left-hand-side is exactly the same as in equation (6.21). However, the right-hand-side contains the time-derivative of the magnetic body source function, as well as the recorded magnetic field strength components. This time differentiation presents some interesting issues:

1) Using the convolution-differentiation theorem of section 2.4, one could apply the time-derivative to the magnetic perturbation $\delta h_i(\mathbf{x}^R,t)$. Then, it is straightforward to develop Fréchet derivatives pertaining to $dh_i(\mathbf{x}^R,t)/dt$ recording.
2) Using the convolution-integration theorem of section 2.4, one could integrate equation (6.26) by convolving with the Heaviside unit step function $H(t)$. Then, Fréchet derivatives pertaining to $h_i(\mathbf{x}^R,t)$ recording will result.

We will illustrate both strategies. Regarding time-derivative magnetic recording (denoted by the symbol $\dot{h}_i(\mathbf{x}^R,t)$), volume integral expression (6.26) implies the four Fréchet derivatives:

$$\mathcal{F}_{\sigma}^{\dot{h}_d}\left(\mathbf{x},t;\mathbf{x}^S,\mathbf{x}^R\right)\equiv e_i\left(\mathbf{x},t;\mathbf{x}^S\right)*\hat{e}_i\left(\mathbf{x},t;\mathbf{x}^R\right), \qquad \text{SI unit: } \frac{(\text{A/m})/\text{s}}{\left(\frac{\text{A/m}}{\text{V}}\right)\text{-m}^3}, \tag{6.27a}$$

$$\mathcal{F}_{\varepsilon}^{\dot{h}_d}\left(\mathbf{x},t;\mathbf{x}^S,\mathbf{x}^R\right)\equiv \frac{\partial e_i(\mathbf{x},t;\mathbf{x}^S)}{\partial t}*\hat{e}_i\left(\mathbf{x},t;\mathbf{x}^R\right), \qquad \text{SI unit: } \frac{(\text{A/m})/\text{s}}{\left(\frac{\text{A/m}}{\text{V/s}}\right)\text{-m}^3}, \tag{6.27b}$$

$$\mathcal{F}_{\mu}^{\dot{h}_d}\left(\mathbf{x},t;\mathbf{x}^S,\mathbf{x}^R\right)\equiv -\frac{\partial h_i(\mathbf{x},t;\mathbf{x}^S)}{\partial t}*\hat{h}_i\left(\mathbf{x},t;\mathbf{x}^R\right), \qquad \text{SI unit: } \frac{(\text{A/m})/\text{s}}{\left(\frac{\text{V/m}}{\text{A/s}}\right)\text{-m}^3}. \tag{6.27c}$$

$$\mathcal{F}_{j^s-i}^{\dot{h}_d}\left(\mathbf{x},t;\mathbf{x}^R,\tau\right)\equiv \hat{e}_i\left(\mathbf{x},\tau-t;\mathbf{x}^R\right), \qquad \text{SI unit: } \frac{(\text{A/m})/\text{s}}{\left(\frac{\text{A}}{\text{m}^2}\right)\text{-m}^3\text{-s}}. \tag{6.27d}$$

These expressions for Fréchet derivatives of time-differentiated magnetic vector observations have exactly the same mathematical forms as the previous expressions (6.24) for derivatives of electric vector recording! However, the receiver-side impulse response wavefields are defined differently, and thus have different SI units:

$$\hat{e}_i\left(\mathbf{x},t;\mathbf{x}^R\right)\equiv \frac{e_i(\mathbf{x},t;\mathbf{x}^R)*w^R(t)^{-1}}{-B^R} \qquad \text{with SI unit } \frac{\text{V/m}}{\text{V-m-s}^2}, \tag{6.28a}$$

$$\hat{h}_i\left(\mathbf{x},t;\mathbf{x}^R\right)\equiv \frac{h_i(\mathbf{x},t;\mathbf{x}^R)*w^R(t)^{-1}}{-B^R} \qquad \text{with SI unit } \frac{\text{A/m}}{\text{V-m-s}^2}. \tag{6.28b}$$

Note in particular that a negative sign is incorporated into the definition of the impulse responses.

With regard to magnetic vector recording (i.e., no time-differentiation), volume integral equation (6.26) implies the four Fréchet derivatives:

$$\mathcal{F}_{\sigma}^{h_d}\left(\mathbf{x},t;\mathbf{x}^S,\mathbf{x}^R\right)\equiv e_i\left(\mathbf{x},t;\mathbf{x}^S\right)*\left[H(t)*\hat{e}_i\left(\mathbf{x},t;\mathbf{x}^R\right)\right], \qquad \text{SI unit: } \frac{\text{A/m}}{\left(\frac{\text{A/m}}{\text{V}}\right)\text{-}\,\text{m}^3}, \tag{6.29a}$$

$$\mathcal{F}_{\varepsilon}^{h_d}\left(\mathbf{x},t;\mathbf{x}^S,\mathbf{x}^R\right)\equiv e_i(\mathbf{x},t;\mathbf{x}^S)*\hat{e}_i\left(\mathbf{x},t;\mathbf{x}^R\right), \qquad \text{SI unit: } \frac{\text{A/m}}{\left(\frac{\text{A/m}}{\text{V/s}}\right)\text{-}\,\text{m}^3}, \tag{6.29b}$$

$$\mathcal{F}_{\mu}^{h_d}\left(\mathbf{x},t;\mathbf{x}^S,\mathbf{x}^R\right)\equiv -h_i(\mathbf{x},t;\mathbf{x}^S)*\hat{h}_i\left(\mathbf{x},t;\mathbf{x}^R\right), \qquad \text{SI unit: } \frac{\text{A/m}}{\left(\frac{\text{V/m}}{\text{A/s}}\right)\text{-}\,\text{m}^3}. \tag{6.29c}$$

$$\mathcal{F}_{j^s-i}^{h_d}\left(\mathbf{x},t;\mathbf{x}^R,\tau\right)\equiv H(t)*\hat{e}_i\left(\mathbf{x},\tau-t;\mathbf{x}^R\right), \qquad \text{SI unit: } \frac{\text{A/m}}{\left(\frac{\text{A}}{\text{m}^2}\right)\text{-}\,\text{m}^3\text{-}\,\text{s}}. \tag{6.29d}$$

The receiver side impulse response wavefields have exactly the same definitions (6.28a and b). In retrospect, it is obvious that these Fréchet derivatives could have been obtained simply by convolving the analogous expressions (6.27a to d) with the Heaviside unit step function. Fréchet derivatives with respect to permittivity ε and permeability μ are simplified. However, the remaining derivatives with respect to conductivity σ and source current density $\mathbf{j}_s$ now require time-integrated impulse response wavefields.

6.2.6 Time-Invariant Sensitivity Equation

A *misfit objective function* is a quantitative measure of the difference between observed and predicted electromagnetic waveform data. A common objective function involves the L2 norm of a residual trace (or traces). In order to develop expressions for the Fréchet derivatives of a misfit objective function with respect to electromagnetic medium parameters and source parameters, we start with the time-varying sensitivity equation (6.22), repeated in modified form as:

$$\int_V \delta\sigma(\mathbf{x})\left[e_i\left(\mathbf{x},t;\mathbf{x}^S\right)*\hat{e}_i\left(\mathbf{x},t;\mathbf{x}^R\right)\right]dV(\mathbf{x})+\int_V \delta\varepsilon(\mathbf{x})\left[\frac{\partial e_i(\mathbf{x},t;\mathbf{x}^S)}{\partial t}*\hat{e}_i\left(\mathbf{x},t;\mathbf{x}^R\right)\right]dV(\mathbf{x})$$

$$+\int_V \delta\mu(\mathbf{x})\left[-\frac{\partial h_i(\mathbf{x},t;\mathbf{x}^S)}{\partial t}*\hat{h}_i\left(\mathbf{x},t;\mathbf{x}^R\right)\right]dV(\mathbf{x})+\int_V \delta j_i^s(\mathbf{x},t)*\hat{e}_i\left(\mathbf{x},t;\mathbf{x}^R\right)dV(\mathbf{x})$$

$$=r\left(\mathbf{x}^R,t;\mathbf{x}_s\right). \tag{6.30}$$

The right-hand-side *residual waveform* is defined as follows:

$$r\left(\mathbf{x}^R,t;\mathbf{x}^S\right)\equiv\begin{cases} d_i^R\,\delta e_i(\mathbf{x}^R,t;\mathbf{x}_s), & \text{electric recording.}\\ \\ d_i^R\,\delta\dot{h}_i(\mathbf{x}^R,t;\mathbf{x}_s), & \text{time derivative magnetic recording.}\end{cases} \tag{6.31}$$

Here $\delta e_i(\mathbf{x}^R,t;\mathbf{x}^S) = e_i^{obs}(\mathbf{x}^R,t;\mathbf{x}^S) - e_i^{prd}(\mathbf{x}^R,t;\mathbf{x}^S)$ is the electric vector component misfit trace associated with source position $\mathbf{x}^S$ and receiver position $\mathbf{x}^R$. An analogous expression and interpretation holds for time-differentiated magnetic vector component recording. Recall that symbols with a superposed "hat" refer to impulse response wavefields activated by a unit magnitude temporal Dirac delta function generated by a point source at the receiver position $\mathbf{x}^R$. We must remember that these possess different definitions and SI units if the point source is an electric current density vector or a magnetic flux density vector!

Now, apply the following sequence of simple mathematical operations to equation (6.31):

1) Multiply by the residual waveform $r(\mathbf{x}^R,t;\mathbf{x}^S)$.
2) Integrate with respect to time t over the infinite interval $(-\infty, +\infty)$.
3) Change the order of temporal and spatial integration.

The result of these manipulations is

$$\begin{aligned}
&\int_V \delta\sigma(\mathbf{x}) \int_{-\infty}^{+\infty} \left[e_i(\mathbf{x},t;\mathbf{x}^S) * \hat{e}_i(\mathbf{x},t;\mathbf{x}^R) \right] r(\mathbf{x}^R,t;\mathbf{x}^S)\, dt\, dV(\mathbf{x}) \\
&+ \int_V \delta\varepsilon(\mathbf{x}) \int_{-\infty}^{+\infty} \left[\frac{\partial e_i(\mathbf{x},t;\mathbf{x}^S)}{\partial t} * \hat{e}_i(\mathbf{x},t;\mathbf{x}^R) \right] r(\mathbf{x}^R,t;\mathbf{x}^S)\, dt\, dV(\mathbf{x}) \\
&+ \int_V \delta\mu(\mathbf{x}) \int_{-\infty}^{+\infty} \left[-\frac{\partial h_i(\mathbf{x},t;\mathbf{x}^S)}{\partial t} * \hat{h}_i(\mathbf{x},t;\mathbf{x}^R) \right] r(\mathbf{x}^R,t;\mathbf{x}^S)\, dt\, dV(\mathbf{x}) \\
&+ \int_V \int_{-\infty}^{+\infty} \left[\delta j_i^s(\mathbf{x},t) * \hat{e}_i(\mathbf{x},t;\mathbf{x}^R) \right] r(\mathbf{x}^R,t;\mathbf{x}^S)\, dt\, dV(\mathbf{x}) \qquad = \int_{-\infty}^{+\infty} r(\mathbf{x}^R,t;\mathbf{x}_s)^2\, dt \equiv \Phi(\mathbf{x}^S,\mathbf{x}^R). \qquad (6.32)
\end{aligned}$$

The scalar quantity $\Phi(\mathbf{x}^S,\mathbf{x}^R)$ is the L2 norm of the misfit trace between observed and predicted (i.e., calculated) electromagnetic data recorded at receiver $\mathbf{x}^R$ and generated by the point source at $\mathbf{x}^S$. It has SI unit $(\text{V/m})^2$-s for electric vector recording and $(\text{A/m})^2$/s for time-differentiated magnetic vector recording, and is always positive-valued.

Next, recall item 7) of the convolution mathematics section (2.4). Convolution and multiplication operations may be exchanged in an integrand, with the central operand reversed in time:

$$\int_{-\infty}^{+\infty} \left[x(t) * y(t) \right] z(t)\, dt = \int_{-\infty}^{+\infty} x(t) \left[\bar{y}(t) * z(t) \right] dt \, .$$

Applying this theorem to the time-integrals on the left-hand-side of the above expression yields

$$\int_V \delta\sigma(\mathbf{x}) \int_{-\infty}^{+\infty} e_i\left(\mathbf{x},t;\mathbf{x}^S\right)\left[\hat{e}_i\left(\mathbf{x},-t;\mathbf{x}^R\right)*r\left(\mathbf{x}^R,t;\mathbf{x}^S\right)\right]dt\, dV(\mathbf{x})$$

$$+\int_V \delta\varepsilon(\mathbf{x}) \int_{-\infty}^{+\infty} \frac{\partial e_i(\mathbf{x},t;\mathbf{x}^S)}{\partial t}\left[\hat{e}_i\left(\mathbf{x},-t;\mathbf{x}^R\right)*r\left(\mathbf{x}^R,t;\mathbf{x}^S\right)\right]dt\, dV(\mathbf{x})$$

$$+\int_V \delta\mu(\mathbf{x}) \int_{-\infty}^{+\infty} -\frac{\partial h_i(\mathbf{x},t;\mathbf{x}^S)}{\partial t}\left[\hat{h}_i\left(\mathbf{x},-t;\mathbf{x}^R\right)*r\left(\mathbf{x}^R,t;\mathbf{x}^S\right)\right]dt\, dV(\mathbf{x})$$

$$+\int_V \int_{-\infty}^{+\infty} \delta j_i^s(\mathbf{x},t)\left[\hat{e}_i\left(\mathbf{x},-t;\mathbf{x}^R\right)*r\left(\mathbf{x}^R,t;\mathbf{x}^S\right)\right]dt\, dV(\mathbf{x}) \qquad = \Phi\left(\mathbf{x}^S,\mathbf{x}^R\right). \tag{6.33}$$

We now define electric and magnetic *adjoint wavefields*, denoted by a superposed tilde (~), as follows:

$$\tilde{e}_i(\mathbf{x},t;\mathbf{x}^R) \equiv \hat{e}_i\left(\mathbf{x},t;\mathbf{x}^R\right)*r(\mathbf{x}^R,-t;\mathbf{x}^S)\,, \tag{6.34a}$$

$$\tilde{h}_i(\mathbf{x},t;\mathbf{x}^R) \equiv \hat{h}_i\left(\mathbf{x},t;\mathbf{x}^R\right)*r(\mathbf{x}^R,-t;\mathbf{x}^S)\,. \tag{6.34b}$$

An adjoint wavefield is generated by a point source located at the receiver position $\mathbf{x}^R$, and activated by the *time-reversed* residual waveform. These adjoint wavefields have the rather curious SI units:

Case 1: Electric recording (point electric current density vector at x^R):

$$\tilde{e}_i(\mathbf{x},t;\mathbf{x}^R) \rightarrow \frac{\mathrm{V}^2}{\mathrm{A\text{-}m}^3} = \frac{\mathrm{V\text{-}\Omega}}{\mathrm{m}^3}\,, \qquad \tilde{h}_i(\mathbf{x},t;\mathbf{x}^R) \rightarrow \frac{\mathrm{V}}{\mathrm{m}^3}\,.$$

Case 2: Time-derivative magnetic recording (point magnetic flux density vector at x^R):

$$\tilde{e}_i(\mathbf{x},t;\mathbf{x}^R) \rightarrow \frac{\mathrm{A}}{\mathrm{s}^2\text{-}\mathrm{m}^3}\,, \qquad \tilde{h}_i(\mathbf{x},t;\mathbf{x}^R) \rightarrow \frac{\mathrm{A}^2}{\mathrm{V\text{-}s}^2\text{-}\mathrm{m}^3} = \frac{\mathrm{A}}{\Omega\text{-}\mathrm{s}^2\text{-}\mathrm{m}^3}\,.$$

Here symbol Ω stands for the SI unit of resistance or impedance: Ω = V/A. Substitute the definitions of the adjoint wavefields into equation (6.33) and utilize the time-reversal convolution theorem $\overline{x(t)*y(t)} = \bar{x}(t)*\bar{y}(t)$ (where an overbar denotes the time-reversal $\bar{x}(t) = x(-t)$). The result is

$$\int_V \delta\sigma(\mathbf{x}) \int_{-\infty}^{+\infty} e_i\left(\mathbf{x},t;\mathbf{x}^S\right)\tilde{e}_i\left(\mathbf{x},-t;\mathbf{x}^R\right)dt\,dV(\mathbf{x})$$

$$+\int_V \delta\varepsilon(\mathbf{x}) \int_{-\infty}^{+\infty} \frac{\partial e_i(\mathbf{x},t;\mathbf{x}^S)}{\partial t}\,\tilde{e}_i\left(\mathbf{x},-t;\mathbf{x}^R\right)dt\,dV(\mathbf{x})$$

$$+\int_V \delta\mu(\mathbf{x}) \int_{-\infty}^{+\infty} -\frac{\partial h_i(\mathbf{x},t;\mathbf{x}^S)}{\partial t}\,\tilde{h}_i\left(\mathbf{x},-t;\mathbf{x}^R\right)dt\,dV(\mathbf{x})$$

$$+\int_V \int_{-\infty}^{+\infty} \delta j_i^s(\mathbf{x},t)\,\tilde{e}_i\left(\mathbf{x},-t;\mathbf{x}^R\right)dt\,dV(\mathbf{x}) \qquad = \Phi\left(\mathbf{x}^S,\mathbf{x}^R\right). \tag{6.35}$$

This is the *time-invariant sensitivity equation*! However, it can be expressed in a different manner by recalling the definition of the cross-correlation of two functions:

$$x(t)\otimes y(t) \equiv \int_{-\infty}^{+\infty} x(\tau)\,y(\tau-t)\,d\tau\,.$$

Then, an alternative version of the time-invariant sensitivity equation (6.35) is

$$\int_V \delta\sigma(\mathbf{x})\left[e_i\left(\mathbf{x},t;\mathbf{x}^S\right)\otimes\tilde{e}_i\left(\mathbf{x},-t;\mathbf{x}^R\right)\right]_{t=0} dV(\mathbf{x})$$

$$+\int_V \delta\varepsilon(\mathbf{x})\left[\frac{\partial e_i(\mathbf{x},t;\mathbf{x}^S)}{\partial t}\otimes\tilde{e}_i\left(\mathbf{x},-t;\mathbf{x}^R\right)\right]_{t=0} dV(\mathbf{x})$$

$$+\int_V \delta\mu(\mathbf{x})\left[-\frac{\partial h_i(\mathbf{x},t;\mathbf{x}^S)}{\partial t}\otimes\tilde{h}_i\left(\mathbf{x},-t;\mathbf{x}^R\right)\right]_{t=0} dV(\mathbf{x})$$

$$+\int_V \left[\delta j_i^s(\mathbf{x},t)\otimes\tilde{e}_i\left(\mathbf{x},-t;\mathbf{x}^R\right)\right]_{t=0} dV(\mathbf{x}) \qquad = \Phi\left(\mathbf{x}^S,\mathbf{x}^R\right). \tag{6.36}$$

The first three time-integrals are seen to be cross-correlations, evaluated at zero time lag, of a forward modeled wavefield (sourced at $\mathbf{x}^S$) with an time-reversed adjoint wavefield (sourced at $\mathbf{x}^R$).

The time-invariant sensitivity equation (6.36) implies Fréchet derivatives of the misfit objective function Φ with respect to the three electromagnetic medium parameters:

$$\mathcal{F}_\sigma^\Phi(\mathbf{x};\mathbf{x}^S,\mathbf{x}^R) \equiv \left[e_i(\mathbf{x},t;\mathbf{x}^S) \otimes \tilde{e}_i(\mathbf{x},-t;\mathbf{x}^R) \right]_{t=0}, \tag{6.37a}$$

$$\mathcal{F}_\varepsilon^\Phi(\mathbf{x};\mathbf{x}^S,\mathbf{x}^R) \equiv \left[\frac{\partial e_i(\mathbf{x},t;\mathbf{x}^S)}{\partial t} \otimes \tilde{e}_i(\mathbf{x},-t;\mathbf{x}^R) \right]_{t=0}, \tag{6.37b}$$

$$\mathcal{F}_\mu^\Phi(\mathbf{x};\mathbf{x}^S,\mathbf{x}^R) \equiv \left[-\frac{\partial h_i(\mathbf{x},t;\mathbf{x}^S)}{\partial t} \otimes \tilde{h}_i(\mathbf{x},-t;\mathbf{x}^R) \right]_{t=0}. \tag{6.37c}$$

Recall that summation over repeated indices is implied. At each location **x** within volume V, a Fréchet derivative is obtained by cross-correlating a forward-modeled wavefield (sourced at $\mathbf{x}^S$) with a time-reversed adjoint wavefield (sourced at $\mathbf{x}^R$), and evaluating the cross-correlation at zero time lag. These derivatives have the SI units:

Case 1: Electric recording **(*point electric current density vector at*** $\boldsymbol{x}^{\boldsymbol{R}}$**)**:

$$\mathcal{F}_\sigma^\Phi(\mathbf{x};\mathbf{x}^S,\mathbf{x}^R) \;\rightarrow\; \frac{(\mathrm{V/m})^2\text{-}\mathrm{s}}{\left(\frac{\mathrm{A}}{\mathrm{V\text{-}m}}\right)\text{-}\mathrm{m}^3},$$

$$\mathcal{F}_\varepsilon^\Phi(\mathbf{x};\mathbf{x}^S,\mathbf{x}^R) \;\rightarrow\; \frac{(\mathrm{V/m})^2\text{-}\mathrm{s}}{\left(\frac{\mathrm{A/m}}{\mathrm{V/s}}\right)\text{-}\mathrm{m}^3},$$

$$\mathcal{F}_\mu^\Phi(\mathbf{x};\mathbf{x}^S,\mathbf{x}^R) \;\rightarrow\; \frac{(\mathrm{V/m})^2\text{-}\mathrm{s}}{\left(\frac{\mathrm{V/m}}{\mathrm{A/s}}\right)\text{-}\mathrm{m}^3}.$$

Case 2: Time derivative magnetic recording **(*point magnetic flux density vector at*** $\boldsymbol{x}^{\boldsymbol{R}}$**)**:

$$\mathcal{F}_\sigma^\Phi(\mathbf{x};\mathbf{x}^S,\mathbf{x}^R) \;\rightarrow\; \frac{(\mathrm{A/m})^2/\mathrm{s}}{\left(\frac{\mathrm{A}}{\mathrm{V\text{-}m}}\right)\text{-}\mathrm{m}^3},$$

$$\mathcal{F}_\varepsilon^\Phi(\mathbf{x};\mathbf{x}^S,\mathbf{x}^R) \;\rightarrow\; \frac{(\mathrm{A/m})^2/\mathrm{s}}{\left(\frac{\mathrm{A/m}}{\mathrm{V/s}}\right)\text{-}\mathrm{m}^3},$$

$$\mathcal{F}_\mu^\Phi(\mathbf{x};\mathbf{x}^S,\mathbf{x}^R) \;\rightarrow\; \frac{(\mathrm{A/m})^2/\mathrm{s}}{\left(\frac{\mathrm{V/m}}{\mathrm{A/s}}\right)\text{-}\mathrm{m}^3}.$$

In each case, the derivative has the proper unit of $\dfrac{\text{misfit objective function}}{\text{medium parameter} \times \text{volume}}$. Once again, a Fréchet derivative may be interpreted as a volume density of a partial derivative.

Finally, the Fréchet derivative of the misfit objective function $\Phi(\mathbf{x}^S,\mathbf{x}^R)$ with respect to the body source current density vector $\mathbf{j}_s$ is obtained from the time-invariant sensitivity expression (6.35) as

$$\mathcal{F}^{\Phi}_{j^s-i}\left(\mathbf{x},t;\mathbf{x}^R\right) = \tilde{e}_i\left(\mathbf{x},-t;\mathbf{x}^R\right). \tag{6.38}$$

The Fréchet derivative (a vector) at location $\mathbf{x}$ and time t equals the i^{th} component of the time-reversed adjoint electric wavefield at that point. SI units are the same as for the adjoint wavefield, or

Case 1: Electric recording at x^R:

$$\mathcal{F}^{\Phi}_{j^s-i}\left(\mathbf{x},t;\mathbf{x}^R\right) \;\rightarrow\; \frac{\mathrm{V}^2}{\mathrm{A\text{-}m^3}} = \frac{(\mathrm{V/m})^2\,\text{-}\,\mathrm{s}}{\left(\dfrac{\mathrm{A}}{\mathrm{m}^2}\right)\text{-}\,\mathrm{s}\,\text{-}\,\mathrm{m}^3},$$

Case 2: Time derivative magnetic recording at x^R:

$$\mathcal{F}^{\Phi}_{j^s-i}\left(\mathbf{x},t;\mathbf{x}^R\right) \;\rightarrow\; \frac{\mathrm{A}}{\mathrm{s^2\text{-}m^3}} = \frac{(\mathrm{A/m})^2/\mathrm{s}}{\left(\dfrac{\mathrm{A}}{\mathrm{m}^2}\right)\text{-}\,\mathrm{s}\,\text{-}\,\mathrm{m}^3}.$$

So, this Frechet derivative is properly interpreted as $\dfrac{\text{misfit objective function}}{\text{current density} \times \text{time} \times \text{volume}}$, similar to a partial derivative per unit time per unit volume. It is interesting to compare this Fréchet derivative with the analogous Fréchet derivatives of recorded data (electric vector and time-differentiated magnetic vector) given by the previous equations (6.24d) and (6.27d). Those derivatives involve the time-reversed *impulse response* wavefield sourced at the receiver location $\mathbf{x}^R$, and contain an additional parameter (the time shift τ).

The derivation of explicit mathematical formulae for Fréchet derivatives (of both recorded data and an L2 norm misfit objective function) completes the analysis intended for this chapter. As indicated, the reciprocity theorem forms the point of departure for the rather lengthy derivation. The importance and utility of Fréchet derivatives resides in the fact that they are basic mathematical tools used for solving the full waveform inverse problem. Generalization of the formulae to accommodate multiple simultaneously-acting sources and receivers, as well as visualization of Fréchet derivatives, constitute necessary next steps.

7.0 CONCLUSIONS

The extensive study of electromagnetic (EM) reciprocity reported herein yields two concrete benefits:

1) We have verified that two geophysical EM forward modeling algorithms in present use satisfy reciprocity, for a variety of different types of point sources and point receivers. Program EMHOLE is a frequency-domain Green function algorithm appropriate for a homogenous and isotropic wholespace. Closed-form mathematical formulae for EM responses are evaluated in the frequency-domain. Time-domain traces are then obtained by inverse numerical Fourier transformation. The mathematical expressions are easily demonstrated to satisfy reciprocity in a theoretical sense (in fact, individual near-field and far-field terms in these formulae are reciprocal as well). Hence, it is no surprise that the numerically-generated time-domain traces for various source-receiver pairs are reciprocal as well. The exercise validates proper implementation of the algorithm source code. EMHOLE enables rapid testing of various "what if" modeling scenarios regarding signal amplitude, polarity, waveform, etc. However, it is restricted to a homogeneous medium. In contrast, algorithm FDEM utilizes an explicit, time-domain, finite-difference approach for calculating EM responses in a 3D heterogeneous (but still isotropic) medium. The first-order "EH" coupled partial differential system is numerically solved on a 3D spatial grid. Reciprocity adherence constitutes a powerful "stress test" for this, as well as many other, numerical algorithms.

2) We have derived space-time domain expressions for Fréchet derivatives of electromagnetic data with respect to the three isotropic EM medium parameters electric permittivity $\varepsilon(\mathbf{x})$, magnetic permeability $\mu(\mathbf{x})$, and current conductivity $\sigma(\mathbf{x})$, as well as a current density body source $\mathbf{j}_s(\mathbf{x},t)$. Fréchet derivatives are rather complicated mathematical objects that quantitatively characterize the sensitivity of synthetic (or calculated, or modeled) data with respect to various parameters. They constitute essential ingredients for modern full waveform inverse solutions. Moreover, they are extremely useful for rational design of geophysical data acquisition experiments. Our development includes both time-varying and time-invariant data sensitivity expressions. The present derivation is deliberately simplified to accommodate only isotropic media, a current density source, and a single source-receiver pair. Generalization to more complicated and realistic situations is intended for the future.

The above two results are enabled by a re-derivation and extension of previously-known electromagnetic reciprocity theorems, due mainly to A. T. de Hoop. Our results treat a slightly broader range of EM media types, body sources, and data receivers. In particular, current conductivity is explicitly incorporated into the derivations. Interfaces (i.e., discontinuity surfaces in material parameters) with surface current sources are allowed. Although the time-convolution reciprocity theorem appears to be the most useful, we have also included a development of the time-correlation reciprocity theorem in an Appendix. Finally, in view of the practical utility of the potential formulation of electromagnetics for problem solving, we include an extensive Appendix culminating in a reciprocity theorem for potentials.

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9.0 APPENDIX A: GENERAL ELECTROMAGNETIC CONSTITUTIVE RELATIONS

The three electromagnetic constitutive relations (2.2) are appropriate for linear, time-invariant, and locally-reacting media. A rigorous derivation of these expressions from fundamental principles is given in this Appendix. We use the conventional constitutive relationship between the electric current density vector $j_i(\mathbf{x},t)$ and the electric field vector $e_i(\mathbf{x},t)$ to demonstrate the derivational procedure.

Consider the difference between the electric current density vector $j_i(\mathbf{x},t)$ and the electric current density source vector $j_i^s(\mathbf{x},t)$:

$$\chi_i(\mathbf{x},t) \equiv j_i(\mathbf{x},t) - j_i^s(\mathbf{x},t). \tag{9.1}$$

Current sources may arise from naturally occurring processes internal to the medium supporting electromagnetic wave propagation (i.e., lightning or magnetotelluic currents). Alternately, currents may be externally imposed upon the medium, as with the artificial electric current sources used for geophysical prospecting. However, the difference $\chi_i(\mathbf{x},t)$ is attributed solely to an electric field extant in the medium. Following terminology used in seismology (i.e., Backus and Mulcahy (1976)), we refer to χ_i as a *mathematical current density* or a *model current density*, because a mathematical model is assumed for its dependence on medium properties. Following Tarantola's (1988) methodology in continuum mechanics, the general form of the electromagnetic constitutive relations is now derived.

The most general *linear* relationship between the mathematical/model current density vector components $\chi_i(\mathbf{x},t)$ and the electric field vector components $e_i(\mathbf{x},t)$ is given by

$$\chi_i(\mathbf{x},t) = \int_V \int_{-\infty}^{+\infty} \eta_{ij}^{(1)}(\mathbf{x},t,\mathbf{x}',t') e_j(\mathbf{x}',t')\, dt'\, dV(\mathbf{x}'), \tag{9.2}$$

where V is the three-dimensional volume occupied by the body. Repeated subscripts imply summation. The second-rank tensor kernels $\eta^{(1)}{}_{ij}$ may be distributions (i.e., containing temporal and/or spatial Dirac delta functions and/or their derivatives). In this and subsequent formulae, superscripts within parentheses denote different versions of the same general response function.

Relation (9.2) is linear in the sense of additive superposition:

$$\text{if } \; e_j(\mathbf{x},t)\big|_1 \; \text{implies} \; \chi_i(\mathbf{x},t)\big|_1 \qquad \text{and} \qquad \text{if } \; e_j(\mathbf{x},t)\big|_2 \; \text{implies} \; \chi_i(\mathbf{x},t)\big|_2,$$

$$\text{then } \; c_1 e_j(\mathbf{x},t)\big|_1 + c_2 e_j(\mathbf{x},t)\big|_2 \quad \text{implies} \quad c_1\chi_i(\mathbf{x},t)\big|_1 + c_2\chi_i(\mathbf{x},t)\big|_2,$$

where c_1 and c_2 are arbitrary constants.

If the relationship between current density and electric field is *local*, then

$$\eta_{ij}^{(1)}(\mathbf{x},t,\mathbf{x}',t') = \eta_{ij}^{(2)}(\mathbf{x},t,t')\delta(\mathbf{x}-\mathbf{x}'). \tag{9.3}$$

Furthermore, if the material properties are *time-invariant*, then

$$\eta_{ij}^{(2)}(\mathbf{x},t,t') = \eta_{ij}^{(3)}(\mathbf{x},t-t'). \tag{9.4}$$

Combining equations (9.2) through (9.4) yields the temporal convolution

$$\chi_i(\mathbf{x},t) = \int_{-\infty}^{+\infty} \eta_{ij}^{(3)}(\mathbf{x},t-t') e_j(\mathbf{x},t')\, dt' = \eta_{ij}^{(3)}(\mathbf{x},t) * e_j(\mathbf{x},t). \tag{9.5}$$

Taking $\eta^{(3)}{}_{ij} = \eta_{ij}$ and substituting $\chi_j = j_i - j_i^s$ yields the desired constitutive relation

$$j_i(\mathbf{x},t) = \eta_{ij}(\mathbf{x},t) * e_j(\mathbf{x},t) + j_i^s(\mathbf{x},t), \tag{9.6}$$

which explicitly exhibits the current source term on the right side. The response function $\eta_{ij}(\mathbf{x},t)$ has SI unit (A/(V-m))/s = (S/m)/s, equivalent to current conductivity per time.

More general (but geophysically unconventional) electromagnetic constitutive relations may be adopted. These relations assume that the conduction current density depends on both the electric vector *and* the magnetic vector. The linear relationship (9.2) generalizes to

$$\chi_i(\mathbf{x},t) = \int_V \int_{-\infty}^{+\infty} \left(\eta_{ij}^{(1)-e}(\mathbf{x},t,\mathbf{x}',t') e_j(\mathbf{x}',t') + \eta_{ij}^{(1)-h}(\mathbf{x},t,\mathbf{x}',t') h_j(\mathbf{x}',t') \right) dt'\, dV(\mathbf{x}'), \tag{9.7}$$

where the (admittedly awkward) superscript notation distinguishes between two classes of response functions. The same derivational procedure yields the linear, local, and time-invariant constitutive relation

$$j_i(\mathbf{x},t) = \eta_{ij}^{e}(\mathbf{x},t) * e_j(\mathbf{x},t) + \eta_{ij}^{h}(\mathbf{x},t) * h_j(\mathbf{x},t) + j_i^s(\mathbf{x},t), \tag{9.8}$$

for the current density vector. Conventional response function $\eta_{ij}^{e}(\mathbf{x},t)$ has the usual SI unit (S/m)/s, whereas the unconventional response function $\eta_{ij}^{h}(\mathbf{x},t)$ has SI unit m^{-1}/s = 1/(m-s).

For physically realistic media, certain requirements are imposed on the constitutive response functions. Thus, the convolutional kernel $\eta_{ij}(\mathbf{x},t)$ in equation (9.6) is both real-valued and causal. The Fourier transform is

$$\mathrm{N}_{ij}(\mathbf{x},\omega) = \int_{-\infty}^{+\infty} \eta_{ij}(\mathbf{x},t) \exp(+i\omega t)\, dt, \tag{9.9}$$

and has the SI unit of conductivity S/m. Since $\eta_{ij}(\mathbf{x},t)$ is real, its Fourier transform has the Hermitian symmetry:

$$\mathrm{N}_{ij}(\mathbf{x},-\omega) = \mathrm{N}_{ij}(\mathbf{x},\omega)^*, \tag{9.10}$$

where the asterisk as exponent denotes complex conjugation. Hence, real and imaginary parts of the Fourier transform are even and odd functions of frequency, respectively. [Ru-Shao Cheo (1965) incorrectly states that $\mathrm{N}_{ij}(\mathbf{x},\omega)$ must be an "even function of frequency".] Since $\eta_{ij}(\mathbf{x},t)$ is causal (meaning $\eta_{ij}(\mathbf{x},t)=0$ for t < 0) the real and imaginary parts of its Fourier transform are a Hilbert transform pair:

$$Hi\left\{\mathrm{Re}\left\{\mathrm{N}_{ij}(\mathbf{x},\omega)\right\}\right\}=-\mathrm{Im}\left\{\mathrm{N}_{ij}(\mathbf{x},\omega)\right\}, \qquad Hi\left\{\mathrm{Im}\left\{\mathrm{N}_{ij}(\mathbf{x},\omega)\right\}\right\}=\mathrm{Re}\left\{\mathrm{N}_{ij}(\mathbf{x},\omega)\right\}, \tag{9.11a,b}$$

(Bracewell, 1965). The Hilbert transform of a real-valued function $g(x)$ is defined as

$$Hi\{g(x)\}\equiv-\frac{1}{\pi x}*g(x)=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{g(x')}{x'-x}dx'. \tag{9.12a}$$

The Hilbert transform of a function has the same dimension (i.e., SI unit) as the function. Evaluation of a Hilbert transform integral is sometimes facilitated by using the equivalent form

$$Hi\{g(x)\}=\frac{2}{\pi}\int_{0}^{+\infty}\frac{xg_e(x')+x'g_o(x')}{(x')^2-x^2}dx', \tag{9.12b}$$

where $g_e(x)$ and $g_o(x)$ are the even and odd parts of $g(x)$, respectively. Thus, the Hilbert transform of an even/odd function is odd/even.

Causality of the convolutional kernel $\eta_{ij}(\mathbf{x},t)$ implies that the current density vector $\mathbf{j}(\mathbf{x},t)$ in constitutive relation (9.6) depends only on present and past values of an electric field $\mathbf{e}(\mathbf{x},t)$. In the absence of a current density source, current flow cannot precede in time an applied electric field. It is equally logical to assume that the inverse constitutive relation is also causal. That is, an electric field cannot precede current flow at a position $\mathbf{x}$. Neglecting the source term, the inverse of constitutive relation (9.6) is

$$e_i(\mathbf{x},t)=\eta_{ij}^{-1}(\mathbf{x},t)*j_j(\mathbf{x},t). \tag{9.13}$$

The inverse tensor function satisfies

$$\eta_{ik}(\mathbf{x},t)*\eta_{kj}^{-1}(\mathbf{x},t)=\delta_{ij}\delta(t), \tag{9.14a}$$

at all positions $\mathbf{x}$, where δ_{ik} is the Kronecker delta symbol and $\delta(t)$ is the temporal Dirac delta function. This implies that $\eta_{ij}^{-1}(\mathbf{x},t)$ has SI unit 1/((S/m)–s). It is straightforward to demonstrate that

$$\eta_{ik}^{-1}(\mathbf{x},t)*\eta_{kj}(\mathbf{x},t)=\delta_{ij}\delta(t), \tag{9.14b}$$

also holds.

If a time function $\eta_{ij}(\mathbf{x},t)$ and its inverse $\eta_{ij}^{-1}(\mathbf{x},t)$ are *both* causal, then that function (and its inverse) are called *minimum delay* (alternately, *minimum phase*). Minimum delay is a stronger property than mere causality. The physical understanding of minimum delay is that the energy content of the waveform is concentrated as closely as possible towards the onset time (at t = 0) as the frequency amplitude spectrum permits. The minimum delay (or minimum phase) property implies that the frequency-domain amplitude and phase spectra of the function are related by the Hilbert transforms

$$Hi\left\{\ln\left(\frac{\mathrm{mod}\{\mathrm{N}_{ij}(\mathbf{x},\omega)\}}{\mathrm{mod}\{N_{ij}(\mathbf{x},\omega_{ref})\}}\right)\right\} = -\arg\{\mathrm{N}_{ij}(\mathbf{x},\omega)\}, \qquad Hi\{\arg\{\mathrm{N}_{ij}(\mathbf{x},\omega)\}\} = \ln\left(\frac{\mathrm{mod}\{\mathrm{N}_{ij}(\mathbf{x},\omega)\}}{\mathrm{mod}\{\mathrm{N}_{ij}(\mathbf{x},\omega_{ref})\}}\right), \tag{9.15a,b}$$

where $\mathrm{N}_{ij}(\mathbf{x},\omega) = \mathrm{mod}\{\mathrm{N}_{ij}(\mathbf{x},\omega)\}\exp(+i\arg\{\mathrm{N}_{ij}(\mathbf{x},\omega)\})$ is the polar representation of the complex-valued Fourier spectrum $\mathrm{N}_{ij}(\mathbf{x},\omega)$. ω_{ref} is a reference angular frequency. Hence, the phase spectrum is determined if the amplitude spectrum is specified, or vice versa. A proof of property (9.15) may be found in Aki and Richards (1980) or in Aldridge (2014).

Finally, Ru-Shao Cheo (1965), reasoning on physical grounds, declares that "all physical (electromagnetic) media should exhibit the properties of free space at infinite frequency". This would appear to imply that the current conductivity $\mathrm{N}_{ij}(\mathbf{x},\pm\infty) = 0$, since the current conductivity of free space (i.e., vacuum) is zero. However, we find this assertion to be somewhat dubious. For the simple case of a point electromagnetic energy source (either an electric dipole or a magnetic dipole) situated in a homogeneous and isotropic wholespace, time-domain electromagnetic responses, applicable in the high-frequency limit, are given by our equations (4.45) and (4.46). The infinite frequency attenuation factor that appears in these expressions is $\alpha_\infty \equiv \omega_t/2c_\infty = (\sigma/2)\sqrt{\mu/\varepsilon}$, and is non-vanishing. The basic reason is that the complex wavenumber $k(\omega)$ approaches

$$k(\omega) \to \frac{\omega}{c_\infty} + i\alpha_\infty,$$

in the limit of high frequency. In our opinion, the imaginary part need not vanish.

10.0 APPENDIX B: TIME-CORRELATION RECIPROCITY

Local and global forms of a time-correlation reciprocity theorem are developed in this Appendix. The procedure is similar to the derivation of the time-convolution theorems in chapter 3, although great care is required to obtain the correct algebraic signs. Following de Hoop (1987) and de Hoop and de Hoop (2000), define a *local interaction quantity of the time-correlation type* as

$$\Psi(\mathbf{x},t) \equiv \frac{\partial}{\partial x_i}\left\{\varepsilon_{ijk}\left(e_j^A(\mathbf{x},t)\otimes h_k^B(\mathbf{x},t)+\overline{e_j^B(\mathbf{x},t)\otimes h_k^A(\mathbf{x},t)}\right)\right\}. \tag{10.1}$$

where $\otimes$ denotes cross-correlation, and summation over repeated subscripts is assumed. Recall that an overbar designates time reversal: $\bar{x}(t)=x(-t)$. Like the time-convolution interaction quantity (3.1), scalar quantity $\Psi(\mathbf{x},t)$ is the divergence of a vector and has physical dimension "energy/volume" (i.e., energy density, equal to pressure) and SI unit: J/m^3 = N/m^2 = P. But note that if the two electromagnetic wavefields A and B are identical, then this interaction quantity does *not* vanish. The reason, as per equation (2.6) of the text, is that cross-correlation is not symmetric with respect to its operand functions.

Using the theorems from convolution mathematics, the cross-correlation functions are replaced by convolutions. The interaction quantity is re-written as follows:

$$\Psi(\mathbf{x},t) = \frac{\partial}{\partial x_i}\left\{\varepsilon_{ijk}\left(e_j^A(\mathbf{x},t)*\bar{h}_k^B(\mathbf{x},t)+\overline{e_j^B(\mathbf{x},t)*\bar{h}_k^A(\mathbf{x},t)}\right)\right\}$$

$$= \frac{\partial}{\partial x_i}\left\{\varepsilon_{ijk}\left(e_j^A(\mathbf{x},t)*\bar{h}_k^B(\mathbf{x},t)+\bar{e}_j^B(\mathbf{x},t)*h_k^A(\mathbf{x},t)\right)\right\}. \tag{10.2}$$

Thus, in the time-correlation interaction quantity, the electromagnetic wavefield associated with state B is time-reversed.

We now follow the same procedure used in section 3.0 to derive the local form of the time-convolution reciprocity theorem. The spatial differentiations in (10.2) distribute over the temporal convolutions, yielding

$$\Psi(\mathbf{x},t) = \varepsilon_{ijk}\left\{\frac{\partial e_j^A(\mathbf{x},t)}{\partial x_i}*\bar{h}_k^B(\mathbf{x},t)+e_j^A(\mathbf{x},t)*\frac{\partial \bar{h}_k^B(\mathbf{x},t)}{\partial x_i}\right\}$$
$$+\,\varepsilon_{ijk}\left\{\frac{\partial \bar{e}_j^B(\mathbf{x},t)}{\partial x_i}*h_k^A(\mathbf{x},t)+\bar{e}_j^B(\mathbf{x},t)*\frac{\partial h_k^A(\mathbf{x},t)}{\partial x_i}\right\}.$$

Regrouping terms gives the equivalent expression:

$$\Psi(\mathbf{x},t) = \left\{\varepsilon_{ijk}\frac{\partial e_j^A(\mathbf{x},t)}{\partial x_i}*\bar{h}_k^B(\mathbf{x},t)+\varepsilon_{ijk}\frac{\partial \bar{e}_j^B(\mathbf{x},t)}{\partial x_i}*h_k^A(\mathbf{x},t)\right\}$$
$$+\left\{e_j^A(\mathbf{x},t)*\varepsilon_{ijk}\frac{\partial \bar{h}_k^B(\mathbf{x},t)}{\partial x_i}+\bar{e}_j^B(\mathbf{x},t)*\varepsilon_{ijk}\frac{\partial h_k^A(\mathbf{x},t)}{\partial x_i}\right\}.$$

If any two indices of the permuting symbol are swapped, then its value changes sign: $\varepsilon_{ijk} = -\varepsilon_{jik}$, $\varepsilon_{ijk} = -\varepsilon_{ikj}$, and $\varepsilon_{ijk} = -\varepsilon_{kji}$. Hence, two successive swaps preserves the sign. Thus, the above interaction quantity is put into the form

$$\Psi(\mathbf{x},t) = \left\{ \varepsilon_{kij} \frac{\partial e_j^A(\mathbf{x},t)}{\partial x_i} * \bar{h}_k^B(\mathbf{x},t) + \varepsilon_{kij} \frac{\partial \bar{e}_j^B(\mathbf{x},t)}{\partial x_i} * h_k^A(\mathbf{x},t) \right\}$$
$$- \left\{ e_j^A(\mathbf{x},t) * \varepsilon_{jik} \frac{\partial \bar{h}_k^B(\mathbf{x},t)}{\partial x_i} + \bar{e}_j^B(\mathbf{x},t) * \varepsilon_{jik} \frac{\partial h_k^A(\mathbf{x},t)}{\partial x_i} \right\}. \tag{10.3}$$

Next, the spatial derivatives are eliminated in favor of temporal derivatives by substituting from the electromagnetic field equations (2.1). Care is exercised to obtain the proper form of the time-reversed time derivatives:

$$\Psi(\mathbf{x},t) = -\left\{ \frac{\partial b_k^A(\mathbf{x},t)}{\partial t} * \bar{h}_k^B(\mathbf{x},t) - \frac{\partial \bar{b}_k^B(\mathbf{x},t)}{\partial t} * h_k^A(\mathbf{x},t) \right\}$$
$$+ \left\{ e_j^A(\mathbf{x},t) * \left(\frac{\partial \bar{d}_j^B(\mathbf{x},t)}{\partial t} - \bar{j}_j^B(\mathbf{x},t) \right) - \bar{e}_j^B(\mathbf{x},t) * \left(\frac{\partial d_j^A(\mathbf{x},t)}{\partial t} + j_j^A(\mathbf{x},t) \right) \right\}. \tag{10.4}$$

Using the convolution-differentiation theorem $d/dt\big(x(t) * y(t)\big) = x'(t) * y(t) = x(t) * y'(t)$ (where a prime indicates differentiation with respect to the argument), the interaction quantity is put into the form:

$$\Psi(\mathbf{x},t) = -\frac{\partial}{\partial t}\left\{ b_k^A(\mathbf{x},t) * \bar{h}_k^B(\mathbf{x},t) - \bar{b}_k^B(\mathbf{x},t) * h_k^A(\mathbf{x},t) \right\}$$
$$+ \frac{\partial}{\partial t}\left\{ e_j^A(\mathbf{x},t) * \left(\bar{d}_j^B(\mathbf{x},t) - \bar{j}_j^B(\mathbf{x},t) * H(t) \right) - \bar{e}_j^B(\mathbf{x},t) * \left(d_j^A(\mathbf{x},t) + j_j^A(\mathbf{x},t) * H(t) \right) \right\}, \tag{10.5}$$

where $H(t)$ is the Heaviside unit step function. Recall that $dH(t)/dt = \delta(t)$ where $\delta(t)$ is the temporal Dirac delta function.

Expression (10.5) contains the two electromagnetic flux density vectors **b** and **d**, as well as the current density vector **j**. These vector components are eliminated in favor of the electromagnetic field variables **e** and **h** by substituting from the (conventional) constitutive relations (2.2). Recall that time-reversed convolutions are given by $\overline{x(t) * y(t)} = \bar{x}(t) * \bar{y}(t)$. Hence:

$$\Psi(\mathbf{x},t) = -\frac{\partial}{\partial t}\left\{ \left(\varphi_{kj}^A(\mathbf{x},t) * h_j^A(\mathbf{x},t) + b_k^{A-s}(\mathbf{x},t) \right) * \bar{h}_k^B(\mathbf{x},t) - \left(\bar{\varphi}_{kj}^B(\mathbf{x},t) * \bar{h}_j^B(\mathbf{x},t) + \bar{b}_k^{B-s}(\mathbf{x},t) \right) * h_k^A(\mathbf{x},t) \right\}$$

$$+ \frac{\partial}{\partial t}\Big\{ e_j^A(\mathbf{x},t) * \left\{ \left(\bar{\psi}_{jk}^B(\mathbf{x},t) * \bar{e}_k^B(\mathbf{x},t) + \bar{d}_j^{B-s}(\mathbf{x},t) \right) - \left(\bar{\eta}_{jk}^B(\mathbf{x},t) * \bar{e}_k^B(\mathbf{x},t) + \bar{j}_j^{B-s}(\mathbf{x},t) \right) * H(t) \right\}$$
$$- \bar{e}_j^B(\mathbf{x},t) * \left\{ \left(\psi_{jk}^A(\mathbf{x},t) * e_k^A(\mathbf{x},t) + d_j^{A-s}(\mathbf{x},t) \right) + \left(\eta_{jk}^A(\mathbf{x},t) * e_k^A(\mathbf{x},t) + j_j^{A-s}(\mathbf{x},t) \right) * H(t) \right\} \Big\}.$$

Separate the terms into two major groups, the first including the medium-dependent response functions, and the second including the electromagnetic wavefield source functions:

$$
\begin{aligned}
\Psi(\mathbf{x},t) = &-\frac{\partial}{\partial t}\left\{ \varphi_{kj}^{A}(\mathbf{x},t) * h_j^A(\mathbf{x},t) * \bar{h}_k^B(\mathbf{x},t) - \bar{\varphi}_{kj}^{B}(\mathbf{x},t) * h_k^A(\mathbf{x},t) * \bar{h}_j^B(\mathbf{x},t) \right\} \\
&+\frac{\partial}{\partial t}\Big\{ \bar{\psi}_{jk}^{B}(\mathbf{x},t) * e_j^A(\mathbf{x},t) * \bar{e}_k^B(\mathbf{x},t) - \psi_{jk}^{A}(\mathbf{x},t) * e_k^A(\mathbf{x},t) * \bar{e}_j^B(\mathbf{x},t) \\
&\quad -\bar{\eta}_{jk}^{B}(\mathbf{x},t) * H(t) * e_j^A(\mathbf{x},t) * \bar{e}_k^B(\mathbf{x},t) - \eta_{jk}^{A}(\mathbf{x},t) * H(t) * e_k^A(\mathbf{x},t) * \bar{e}_j^B(\mathbf{x},t) \Big\} \\
&-\left\{ e_j^A(\mathbf{x},t) * \bar{j}_j^{B-s}(\mathbf{x},t) + \bar{e}_j^B(\mathbf{x},t) * j_j^{A-s}(\mathbf{x},t) \right\} \\
&+\left\{ h_k^A(\mathbf{x},t) * \frac{\partial \bar{b}_k^{B-s}(\mathbf{x},t)}{\partial t} - \bar{h}_k^B(\mathbf{x},t) * \frac{\partial b_k^{A-s}(\mathbf{x},t)}{\partial t} \right\} \\
&+\left\{ e_j^A(\mathbf{x},t) * \frac{\partial \bar{d}_j^{B-s}(\mathbf{x},t)}{\partial t} - \bar{e}_j^B(\mathbf{x},t) * \frac{\partial d_j^{A-s}(\mathbf{x},t)}{\partial t} \right\}.
\end{aligned} \tag{10.6}
$$

Note that the time-differentiations are brought back into the convolutions in the second group of terms. This motivates defining the *source magnetic current density* (SI unit: T/s = V/m^2) and *source displacement current density* (SI unit: (C/m^2)/s = A/m^2) as

$$
k_i^s(\mathbf{x},t) \equiv \frac{\partial b_i^s(\mathbf{x},t)}{\partial t}, \qquad l_i^s(\mathbf{x},t) \equiv \frac{\partial d_i^s(\mathbf{x},t)}{\partial t}, \tag{10.7a,b}
$$

respectively. Clearly, the source displacement current has the same dimension (and SI unit) as the *source conduction current density* $j_i^s(\mathbf{x},t)$. Then, engaging in some re-indexing, equation (10.6) is put into the more compact form

$$
\begin{aligned}
\Psi(\mathbf{x},t) = &-\frac{\partial}{\partial t}\left\{ \left(\varphi_{ji}^{A}(\mathbf{x},t) - \bar{\varphi}_{ij}^{B}(\mathbf{x},t)\right) * h_i^A(\mathbf{x},t) * \bar{h}_j^B(\mathbf{x},t) \right\} \\
&-\frac{\partial}{\partial t}\left\{ \left\{ \left(\psi_{ji}^{A}(\mathbf{x},t) - \bar{\psi}_{ij}^{B}(\mathbf{x},t)\right) + \left(\eta_{ji}^{A}(\mathbf{x},t) + \bar{\eta}_{ij}^{B}(\mathbf{x},t)\right) * H(t) \right\} * e_i^A(\mathbf{x},t) * \bar{e}_j^B(\mathbf{x},t) \right\} \\
&-\left\{ e_i^A(\mathbf{x},t) * \bar{j}_i^{B-s}(\mathbf{x},t) + \bar{e}_i^B(\mathbf{x},t) * j_i^{A-s}(\mathbf{x},t) \right\} \\
&-\left\{ h_i^A(\mathbf{x},t) * \bar{k}_i^{B-s}(\mathbf{x},t) + \bar{h}_i^B(\mathbf{x},t) * k_i^{A-s}(\mathbf{x},t) \right\} \\
&-\left\{ e_i^A(\mathbf{x},t) * \bar{l}_i^{B-s}(\mathbf{x},t) + \bar{e}_i^B(\mathbf{x},t) * l_i^{A-s}(\mathbf{x},t) \right\},
\end{aligned} \tag{10.8}
$$

in which the signs of time-reversed time derivatives are again handled with care. Finally, inserting the definition (10.1) of the interaction quantity on the left-hand-side yields the *local form* of the time-correlation reciprocity theorem (next page):

$$
\begin{aligned}
&\frac{\partial}{\partial x_i}\left\{\varepsilon_{ijk}\left(e_j^A(\mathbf{x},t)*\bar{h}_k^B(\mathbf{x},t)+\bar{e}_j^B(\mathbf{x},t)*h_k^A(\mathbf{x},t)\right)\right\}= \\
&\quad -\frac{\partial}{\partial t}\left\{\left(\varphi_{ji}^A(\mathbf{x},t)-\bar{\varphi}_{ij}^B(\mathbf{x},t)\right)*h_i^A(\mathbf{x},t)*\bar{h}_j^B(\mathbf{x},t)\right\} \\
&\quad -\frac{\partial}{\partial t}\left\{\left\{\left(\psi_{ji}^A(\mathbf{x},t)-\bar{\psi}_{ij}^B(\mathbf{x},t)\right)+\left(\eta_{ji}^A(\mathbf{x},t)+\bar{\eta}_{ij}^B(\mathbf{x},t)\right)*H(t)\right\}*e_i^A(\mathbf{x},t)*\bar{e}_j^B(\mathbf{x},t)\right\} \\
&\quad -\left\{e_i^A(\mathbf{x},t)*\bar{j}_i^{B-s}(\mathbf{x},t)+\bar{e}_i^B(\mathbf{x},t)*j_i^{A-s}(\mathbf{x},t)\right\} \\
&\quad -\left\{h_i^A(\mathbf{x},t)*\bar{k}_i^{B-s}(\mathbf{x},t)+\bar{h}_i^B(\mathbf{x},t)*k_i^{A-s}(\mathbf{x},t)\right\} \\
&\quad -\left\{e_i^A(\mathbf{x},t)*\bar{l}_i^{B-s}(\mathbf{x},t)+\bar{e}_i^B(\mathbf{x},t)*l_i^{A-s}(\mathbf{x},t)\right\}. \qquad (10.9)
\end{aligned}
$$

This agrees with equation (29) in de Hoop (1987), provided the conductivity response functions η_{ij} and displacement current sources l_i are neglected. Note that *all* quantities (i.e., electromagnetic fields, response functions, body sources) associated with state B are time-reversed.

The first group of terms on the right-hand-side of (10.9) depends on electromagnetic medium properties in the two states A and B, via the three constitutive relation response functions. If the three conditions

$$\varphi_{ij}^A(\mathbf{x},t)=\bar{\varphi}_{ji}^B(\mathbf{x},t), \qquad \psi_{ij}^A(\mathbf{x},t)=\bar{\psi}_{ji}^B(\mathbf{x},t), \qquad \eta_{ij}^A(\mathbf{x},t)=-\bar{\eta}_{ji}^B(\mathbf{x},t), \qquad (10.10a,b,c)$$

hold, then this first group of terms vanishes. In this case, media A and B are referred to as *time-reverse adjoints* of each other. Note the negative sign that appears in equation (10.10c), which differs from the analogous case with the *time-convolution* reciprocity theorem! If medium A is causal (i.e., its response functions vanish for $t < 0$), then these time-reverse adjoint conditions imply that medium B is non-causal. Hence, the reciprocity theorem of the time-correlation type is appropriate for numerical or computational media, rather than physical media (where causality must hold).

If media A and B are identical (in which case the state superscripts are omitted), then the time-reversed adjoint condition implies that the response function tensors satisfy:

$$\varphi_{ij}(\mathbf{x},t)=\bar{\varphi}_{ji}(\mathbf{x},t), \qquad \psi_{ij}(\mathbf{x},t)=\bar{\psi}_{ji}(\mathbf{x},t), \qquad \eta_{ij}(\mathbf{x},t)=-\bar{\eta}_{ji}(\mathbf{x},t). \qquad (10.11a,b,c)$$

Thus, the magnetic permeability and electric permittivity response tensors have a kind of "time reversed symmetry". The current conductivity response tensor is "time reversed anti-symmetric".

The second group of terms in the local reciprocity theorem (10.9) depends on electromagnetic body sources. Interestingly, these terms involve time convolutions of the "A" electromagnetic wavefield with the "B" electromagnetic body sources, and vice versa.

If the *unconventional* electromagnetic constitutive relations (2.3) (i.e., containing "cross response" functions with identifying superscripts) are substituted into the local interaction quantity expression (10.5), then *three* major groups of terms are obtained:

$$\Psi(\mathbf{x},t) = -\frac{\partial}{\partial t}\left\{ \left(\varphi_{ji}^{A-h}(\mathbf{x},t) - \overline{\varphi}_{ij}^{B-h}(\mathbf{x},t)\right) * h_i^A(\mathbf{x},t) * \overline{h}_j^B(\mathbf{x},t) \right\}$$
$$- \frac{\partial}{\partial t}\left\{ \left\{ \left(\psi_{ji}^{A-e}(\mathbf{x},t) - \overline{\psi}_{ij}^{B-e}(\mathbf{x},t)\right) + \left(\eta_{ji}^{A-e}(\mathbf{x},t) + \overline{\eta}_{ij}^{B-e}(\mathbf{x},t)\right) * H(t) \right\} * e_i^A(\mathbf{x},t) * \overline{e}_j^B(\mathbf{x},t) \right\}$$

$$- \frac{\partial}{\partial t}\left\{ \left(\varphi_{ji}^{A-e}(\mathbf{x},t) - \overline{\psi}_{ij}^{B-h}(\mathbf{x},t) + \overline{\eta}_{ij}^{B-h}(\mathbf{x},t) * H(t)\right) * e_i^A(\mathbf{x},t) * \overline{h}_j^B(\mathbf{x},t) \right\}$$
$$- \frac{\partial}{\partial t}\left\{ \left(\psi_{ji}^{A-h}(\mathbf{x},t) + \eta_{ji}^{A-h}(\mathbf{x},t) * H(t) - \overline{\varphi}_{ij}^{B-e}(\mathbf{x},t)\right) * h_i^A(\mathbf{x},t) * \overline{e}_j^B(\mathbf{x},t) \right\}$$

$$- \left\{ e_i^A(\mathbf{x},t) * \overline{j}_i^{B-s}(\mathbf{x},t) + \overline{e}_i^B(\mathbf{x},t) * j_i^{A-s}(\mathbf{x},t) \right\}$$
$$- \left\{ h_i^A(\mathbf{x},t) * \overline{k}_i^{B-s}(\mathbf{x},t) + \overline{h}_i^B(\mathbf{x},t) * k_i^{A-s}(\mathbf{x},t) \right\}$$
$$- \left\{ e_i^A(\mathbf{x},t) * \overline{l}_i^{B-s}(\mathbf{x},t) + \overline{e}_i^B(\mathbf{x},t) * l_i^{A-s}(\mathbf{x},t) \right\}. \tag{10.12}$$

The first group of terms involves convolutions between the magnetic fields of states A and B, as well as convolutions between the electric fields of states A and B. These terms are identical to the corresponding terms in equation (10.8). The third group of terms, involving convolutions between electromagnetic wavefields and body sources (of opposing states), is also identical to equation (10.8). The second group of terms is new, and involves convolutions between the electric field of state A and the magnetic field of state B, and vice versa. These terms vanish if the additional time-reversal adjoint conditions hold:

$$\varphi_{ij}^{A-e}(\mathbf{x},t) = \overline{\psi}_{ji}^{B-h}(\mathbf{x},t) - \overline{\eta}_{ji}^{B-h}(\mathbf{x},t) * H(t)\,, \tag{10.13a}$$

$$\psi_{ij}^{A-h}(\mathbf{x},t) + \eta_{ij}^{A-h}(\mathbf{x},t) * H(t) = \overline{\varphi}_{ji}^{B-e}(\mathbf{x},t)\,. \tag{10.13b}$$

If the two media are identical, then these reduce to the pair of expressions

$$\varphi_{ij}^{e}(\mathbf{x},t) = \overline{\psi}_{ji}^{h}(\mathbf{x},t) - \overline{\eta}_{ji}^{h}(\mathbf{x},t) * H(t)\,, \tag{10.14a}$$

$$\psi_{ij}^{h}(\mathbf{x},t) + \eta_{ij}^{h}(\mathbf{x},t) * H(t) = \overline{\varphi}_{ji}^{e}(\mathbf{x},t)\,. \tag{10.14b}$$

The *global* reciprocity theorem of the time-correlation type is obtained by integrating the local form (10.9) (or subsequently (10.12)) over the three-dimensional volume V bounded by closed surface S supporting electromagnetic wave propagation. Thus, working with equation (10.9) first gives (next page):

$$\int_V \left\{ e_i^A(\mathbf{x},t) * \bar{j}_i^{B-s}(\mathbf{x},t) + \bar{e}_i^B(\mathbf{x},t) * j_i^{A-s}(\mathbf{x},t) \right\} dV(\mathbf{x})$$

$$+ \int_V \left\{ h_i^A(\mathbf{x},t) * \bar{k}_i^{B-s}(\mathbf{x},t) + \bar{h}_i^B(\mathbf{x},t) * k_i^{A-s}(\mathbf{x},t) \right\} dV(\mathbf{x})$$

$$+ \int_V \left\{ e_i^A(\mathbf{x},t) * \bar{l}_i^{B-s}(\mathbf{x},t) + \bar{e}_i^B(\mathbf{x},t) * l_i^{A-s}(\mathbf{x},t) \right\} dV(\mathbf{x})$$

$$+ \int_V \frac{\partial}{\partial t} \left\{ \left(\varphi_{ji}^A(\mathbf{x},t) - \bar{\varphi}_{ij}^B(\mathbf{x},t) \right) * h_i^A(\mathbf{x},t) * \bar{h}_j^B(\mathbf{x},t) \right\} dV(\mathbf{x})$$

$$+ \int_V \frac{\partial}{\partial t} \left\{ \left\{ \left(\psi_{ji}^A(\mathbf{x},t) - \bar{\psi}_{ij}^B(\mathbf{x},t) \right) + \left(\eta_{ji}^A(\mathbf{x},t) + \bar{\eta}_{ij}^B(\mathbf{x},t) \right) * H(t \right\} * e_i^A(\mathbf{x},t) * \bar{e}_j^B(\mathbf{x},t) \right\} dV(\mathbf{x})$$

$$= -\varepsilon_{ijk} \int_V \frac{\partial}{\partial x_i} \left\{ e_j^A(\mathbf{x},t) * \bar{h}_k^B(\mathbf{x},t) + \bar{e}_j^B(\mathbf{x},t) * h_k^A(\mathbf{x},t) \right\} dV(\mathbf{x}) \,. \qquad (10.15)$$

The right-hand-side of equation (10.15) is now simplified in exactly the same manner as with the time-convolution reciprocity theorem developed in chapter 3. In particular, we use the generalized divergence theorem (2.7) and the interface conditions (3.17) involving jump discontinuities of the electromagnetic wavefield. The result is

$$\int_V \left\{ e_i^A(\mathbf{x},t) * \bar{j}_i^{B-s}(\mathbf{x},t) + \bar{e}_i^B(\mathbf{x},t) * j_i^{A-s}(\mathbf{x},t) \right\} dV(\mathbf{x})$$

$$+ \int_V \left\{ h_i^A(\mathbf{x},t) * \bar{k}_i^{B-s}(\mathbf{x},t) + \bar{h}_i^B(\mathbf{x},t) * k_i^{A-s}(\mathbf{x},t) \right\} dV(\mathbf{x})$$

$$+ \int_V \left\{ e_i^A(\mathbf{x},t) * \bar{l}_i^{B-s}(\mathbf{x},t) + \bar{e}_i^B(\mathbf{x},t) * l_i^{A-s}(\mathbf{x},t) \right\} dV(\mathbf{x})$$

$$+ \int_V \frac{\partial}{\partial t} \left\{ \left(\varphi_{ji}^A(\mathbf{x},t) - \bar{\varphi}_{ij}^B(\mathbf{x},t) \right) * h_i^A(\mathbf{x},t) * \bar{h}_j^B(\mathbf{x},t) \right\} dV(\mathbf{x})$$

$$+ \int_V \frac{\partial}{\partial t} \left\{ \left\{ \left(\psi_{ji}^A(\mathbf{x},t) - \bar{\psi}_{ij}^B(\mathbf{x},t) \right) + \left(\eta_{ji}^A(\mathbf{x},t) + \bar{\eta}_{ij}^B(\mathbf{x},t) \right) * H(t) \right\} * e_i^A(\mathbf{x},t) * \bar{e}_j^B(\mathbf{x},t) \right\} dV(\mathbf{x})$$

$$+ \sum_{n=1}^{N} \int_{S_n} \left\{ \left\langle e_i^A(\mathbf{x},t) \right\rangle * \bar{s}_i^B(\mathbf{x},t) + \left\langle \bar{e}_i^B(\mathbf{x},t) \right\rangle * s_i^A(\mathbf{x},t) \right\} dS(\mathbf{x})$$

$$+ \varepsilon_{ijk} \int_S m_i(\mathbf{x}) \left\{ e_j^A(\mathbf{x},t) * \bar{h}_k^B(\mathbf{x},t) + \bar{e}_j^B(\mathbf{x},t) * h_k^A(\mathbf{x},t) \right\} dS(\mathbf{x}) = 0 \,. \qquad (10.16)$$

Recall that $\mathbf{m}(\mathbf{x})$ is an outward-directed unit normal to the outer bounding surface S, and that summation over repeated indices is implied. Also, the bracket notation $\langle\ \rangle$ refers to the average value of a quantity as the interface is approached from each side. Each major term has physical dimension "energy", with SI unit J. This expression corresponds to equation (30) in de Hoop (1987) when i) displacement current body sources $\mathbf{l}^s(\mathbf{x},t)$ are neglected, ii) conductivity response functions $\eta_{ij}(\mathbf{x},t)$ are neglected, and iii) surface currents $\mathbf{s}(\mathbf{x},t)$ are neglected. De Hoop (1992) does not give a time-correlation reciprocity theorem.

Global reciprocity theorem (10.16) is written in terms of convolutions and time-reversed functions associated with state B. It can be re-written in terms of cross-correlations and without time-reversed state B functions as follows:

$$\begin{aligned}
&\int_V \left\{ e_i^A(\mathbf{x},t) \otimes j_i^{B-s}(\mathbf{x},t) + j_i^{A-s}(\mathbf{x},t) \otimes e_i^B(\mathbf{x},t) \right\} dV(\mathbf{x}) \\
&+ \int_V \left\{ h_i^A(\mathbf{x},t) \otimes k_i^{B-s}(\mathbf{x},t) + k_i^{A-s}(\mathbf{x},t) \otimes h_i^B(\mathbf{x},t) \right\} dV(\mathbf{x}) \\
&+ \int_V \left\{ e_i^A(\mathbf{x},t) \otimes l_i^{B-s}(\mathbf{x},t) + l_i^{A-s}(\mathbf{x},t) \otimes e_i^B(\mathbf{x},t) \right\} dV(\mathbf{x}) \\
&+ \int_V \frac{\partial}{\partial t} \left\{ \left(\varphi_{ji}^A(\mathbf{x},t) * h_i^A(\mathbf{x},t) \right) \otimes h_j^B(\mathbf{x},t) - h_i^A(\mathbf{x},t) \otimes \left(\varphi_{ij}^B(\mathbf{x},t) * h_j^B(\mathbf{x},t) \right) \right\} dV(\mathbf{x}) \\
&+ \int_V \frac{\partial}{\partial t} \left\{ \left(\left(\psi_{ji}^A(\mathbf{x},t) + \eta_{ji}^A(\mathbf{x},t) * H(t) \right) * e_i^A(\mathbf{x},t) \right) \otimes e_j^B(\mathbf{x},t) \right. \\
&\qquad\qquad \left. - e_i^A(\mathbf{x},t) \otimes \left(\left(\psi_{ij}^B(\mathbf{x},t) - \eta_{ij}^B(\mathbf{x},t) * \overline{H}(t) \right) * e_j^B(\mathbf{x},t) \right) \right\} dV(\mathbf{x}) \\
&+ \sum_{n=1}^{N} \int_{S_n} \left\{ \left\langle e_i^A(\mathbf{x},t) \right\rangle \otimes s_i^B(\mathbf{x},t) + s_i^A(\mathbf{x},t) \otimes \left\langle e_i^B(\mathbf{x},t) \right\rangle \right\} dS(\mathbf{x}) \\
&+ \varepsilon_{ijk} \int_S m_i(\mathbf{x}) \left\{ e_j^A(\mathbf{x},t) \otimes h_k^B(\mathbf{x},t) + h_k^A(\mathbf{x},t) \otimes e_j^B(\mathbf{x},t) \right\} dS(\mathbf{x}) = 0 . \qquad (10.17)
\end{aligned}$$

However, it is not obvious that this form is particularly useful. Finally, under the simplifying assumptions i) medium parameters A and B satisfy time-reversed adjoint conditions, ii) no surface currents, and iii) EM wavefields A and B satisfy radiation conditions as the outer bounding surface S is removed to infinity, then the global reciprocity theorem of the time-correlation type reduces to the simpler form (next page):

$$\int_V \left\{ e_i^A(\mathbf{x},t) * \bar{j}_i^{B-s}(\mathbf{x},t) + \bar{e}_i^B(\mathbf{x},t) * j_i^{A-s}(\mathbf{x},t) \right\} dV(\mathbf{x})$$

$$+ \int_V \left\{ h_i^A(\mathbf{x},t) * \bar{k}_i^{B-s}(\mathbf{x},t) + \bar{h}_i^B(\mathbf{x},t) * k_i^{A-s}(\mathbf{x},t) \right\} dV(\mathbf{x})$$

$$+ \int_V \left\{ e_i^A(\mathbf{x},t) * \bar{l}_i^{B-s}(\mathbf{x},t) + \bar{e}_i^B(\mathbf{x},t) * l_i^{A-s}(\mathbf{x},t) \right\} dV(\mathbf{x}) = 0 . \tag{10.18}$$

This form involves only body sources of the electromagnetic wavefields of states A and B. The reason for writing the reciprocity theorem in this simplified manner is to compare with the analogous results developed by Welch (1960). de Hoop (1987) states that Welch's (1960) reciprocity theorem is of the time-correlation type. [Interestingly, de Hoop (1987) also asserts that Welch's reciprocity relation applies to homogeneous media, and is derived by using Fourier transforms with respect to time. However, these two statements are incorrect. Welch's "Theorem II" is derived entirely in the space-time domain. Moreover, Welch considers permittivity $\varepsilon(\mathbf{x})$ and permeability $\mu(\mathbf{x})$ to be "simple scalar functions of position". But, current conductivity $\sigma(\mathbf{x})$ is assumed to vanish, which commonly implies the electromagnetic fields propagate within a vacuum of uniform permittivity ε_0 and permeability μ_0. Nevertheless, there *might* be media with vanishing conductivity $\sigma = 0$ and spatially variable permittivity $\varepsilon(\mathbf{x})$ and permeability $\mu(\mathbf{x})$.]

Neglecting displacement current sources in (10.18), and writing the temporal convolutions as integrals yields

$$\int_V \left\{ \int_{-\infty}^{+\infty} e_i^A(\mathbf{x},\tau) \bar{j}_i^{B-s}(\mathbf{x},t-\tau)\, d\tau + \int_{-\infty}^{+\infty} \bar{e}_i^B(\mathbf{x},t-\tau) j_i^{A-s}(\mathbf{x},\tau)\, d\tau \right\} dV(\mathbf{x})$$

$$+ \int_V \left\{ \int_{-\infty}^{+\infty} h_i^A(\mathbf{x},\tau) \bar{k}_i^{B-s}(\mathbf{x},t-\tau)\, d\tau + \int_{-\infty}^{+\infty} \bar{h}_i^B(\mathbf{x},t-\tau) k_i^{A-s}(\mathbf{x},\tau)\, d\tau \right\} dV(\mathbf{x}) = 0 .$$

Our time-correlation reciprocity theorem is a function of time t, whereas Welch's (1960) theorem (his equation (18)) is independent of time. Thus, setting $t = 0$ in the above expression gives

$$\int_V \left\{ \int_{-\infty}^{+\infty} e_i^A(\mathbf{x},\tau) \bar{j}_i^{B-s}(\mathbf{x},-\tau)\, d\tau + \int_{-\infty}^{+\infty} \bar{e}_i^B(\mathbf{x},-\tau) j_i^{A-s}(\mathbf{x},\tau)\, d\tau \right\} dV(\mathbf{x})$$

$$+ \int_V \left\{ \int_{-\infty}^{+\infty} h_i^A(\mathbf{x},\tau) \bar{k}_i^{B-s}(\mathbf{x},-\tau)\, d\tau + \int_{-\infty}^{+\infty} \bar{h}_i^B(\mathbf{x},-\tau) k_i^{A-s}(\mathbf{x},\tau)\, d\tau \right\} dV(\mathbf{x}) = 0 .$$

However, a doubly time-reversed function equals the original function, as per $\bar{\bar{x}}(t) = x(t)$. Thus, the above expression is re-written as

$$\int_V \left\{ \int_{-\infty}^{+\infty} e_i^A(\mathbf{x},\tau) j_i^{B-s}(\mathbf{x},\tau)\, d\tau + \int_{-\infty}^{+\infty} e_i^B(\mathbf{x},\tau) j_i^{A-s}(\mathbf{x},\tau)\, d\tau \right\} dV(\mathbf{x})$$

$$+ \int_V \left\{ \int_{-\infty}^{+\infty} h_i^A(\mathbf{x},\tau) k_i^{B-s}(\mathbf{x},\tau)\, d\tau + \int_{-\infty}^{+\infty} h_i^B(\mathbf{x},\tau) k_i^{A-s}(\mathbf{x},\tau)\, d\tau \right\} dV(\mathbf{x}) = 0\,.$$

This form bears a strong resemblance to Welch's (1960) equation (18)! The only difference is that he superposes a tilde "~" on the electromagnetic wavefields (but *not* the sources) of state B. Welch indicates that the tilde notation denotes "time-advanced" wavefields, which is not the same as "time-reversed" wavefields (our superposed bar "—" notation).

11.0 APPENDIX C: POTENTIAL FUNCTION FORMULATION

The reciprocity theorems developed in this report pertain to the electric field vector $\mathbf{e}(\mathbf{x},t)$ and the magnetic field vector $\mathbf{h}(\mathbf{x},t)$. However, it is well known that solution of certain electromagnetic problems is facilitated by introducing auxiliary functions called "potential functions" from which these physical fields may be derived (e.g., by spatial and temporal differentiations). This situation motivates investigating whether reciprocity theorems for the potential functions can be derived. Welch (1960) and Bojarski (1983) develop reciprocity relations involving the potential functions, but under the rather restricting condition that the medium supporting electromagnetic wave propagation is a homogeneous and isotropic wholespace with vanishing conductivity. In this Appendix, an attempt is made to generalize their results to heterogeneous media characterized by linear, time-invariant, and local constitutive relations. But first, a simple development of the potential formulation of electromagnetism is outlined. The approach is similarly restricted to an isotropic medium. Particular emphasis is placed on developing space-time, rather than space-frequency, mathematical formulae.

11.1 Maxwell's Equations and Constitutive Relations

The three fundamental (Maxwell?) equations governing electromagnetic phenomena are

$$\frac{\partial \mathbf{b}(\mathbf{x},t)}{\partial t} + \mathbf{curl}\ \mathbf{e}(\mathbf{x},t) = \mathbf{0}, \qquad \text{(Faraday law)} \qquad (11.1a)$$

$$\frac{\partial \mathbf{d}(\mathbf{x},t)}{\partial t} + \mathbf{j}(\mathbf{x},t) - \mathbf{curl}\ \mathbf{h}(\mathbf{x},t) = \mathbf{0}, \qquad \text{(Ampere-Maxwell law)} \qquad (11.1b)$$

$$\frac{\partial \theta(\mathbf{x},t)}{\partial t} + \mathrm{div}\ \mathbf{j}(\mathbf{x},t) = 0. \qquad \text{(Charge continuity law)} \qquad (11.1c)$$

The six dependent variables appearing in these equations are named

$\mathbf{e}(\mathbf{x},t)$ – electric vector (SI unit: V/m),

$\mathbf{h}(\mathbf{x},t)$ – magnetic vector (A/m),

$\mathbf{d}(\mathbf{x},t)$ – electric flux density ((A-s)/m^2),

$\mathbf{b}(\mathbf{x},t)$ – magnetic flux density ((V-s)/m^2),

$\mathbf{j}(\mathbf{x},t)$ – conduction current density (A/m^2 = (A-m)/m^3),

$\theta(\mathbf{x},t)$ – mobile charge density (C/m^3 = (A-s)/m^3).

From the above three expressions, it is straightforward to derive the two remaining equations of Maxwell electromagnetic theory (as in Tai (1971) or Aldridge (2013)). The two Gauss laws are

$$\mathrm{div}\ \mathbf{b}(\mathbf{x},t) = 0, \qquad \text{(Gauss magnetic law)} \qquad (11.1d)$$

$$\mathrm{div}\ \mathbf{d}(\mathbf{x},t) - \theta(\mathbf{x},t) = 0. \qquad \text{(Gauss electric law)} \qquad (11.1e)$$

In our opinion, all quantities appearing in Maxwell's equations are dependent variables. In particular, body sources of electromagnetic wavefields do not appear (as they apparently *do* in equations (15) in Welch (1960)). Both medium properties and body sources are introduced via constitutive relations. For the present, we consider the medium supporting the electromagnetic fields to be linear, time-invariant, local, instantaneously-reacting, and isotropic. Hence, the relevant constitutive relations, with body source terms, are

$$\mathbf{b}(\mathbf{x},t) = \mu(\mathbf{x})\,\mathbf{h}(\mathbf{x},t) + \mathbf{b}_s(\mathbf{x},t)\,, \tag{11.2a}$$

$$\mathbf{d}(\mathbf{x},t) = \varepsilon(\mathbf{x})\,\mathbf{e}(\mathbf{x},t) + \mathbf{d}_s(\mathbf{x},t)\,, \tag{11.2b}$$

$$\mathbf{j}(\mathbf{x},t) = \sigma(\mathbf{x})\,\mathbf{e}(\mathbf{x},t) + \mathbf{j}_s(\mathbf{x},t)\,, \tag{11.2c}$$

where subscript "*s*" denotes a body source of the pertaining quantity. The medium is characterized by the three scalar parameters

$\varepsilon(\mathbf{x})$: electric permittivity, SI unit: $\dfrac{\text{C/V}}{\text{m}} = \dfrac{\text{A/m}}{\text{V/s}} = \dfrac{\text{F}}{\text{m}}$,

$\mu(\mathbf{x})$: magnetic permeability, SI unit: $\dfrac{\text{N}}{\text{A}^2} = \dfrac{\text{V/m}}{\text{A/s}} = \dfrac{\text{T}}{\text{A/m}} = \dfrac{\text{H}}{\text{m}}$,

$\sigma(\mathbf{x})$: current conductivity, SI unit: $\dfrac{\text{A/V}}{\text{m}} = \dfrac{\text{S}}{\text{m}}$,

We take the electric permittivity and magnetic permeability to be intrinsically positive (and bounded from below by the corresponding free space values ε_0 and μ_0). However, the current conductivity may equal zero, as in an absolute vacuum containing no electric charges.

11.2 Potential Formulation

The Gauss magnetic law (11.1d) suggests that the magnetic induction vector **b(x,**t**)** may be expressed as the **curl** of a vector potential function via

$$\mathbf{b}(\mathbf{x},t) = \mathbf{curl}\;\mathbf{a}(\mathbf{x},t)\,, \tag{11.3}$$

where $\mathbf{a}(\mathbf{x},t)$ has SI unit T-m = (V-s)/m = (V/m)/Hz. Substituting this into the Faraday law (11.1a) yields

$$\mathbf{curl}\left[\frac{\partial \mathbf{a}(\mathbf{x},t)}{\partial t} + \mathbf{e}(\mathbf{x},t)\right] = \mathbf{0}\,.$$

In turn, this equation suggests that the quantity in square brackets can be expressed as the gradient of a scalar potential function

$$\frac{\partial \mathbf{a}(\mathbf{x},t)}{\partial t} + \mathbf{e}(\mathbf{x},t) = -\mathbf{grad}\;\phi(\mathbf{x},t)\,,$$

where a negative sign is inserted on the right side by convention. [In the electrostatic case where $\partial\mathbf{a}/\partial t = \mathbf{0}$, the electric force on a positive charge q is $\mathbf{f} = q\mathbf{e}$, and this should point in the direction of *decreasing* electric potential ϕ.] Scalar potential function $\phi(\mathbf{x},t)$ has SI unit V. Solving for the electric vector yields

$$\mathbf{e}(\mathbf{x},t) = -\frac{\partial\mathbf{a}(\mathbf{x},t)}{\partial t} - \mathbf{grad}\,\phi(\mathbf{x},t)\,. \tag{11.4}$$

Equations (11.3) and (11.4) indicate that the magnetic flux density vector **b** and electric vector **e** may be obtained from the vector and scalar potential functions. It is emphasized that, at this point, the development of the potential formulation does *not* rely on any assumption of homogeneity or isotropy of the three medium parameters.

11.3 Ampere-Maxwell Law in Potentials

We have used the Gauss magnetic law (11.1d) and the Faraday law (11.1a) to express the magnetic flux density $\mathbf{b}(\mathbf{x},t)$ and the electric vector $\mathbf{e}(\mathbf{x},t)$ in terms of potential functions. We now attempt to re-formulate the Ampere-Maxwell law (11.1b) similarly. Substituting the isotropic constitutive relations (11.2a,b,c) into the Ampere-Maxwell law yields

$$\varepsilon(\mathbf{x})\frac{\partial\mathbf{e}(\mathbf{x},t)}{\partial t} + \sigma(\mathbf{x})\mathbf{e}(\mathbf{x},t) - \mathbf{curl}\left(\mu(\mathbf{x})^{-1}\mathbf{b}(\mathbf{x},t)\right) = -\left(\mathbf{j}_s(\mathbf{x},t) + \frac{\partial\mathbf{d}_s(\mathbf{x},t)}{\partial t}\right) - \mathbf{curl}\left(\mu(\mathbf{x})^{-1}\mathbf{b}_s(\mathbf{x},t)\right).$$

Next, use the vector differential identity $\mathbf{curl}\left(\alpha\mathbf{f}\right) = \alpha\,\mathbf{curl}\,\mathbf{f} + \mathbf{grad}\,\alpha \times \mathbf{f}$ to obtain

$$\begin{aligned}\varepsilon(\mathbf{x})\frac{\partial\mathbf{e}(\mathbf{x},t)}{\partial t} &+ \sigma(\mathbf{x})\mathbf{e}(\mathbf{x},t) - \mu(\mathbf{x})^{-1}\mathbf{curl}\,\mathbf{b}(\mathbf{x},t) - \mathbf{grad}\left(\mu(\mathbf{x})^{-1}\right)\times\mathbf{b}(\mathbf{x},t)\\ &= -\left(\mathbf{j}_s(\mathbf{x},t) + \frac{\partial\mathbf{d}_s(\mathbf{x},t)}{\partial t}\right) - \mu(\mathbf{x})^{-1}\mathbf{curl}\,\mathbf{b}_s(\mathbf{x},t) - \mathbf{grad}\left(\mu(\mathbf{x})^{-1}\right)\times\mathbf{b}_s(\mathbf{x},t)\,.\end{aligned}$$

However, $\mathbf{grad}\left(\mu(\mathbf{x})^{-1}\right) = -\mu(\mathbf{x})^{-1}\mathbf{grad}\ln\kappa_\mu(\mathbf{x})$ where $\kappa_\mu(\mathbf{x}) \equiv \mu(\mathbf{x})/\mu_0$ is the relative magnetic permeability (a dimensionless quantity). The above equation becomes

$$\begin{aligned}\varepsilon(\mathbf{x})\mu(\mathbf{x})\frac{\partial\mathbf{e}(\mathbf{x},t)}{\partial t} &+ \sigma(\mathbf{x})\mu(\mathbf{x})\mathbf{e}(\mathbf{x},t) - \mathbf{curl}\,\mathbf{b}(\mathbf{x},t) + \mathbf{grad}\ln\kappa_\mu(\mathbf{x})\times\mathbf{b}(\mathbf{x},t)\\ &= -\mu(\mathbf{x})\left(\mathbf{j}_s(\mathbf{x},t) + \frac{\partial\mathbf{d}_s(\mathbf{x},t)}{\partial t}\right) - \mathbf{curl}\,\mathbf{b}_s(\mathbf{x},t) + \mathbf{grad}\ln\kappa_\mu(\mathbf{x})\times\mathbf{b}_s(\mathbf{x},t)\,,\end{aligned}$$

where multiplying through by $\mu(\mathbf{x})$ results in a simplification. Substitute equations (11.3) and (11.4) for the magnetic vector $\mathbf{b}(\mathbf{x},t)$ and electric vector $\mathbf{e}(\mathbf{x},t)$ in terms of potential functions, and use the vector differential operator identity $\mathbf{curl}\,\mathbf{curl} = \mathbf{grad}\,\mathrm{div} - \nabla^2$ to obtain (next page):

$$\nabla^2\mathbf{a}(\mathbf{x},t)-\sigma(\mathbf{x})\mu(\mathbf{x})\frac{\partial\mathbf{a}(\mathbf{x},t)}{\partial t}-\varepsilon(\mathbf{x})\mu(\mathbf{x})\frac{\partial^2\mathbf{a}(\mathbf{x},t)}{\partial t^2}-\mathbf{grad}\,\mathrm{div}\,\mathbf{a}(\mathbf{x},t)+\mathbf{grad}\ln\kappa_\mu(\mathbf{x})\times\mathbf{curl}\,\mathbf{a}(\mathbf{x},t)$$
$$-\sigma(\mathbf{x})\mu(\mathbf{x})\,\mathbf{grad}\,\phi(\mathbf{x},t)-\varepsilon(\mathbf{x})\mu(\mathbf{x})\,\mathbf{grad}\left(\frac{\partial\phi(\mathbf{x},t)}{\partial t}\right)$$
$$=-\mu(\mathbf{x})\left(\mathbf{j}_s(\mathbf{x},t)+\frac{\partial\mathbf{d}_s(\mathbf{x},t)}{\partial t}\right)-\mathbf{curl}\,\mathbf{b}_s(\mathbf{x},t)+\mathbf{grad}\ln\kappa_\mu(\mathbf{x})\times\mathbf{b}_s(\mathbf{x},t).$$

This may be written in an equivalent, and perhaps more illustrative form, by grouping terms as

$$\left\{\nabla^2\mathbf{a}(\mathbf{x},t)-\sigma(\mathbf{x})\mu(\mathbf{x})\frac{\partial\mathbf{a}(\mathbf{x},t)}{\partial t}-\varepsilon(\mathbf{x})\mu(\mathbf{x})\frac{\partial^2\mathbf{a}(\mathbf{x},t)}{\partial t^2}\right\}$$
$$-\mathbf{grad}\left\{\mathrm{div}\,\mathbf{a}(\mathbf{x},t)+\sigma(\mathbf{x})\mu(\mathbf{x})\phi(\mathbf{x},t)+\varepsilon(\mathbf{x})\mu(\mathbf{x})\frac{\partial\phi(\mathbf{x},t)}{\partial t}\right\}$$
$$+\mathbf{grad}\left(\sigma(\mathbf{x})\mu(\mathbf{x})\right)\phi(\mathbf{x},t)+\mathbf{grad}\left(\varepsilon(\mathbf{x})\mu(\mathbf{x})\right)\frac{\partial\phi(\mathbf{x},t)}{\partial t}+\mathbf{grad}\ln\kappa_\mu(\mathbf{x})\times\mathbf{curl}\,\mathbf{a}(\mathbf{x},t)$$
$$=-\mu(\mathbf{x})\left(\mathbf{j}_s(\mathbf{x},t)+\frac{\partial\mathbf{d}_s(\mathbf{x},t)}{\partial t}\right)-\mathbf{curl}\,\mathbf{b}_s(\mathbf{x},t)+\mathbf{grad}\ln\kappa_\mu(\mathbf{x})\times\mathbf{b}_s(\mathbf{x},t).$$

Define the *infinite frequency phase speed* and the *transition frequency* as

$$c_\infty(\mathbf{x})=\frac{1}{\sqrt{\varepsilon(\mathbf{x})\mu(\mathbf{x})}},\qquad\qquad \omega_t(\mathbf{x})=\frac{\sigma(\mathbf{x})}{\varepsilon(\mathbf{x})},\qquad\qquad (11.5\text{a,b})$$

respectively. Each depends on spatial position $\mathbf{x}$. Then we obtain the alternate form

$$\left\{\nabla^2\mathbf{a}(\mathbf{x},t)-\frac{\omega_t(\mathbf{x})}{c_\infty^2(\mathbf{x})}\frac{\partial\mathbf{a}(\mathbf{x},t)}{\partial t}-\frac{1}{c_\infty^2(\mathbf{x})}\frac{\partial^2\mathbf{a}(\mathbf{x},t)}{\partial t^2}\right\}-\mathbf{grad}\left\{\mathrm{div}\,\mathbf{a}(\mathbf{x},t)+\frac{\omega_t(\mathbf{x})}{c_\infty^2(\mathbf{x})}\phi(\mathbf{x},t)+\frac{1}{c_\infty^2(\mathbf{x})}\frac{\partial\phi(\mathbf{x},t)}{\partial t}\right\}$$
$$+\mathbf{grad}\left(\frac{\omega_t(\mathbf{x})}{c_\infty^2(\mathbf{x})}\right)\phi(\mathbf{x},t)+\mathbf{grad}\left(\frac{1}{c_\infty^2(\mathbf{x})}\right)\frac{\partial\phi(\mathbf{x},t)}{\partial t}+\mathbf{grad}\ln\kappa_\mu(\mathbf{x})\times\mathbf{curl}\,\mathbf{a}(\mathbf{x},t)$$
$$=-\mu(\mathbf{x})\left(\mathbf{j}_s(\mathbf{x},t)+\frac{\partial\mathbf{d}_s(\mathbf{x},t)}{\partial t}\right)-\mathbf{curl}\,\mathbf{b}_s(\mathbf{x},t)+\mathbf{grad}\ln\kappa_\mu(\mathbf{x})\times\mathbf{b}_s(\mathbf{x},t).$$

Now, it is straightforward to demonstrate that

$$\mathbf{grad}\left(\frac{1}{c_\infty^2}\right)=\frac{1}{c_\infty^2}\,\mathbf{grad}\left(\ln\kappa_\varepsilon\kappa_\mu\right),$$

where $\kappa_\varepsilon(\mathbf{x})=\varepsilon(\mathbf{x})/\varepsilon_0$ is the relative electric permittivity. Also

$$\mathbf{grad}\left(\frac{\omega_t}{c_\infty^2}\right) = \frac{1}{c_\infty^2}\left[\mathbf{grad}\;\omega_t + \omega_t\;\mathbf{grad}\left(\ln \kappa_\varepsilon \kappa_\mu\right)\right].$$

Introducing these into the above expression yields

$$\left\{\nabla^2 \mathbf{a}(\mathbf{x},t) - \frac{\omega_t(\mathbf{x})}{c_\infty^2(\mathbf{x})}\frac{\partial \mathbf{a}(\mathbf{x},t)}{\partial t} - \frac{1}{c_\infty^2(\mathbf{x})}\frac{\partial^2 \mathbf{a}(\mathbf{x},t)}{\partial t^2}\right\} - \mathbf{grad}\left\{\operatorname{div}\mathbf{a}(\mathbf{x},t) + \frac{\omega_t(\mathbf{x})}{c_\infty^2(\mathbf{x})}\phi(\mathbf{x},t) + \frac{1}{c_\infty^2(\mathbf{x})}\frac{\partial \phi(\mathbf{x},t)}{\partial t}\right\}$$

$$+ \frac{\mathbf{grad}\;\omega_t(\mathbf{x})}{c_\infty^2(\mathbf{x})}\phi(\mathbf{x},t) + \frac{\mathbf{grad}\ln\left(\kappa_\varepsilon(\mathbf{x})\kappa_\mu(\mathbf{x})\right)}{c_\infty^2(\mathbf{x})}\left(\frac{\partial \phi(\mathbf{x},t)}{\partial t} + \omega_t(\mathbf{x})\phi(\mathbf{x},t)\right) + \mathbf{grad}\ln\kappa_\mu(\mathbf{x}) \times \mathbf{curl}\;\mathbf{a}(\mathbf{x},t)$$

$$= -\mu(\mathbf{x})\left(\mathbf{j}_s(\mathbf{x},t) + \frac{\partial \mathbf{d}_s(\mathbf{x},t)}{\partial t}\right) - \mathbf{curl}\;\mathbf{b}_s(\mathbf{x},t) + \mathbf{grad}\ln\kappa_\mu(\mathbf{x}) \times \mathbf{b}_s(\mathbf{x},t). \qquad (11.6)$$

Equation (11.6) is an expression of the Ampere-Maxwell law in terms of the vector and scalar potential functions. Clearly, it is a rather complicated second-order (in space and time) inhomogeneous linear partial differential equation with variable (i.e., space-dependent) coefficients. If the medium is homogeneous in all three electromagnetic medium parameters σ, ε, and μ, then the second line (plus a body source term on the right-hand-side) vanishes. However, equation (11.6) is a *single* PDE containing the *two* unknown potential functions. To proceed further, another equation linking these two dependent variables is required.

11.4 Lorenz Gauge Condition

The potential formulation of electromagnetism is not unique, in that the vector and scalar potentials may be transformed without affecting the physical fields $\mathbf{e}(\mathbf{x},t)$ and $\mathbf{b}(\mathbf{x},t)$. For example, consider two new potential functions defined as follows

$$\mathbf{a}' = \mathbf{a} - \mathbf{grad}\;\psi, \qquad \phi' = \phi + \frac{\partial \psi}{\partial t},$$

where $\psi(\mathbf{x},t)$ (SI unit V/Hz) is a differentiable, but otherwise arbitrary, function of position $\mathbf{x}$ and time t. [For notational simplicity, we suppress explicit dependence on $\mathbf{x}$ and t immediately below.] The magnetic and electric vectors implied by these potentials are

$$\mathbf{b}' = \mathbf{curl}\;\mathbf{a}' = \mathbf{curl}\left(\mathbf{a} - \mathbf{grad}\;\phi\right) = \mathbf{curl}\;\mathbf{a} = \mathbf{b},$$

$$\mathbf{e}' = -\frac{\partial \mathbf{a}'}{\partial t} - \mathbf{grad}\;\phi' = -\frac{\partial}{\partial t}\left(\mathbf{a} - \mathbf{grad}\;\psi\right) - \mathbf{grad}\left(\phi + \frac{\partial \psi}{\partial t}\right) = -\frac{\partial \mathbf{a}}{\partial t} - \mathbf{grad}\;\phi = \mathbf{e}.$$

Thus, the magnetic and electric vectors remain invariant under this potential function transformation.

In order to treat the issue of non-uniqueness in the choice of electromagnetic potential functions, a *gauge condition* is usually introduced into the formalism. It is not our purpose here to rigorously examine gauge theory. Rather, we merely state the particular gauge condition that simplifies the problem at hand. The so-called *Lorenz* (not Lorentz!) *gauge* imposes the condition that the argument of the gradient operator in the first line of equation (11.6) vanishes, yielding

$$\frac{\partial\phi(\mathbf{x},t)}{\partial t}+\omega_t(\mathbf{x})\phi(\mathbf{x},t)+c_\infty^2(\mathbf{x})\operatorname{div}\mathbf{a}(\mathbf{x},t)=0\,. \tag{11.7}$$

For $\sigma(\mathbf{x})$ = 0 (implying the transition frequency $\omega_t(\mathbf{x})$ vanishes) expression (11.7) reduces to the gauge condition stated by Welch (1960). [However, Welch also apparently takes $\mu(\mathbf{x}) = \mu_0$ and $\varepsilon(\mathbf{x}) = \varepsilon_0$ (i.e., the free space values) implying that the infinite-frequency phase speed $c_\infty(\mathbf{x}) = 1\big/\sqrt{\varepsilon_0\mu_0} = c_{vac}$ equals the (fixed) speed of light in vacuum.] Introducing the gauge condition into equation (11.6) gives

$$\begin{aligned}\nabla^2\mathbf{a}(\mathbf{x},t)&-\frac{\omega_t(\mathbf{x})}{c_\infty^2(\mathbf{x})}\frac{\partial\mathbf{a}(\mathbf{x},t)}{\partial t}-\frac{1}{c_\infty^2(\mathbf{x})}\frac{\partial^2\mathbf{a}(\mathbf{x},t)}{\partial t^2}\\ &-\mathbf{grad}\ln\big(\kappa_\varepsilon(\mathbf{x})\kappa_\mu(\mathbf{x})\big)\operatorname{div}\mathbf{a}(\mathbf{x},t)+\mathbf{grad}\ln\kappa_\mu(\mathbf{x})\times\mathbf{curl}\,\mathbf{a}(\mathbf{x},t)+\frac{\mathbf{grad}\,\omega_t(\mathbf{x})}{c_\infty^2(\mathbf{x})}\phi(\mathbf{x},t)\\ &=-\mu(\mathbf{x})\left(\mathbf{j}_s(\mathbf{x},t)+\frac{\partial\mathbf{d}_s(\mathbf{x},t)}{\partial t}\right)-\mathbf{curl}\,\mathbf{b}_s(\mathbf{x},t)+\mathbf{grad}\ln\kappa_\mu(\mathbf{x})\times\mathbf{b}_s(\mathbf{x},t)\,.\end{aligned} \tag{11.8}$$

Equations (11.7) and (11.8) are a pair of coupled linear PDEs containing the two potential functions $\mathbf{a}(\mathbf{x},t)$ and $\phi(\mathbf{x},t)$. In principle, the coupled equations can be solved after medium parameters, body sources, and boundary and initial conditions are prescribed.

We remark that the *space-time* equations (11.7) and (11.8) above are compatible with the *space-frequency* equation (4) in Weiss (2013), after some simplifying assumptions (mainly $\mu(\mathbf{x}) = \mu_0$) and many notational changes are made.

11.5 Homogeneous Medium

For a homogeneous medium (in all three material parameters σ, ε, and μ) equation (11.8) reduces to

$$\nabla^2\mathbf{a}(\mathbf{x},t)-\frac{\omega_t}{c_\infty^2}\frac{\partial\mathbf{a}(\mathbf{x},t)}{\partial t}-\frac{1}{c_\infty^2}\frac{\partial^2\mathbf{a}(\mathbf{x},t)}{\partial t^2}=-\mu\left(\mathbf{j}_s(\mathbf{x},t)+\frac{\partial\mathbf{d}_s(\mathbf{x},t)}{\partial t}\right)-\mathbf{curl}\,\mathbf{b}_s(\mathbf{x},t)\,. \tag{11.9}$$

This is a single PDE for the vector potential $\mathbf{a}(\mathbf{x},t)$ only! After solution, the scalar potential $\phi(\mathbf{x},t)$ may be obtained by writing the gauge condition (11.7) in the form of the inhomogeneous *ordinary* differential equation

$$\frac{\partial\phi(\mathbf{x},t)}{\partial t}+\omega_t\phi(\mathbf{x},t)=-c_\infty^2\operatorname{div}\mathbf{a}(\mathbf{x},t)\,.$$

The well-known solution (for vanishing initial conditions) is

$$\phi(\mathbf{x},t)=-c_\infty^2H(t)\exp\left(-\omega_t t\right)*\operatorname{div}\mathbf{a}(\mathbf{x},t)\,, \tag{11.10}$$

where $H(t)$ is the Heaviside unit step function, and the asterisk denotes temporal convolution. Thus, the scalar potential $\phi(\mathbf{x},t)$ is directly linked to the vector potential divergence div $\mathbf{a}(\mathbf{x},t)$ via convolution with a one-sided decaying exponential.

Although this solution methodology is rigorously correct, the convolution integral in (11.10) may be difficult to evaluate in practice. Hence, we seek another PDE that governs the scalar potential function. Take the divergence of the vector equation (11.9) and use $\mathrm{div}\nabla^2\mathbf{f} = \nabla^2\mathrm{div}\,\mathbf{f}$ (which is easily proved in rectangular coordinates) to obtain

$$\nabla^2\mathrm{div}\,\mathbf{a}(\mathbf{x},t) - \frac{\omega_t}{c_\infty^2}\frac{\partial\mathrm{div}\,\mathbf{a}(\mathbf{x},t)}{\partial t} - \frac{1}{c_\infty^2}\frac{\partial^2\mathrm{div}\,\mathbf{a}(\mathbf{x},t)}{\partial t^2} = -\mu\,\mathrm{div}\left(\mathbf{j}_s(\mathbf{x},t) + \frac{\partial\mathbf{d}_s(\mathbf{x},t)}{\partial t}\right).$$

Next, convolve this expression with the space-independent kernel $-c_\infty^2 H(t)\exp(-\omega_t t)$, use the differentiation-convolution theorem of section (2.4), and appeal to (11.10) to obtain

$$\nabla^2\phi(\mathbf{x},t) - \frac{\omega_t}{c_\infty^2}\frac{\partial\phi(\mathbf{x},t)}{\partial t} - \frac{1}{c_\infty^2}\frac{\partial^2\phi(\mathbf{x},t)}{\partial t^2} = \frac{1}{\varepsilon}H(t)\exp(-\omega_t t)*\mathrm{div}\left(\mathbf{j}_s(\mathbf{x},t) + \frac{\partial\mathbf{d}_s(\mathbf{x},t)}{\partial t}\right). \tag{11.11}$$

This is a second-order (in space and time), inhomogeneous PDE for the scalar potential function $\phi(\mathbf{x},t)$. Interestingly, the left-hand-side has the same mathematical structure as PDE (11.9) for the vector potential function $\mathbf{a}(\mathbf{x},t)$.

If current conductivity σ vanishes (implying the transition frequency $\omega_t = 0$), and source electric and magnetic flux densities $\mathbf{d}_s$ and $\mathbf{b}_s$ are ignored, then the separated PDEs (11.9) and (11.11) reduce to

$$\nabla^2\mathbf{a}(\mathbf{x},t) - \frac{1}{c_\infty^2}\frac{\partial^2\mathbf{a}(\mathbf{x},t)}{\partial t^2} = -\mu\mathbf{j}_s(\mathbf{x},t), \qquad \text{and} \qquad \nabla^2\phi(\mathbf{x},t) - \frac{1}{c_\infty^2}\frac{\partial^2\phi(\mathbf{x},t)}{\partial t^2} = \frac{1}{\varepsilon}H(t)*\mathrm{div}\,\mathbf{j}_s(\mathbf{x},t),$$

respectively. These expressions are consistent with equations (VII) and (VIII) in Welch (1960), although the assumption of homogeneity in ε and μ is not explicitly stated there. From the charge continuity law equation (12.1c) (but restricted to a vanishing conductivity $\sigma = 0$ medium) note that $H(t)*\mathrm{div}\,\mathbf{j}_s(\mathbf{x},t) = -H(t)*\partial\theta(\mathbf{x},t)/\partial t = -\theta(\mathbf{x},t)$ where $\theta(\mathbf{x},t)$ is the mobile charge density. Bojarski's (1983) equations (19) and (20) are Fourier transforms of the above two inhomogeneous wave equations (but note a missing negative sign on the right-hand-side of his equation (20)!). Similar to Welch (1960), Bojarski (1983) does not clearly state the underlying assumption of vanishing conductivity.

The common mathematical structure of the left-hand-sides of equations (11.9) and (11.11) is identical to that of the PDEs governing the electric vector $\mathbf{e}(\mathbf{x},t)$ and magnetic vector $\mathbf{h}(\mathbf{x},t)$ in a homogeneous wholespace. From Aldridge (2013, equations (2.9a,b) on pages 13-14) we have

$$\nabla^2 \mathbf{e}(\mathbf{x},t) - \frac{\omega_t}{c_\infty^2}\frac{\partial\, \mathbf{e}(\mathbf{x},t)}{\partial t} - \frac{1}{c_\infty^2}\frac{\partial^2 \mathbf{e}(\mathbf{x},t)}{\partial t^2}$$
$$= \mu \frac{\partial\, \mathbf{j}_s(\mathbf{x},t)}{\partial t} + \frac{\partial\, \mathbf{curl}\, \mathbf{b}_s(\mathbf{x},t)}{\partial t} + \mu \frac{\partial^2 \mathbf{d}_s(\mathbf{x},t)}{\partial t^2}$$
$$- \mu c_\infty^2 H(t) \exp(-\omega_t t) * \mathbf{grad}\, \mathrm{div} \left\{ \mathbf{j}_s(\mathbf{x},t) + \frac{\partial \mathbf{d}_s(\mathbf{x},t)}{\partial t} \right\}, \tag{11.12a}$$

and

$$\nabla^2 \mathbf{h}(\mathbf{x},t) - \frac{\omega_t}{c_\infty^2}\frac{\partial\, \mathbf{h}(\mathbf{x},t)}{\partial t} - \frac{1}{c_\infty^2}\frac{\partial^2 \mathbf{h}(\mathbf{x},t)}{\partial t^2}$$
$$= -\mathbf{curl}\, \mathbf{j}_s(\mathbf{x},t) - \frac{1}{\mu}\left[\mathbf{grad}\, \mathrm{div}\, \mathbf{b}_s(\mathbf{x},t) - \frac{\omega_t}{c_\infty^2}\frac{\partial\, \mathbf{b}_s(\mathbf{x},t)}{\partial t} - \frac{1}{c_\infty^2}\frac{\partial^2 \mathbf{b}_s(\mathbf{x},t)}{\partial t^2} \right] - \mathbf{curl} \frac{\partial\, \mathbf{d}_s(\mathbf{x},t)}{\partial t}. \tag{11.12b}$$

Although left-hand-sides are identical in all of these PDEs, the right-hand-sides (i.e., body source terms) differ.

11.5.1 Point Sources

The solution methodology used in Aldridge (2013) to obtain the electric and magnetic fields generated by various *point* sources may also be applied to equations (11.9) and (11.11) for the vector and scalar potentials. Consider first a point electric current source, represented by the current density vector

$$\mathbf{j}_s(\mathbf{x},t) = J\mathbf{d}w(t)\delta(\mathbf{x}-\mathbf{x}_s), \tag{11.13}$$

where J (SI unit: A-m) is a magnitude scalar, $\mathbf{d}$ is a dimensionless unit orientation vector, $w(t)$ is a dimensionless (and unit maximum absolute amplitude) waveform, and $\delta(\mathbf{x}\text{-}\mathbf{x_s})$ is the 3D spatial Dirac delta function centered on source position $\mathbf{x}_s$. In this case, Aldridge (2013) obtains a frequency-domain expression for the Fourier transformed electric vector $\mathbf{E}(\mathbf{x},\omega)$ as

$$\mathbf{E}(\mathbf{x},\omega) = \left(\frac{J\mu}{4\pi r}\right)\left[(-i\omega)W(\omega)\right]\exp\left[+ik(\omega)r\right] \times$$
$$\left\{ \left[(\mathbf{d}\cdot\mathbf{e}_r)\mathbf{e}_r - \mathbf{d}\right] - \left[3(\mathbf{d}\cdot\mathbf{e}_r)\mathbf{e}_r - \mathbf{d}\right]\left(\frac{1}{ik(\omega)r} - \frac{1}{(ik(\omega)r)^2}\right)\right\}. \tag{11.14}$$

Here $\mathbf{e}_r$ is a unit vector pointing from the source point to a field point at radial distance r, and $k(\omega)$ is the *complex wavenumber* defined by

$$k(\omega)^2 = \frac{\omega^2 + i\omega_t\omega}{c_\infty^2}. \tag{11.15}$$

Expression (11.14) is mathematically exact; no low-frequency or high-frequency approximation is adopted in the derivation. The same derivational procedure applied to PDEs (11.9) and (11.11) yields solutions for the Fourier-transformed vector and scalar potentials as:

$$\mathbf{A}(\mathbf{x},\omega) = \left(\frac{J\mu W(\omega)}{4\pi r}\right) \exp\left(+ik(\omega)r\right) \mathbf{d}\,, \tag{11.16}$$

and

$$\Phi(\mathbf{x},\omega) = \left(\frac{JW(\omega)}{4\pi\varepsilon r^2(\omega_t - i\omega)}\right) \left(1 - ik(\omega)r\right) \exp\left(+ik(\omega)r\right)\left(\mathbf{d}\cdot\mathbf{e}_r\right). \tag{11.17a}$$

The Fourier-transformed vector potential **A** has SI unit (V/m)/Hz^2, and the Fourier-transformed scalar potential Φ has SI unit V/Hz. Interestingly, the vector potential is parallel to the source current density direction **d**. Also, note that **A** has far-field $1/r$ range dependence, whereas Φ has both far-field $1/r$ and near-field $1/r^2$ dependence. An alternative way of writing the scalar potential is

$$\Phi(\mathbf{x},\omega) = \left(\frac{JW(\omega)}{4\pi\sigma r^2}\right)\left(\frac{1}{1 - i\,\omega/\omega_t}\right)\left(1 - ik(\omega)r\right) \exp\left(+ik(\omega)r\right)\left(\mathbf{d}\cdot\mathbf{e}_r\right). \tag{11.17b}$$

For $\omega << \omega_t$, the term in brackets is approximately unity. As a check, note that

$$\mathbf{curl}\ \mathbf{A}(\mathbf{x},\omega) = \left(\frac{J\mu W(\omega)}{4\pi r^2}\right) \exp\left[+ik(\omega)r\right]\left(1 - ik(\omega)r\right)(\mathbf{d}\times\mathbf{e}_r) = \mathbf{B}(\mathbf{x},\omega)\,,$$

which is identical to the expression for $\mathbf{B}(\mathbf{x},\omega) = \mathbf{H}(\mathbf{x},\omega)/\mu$ derived in Aldridge (2013). Also, the Fourier transform of the gauge condition (11.7) is

$$\mathrm{div}\ \mathbf{A}(\mathbf{x},\omega) + \frac{\omega_t - i\omega}{c_\infty^2}\ \Phi(\mathbf{x},\omega) = 0\,.$$

The derived expressions for the potentials (11.16) and (11.17a) satisfy this gauge condition exactly. Next, we calculate the separate contributions to the (Fourier-transformed) electric vector $\mathbf{E}(\mathbf{x},\omega)$ arising from the vector and scalar potential functions:

$$\mathbf{E}(\mathbf{x},\omega)\Big|_{\text{vectorpot}} = -(-i\omega)\mathbf{A}(\mathbf{x},\omega) = -\left(\frac{J\mu}{4\pi r}\right)\left[(-i\omega)W(\omega)\right]\exp\left[+ik(\omega)r\right]\mathbf{d}\,, \tag{11.18a}$$

and

$$\mathbf{E}(\mathbf{x},\omega)\Big|_{\text{scalar pot}} = -\mathbf{grad}\ \Phi(\mathbf{x},\omega) = \left(\frac{J\mu}{4\pi r}\right)\left[(-i\omega)W(\omega)\right]\exp\left[+ik(\omega)r\right]\times$$
$$\left\{(\mathbf{d}\cdot\mathbf{e}_r)\mathbf{e}_r - \left[3(\mathbf{d}\cdot\mathbf{e}_r)\mathbf{e}_r - \mathbf{d}\right]\left(\frac{1}{ik(\omega)r} - \frac{1}{\left(ik(\omega)r\right)^2}\right)\right\}. \tag{11.18b}$$

Finally, adding these via

$$\mathbf{E}(\mathbf{x},\omega) = \mathbf{E}(\mathbf{x},\omega)\Big|_{\text{vectorpot}} + \mathbf{E}(\mathbf{x},\omega)\Big|_{\text{scalar pot}} = -(-i\omega)\mathbf{A}(\mathbf{x},\omega) - \mathbf{grad}\ \Phi(\mathbf{x},\omega),$$

yields the correct expression (11.14) above (from Aldridge, 2013) for the total electric vector generated by a point current density source. Interestingly, the vector potential contributes a source-parallel far-field component, whereas the scalar potential contributes a radial far-field component. Additionally, the scalar potential contributes radial and source-parallel near-field components. Obviously both potentials play a significant role for the total electric vector.

The following figure 11.1 displays time-domain electric field responses corresponding to the frequency-domain equations (11.18a) and (11.18b) above. These traces are recorded by a horizontal line array of receivers situated within a homogeneous and isotropic electromagnetic wholespace. Medium parameters are $\varepsilon = 10\varepsilon_0$, $\mu = \mu_0$, and $\sigma = 0.1$ S/m. A horizontally-directed point current source is located 2000 m below the center trace of the array at $x = 0$ m. The source waveform is an alternating polarity square pulse sequence of 2 s on (positive) time, 2 s off (zero) time, 2 s on (negative) time, and 2 s off (zero) time. The plotted trace length of 8 s corresponds to a single period of source performance. Although the top panel ($-da_x(t)/dt$) and bottom panel ($-d\phi(x)/dx$) traces exhibit a markedly different character, their maximum absolute amplitudes are just about the same. Note that the vector/scalar potential-derived traces descend/ascend to zero/non-zero values at large times.

Figure 11.2 (top panel) displays the sum of the traces illustrated in the previous figure 11.1. For comparison, the traces in the bottom panel are the *x*-component of the electric vector calculated via algorithm EMHOLE (Aldridge, 2013), corresponding to equation (11.14) above. As expected, there is very close agreement between the two trace sets, in both waveform shape and amplitude. Note in particular that the $-d\phi(x)/dx$ traces in figure 11.1 (bottom panel) are *not* a good approximation to the *x*-component of the electric vector.

Finally, figures 11.3 and 11.4 depict "voltage difference traces" generated by subtracting the scalar potential $\phi(\mathbf{x},t)$ (calculated from equation (11.17a or b) above) measured at two distinct points that are symmetrically-placed about the receiver station. This discrete voltage difference is divided by the distance between the two electrode points in order to put responses into the proper unit (V/m) for an electric vector component. Note that the reference point for the scalar potential of equations (11.17) is infinitely far away ($r \to +\infty$). As the electrode separation distance reduces from $d = 100$ m to $d = 1$ m, the difference traces converge, as expected, to the $d\phi(x)/dx$ traces of the previous figure 11.2. The utility of these voltage difference traces is that they may represent an observed quantity in a geophysical electromagnetic field experiment.

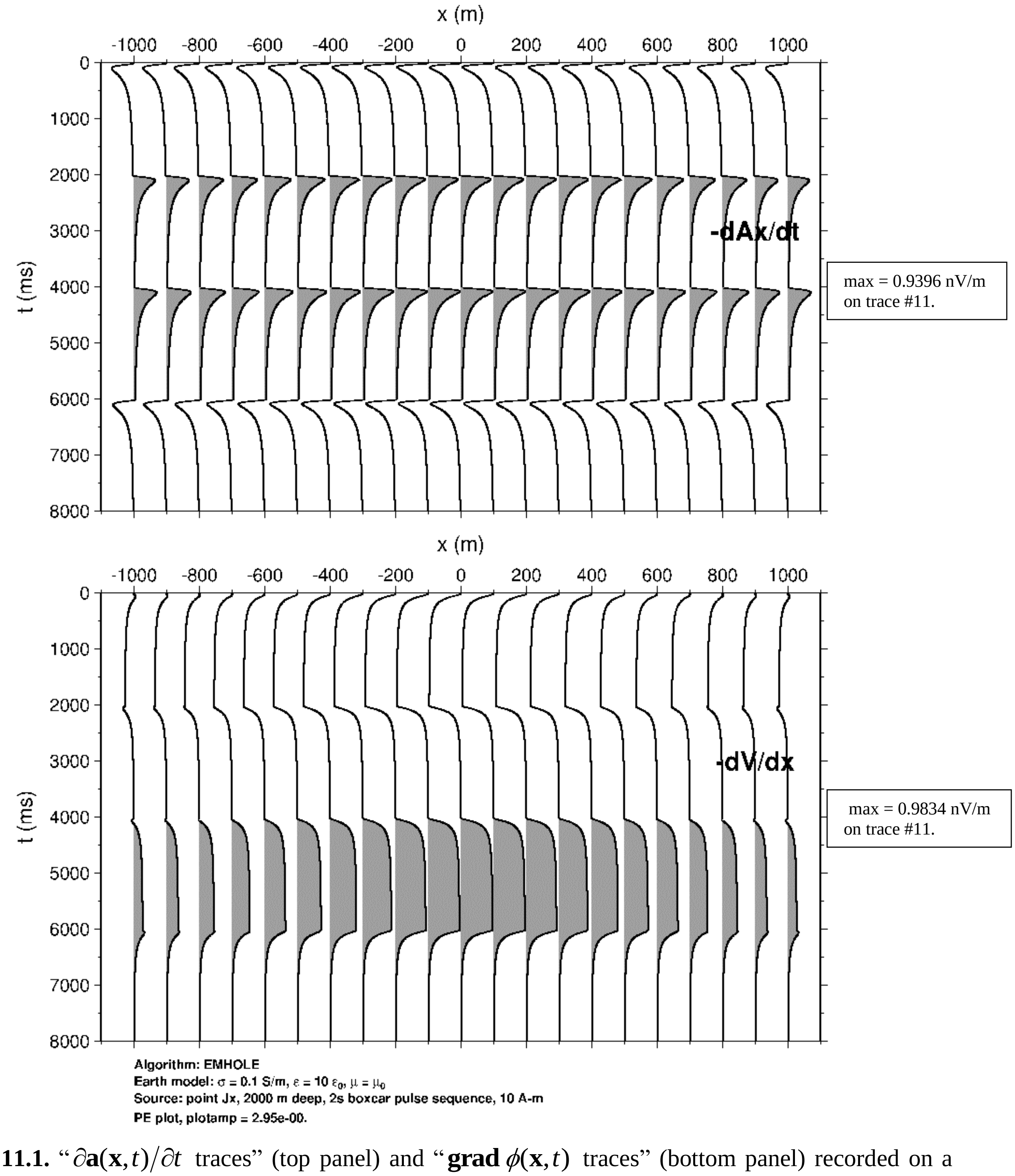

Figure 11.1. "$\partial \mathbf{a}(\mathbf{x},t)/\partial t$ traces" (top panel) and "$\mathbf{grad}\,\phi(\mathbf{x},t)$ traces" (bottom panel) recorded on a horizontal line array of receivers situated in a homogeneous and isotropic electromagnetic wholespace. The EM body source is a horizontally-directed point current density source located 2000 m below the center point of the receiver array at $x = 0$ m. Source activation waveform is an alternating-polarity boxcar pulse sequence (2 s positive, 2 s zero, 2 s negative, 2 s zero) of amplitude 10 A-m. Trace length is 8 s, with positive lobes shaded gray. The maximum absolute amplitude within each panel (occurring on the center trace at $x = 0$ m) is plotted at one trace spacing.

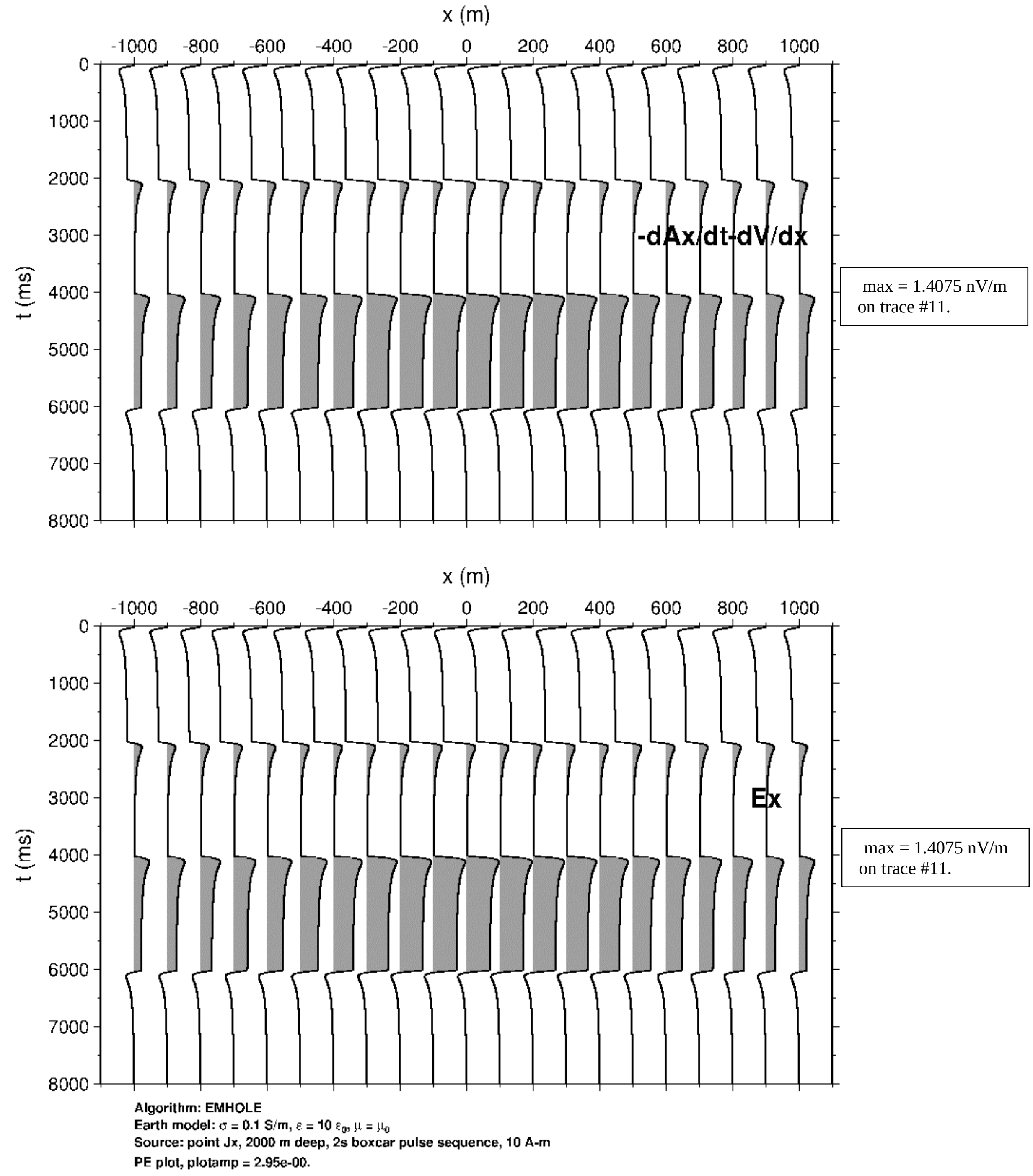

Figure 11.2. Top panel displays the sum of the two sets of traces depicted in figure 11.1. Bottom panel displays the x-component of the electric vector $e_x(\mathbf{x},t)$ calculated by numerical algorithm EMHOLE. The maximum absolute amplitude within each panel (occurring on the center trace at $x = 0$ m) is plotted at one trace spacing. Close agreement between the two panels is obviously obtained.

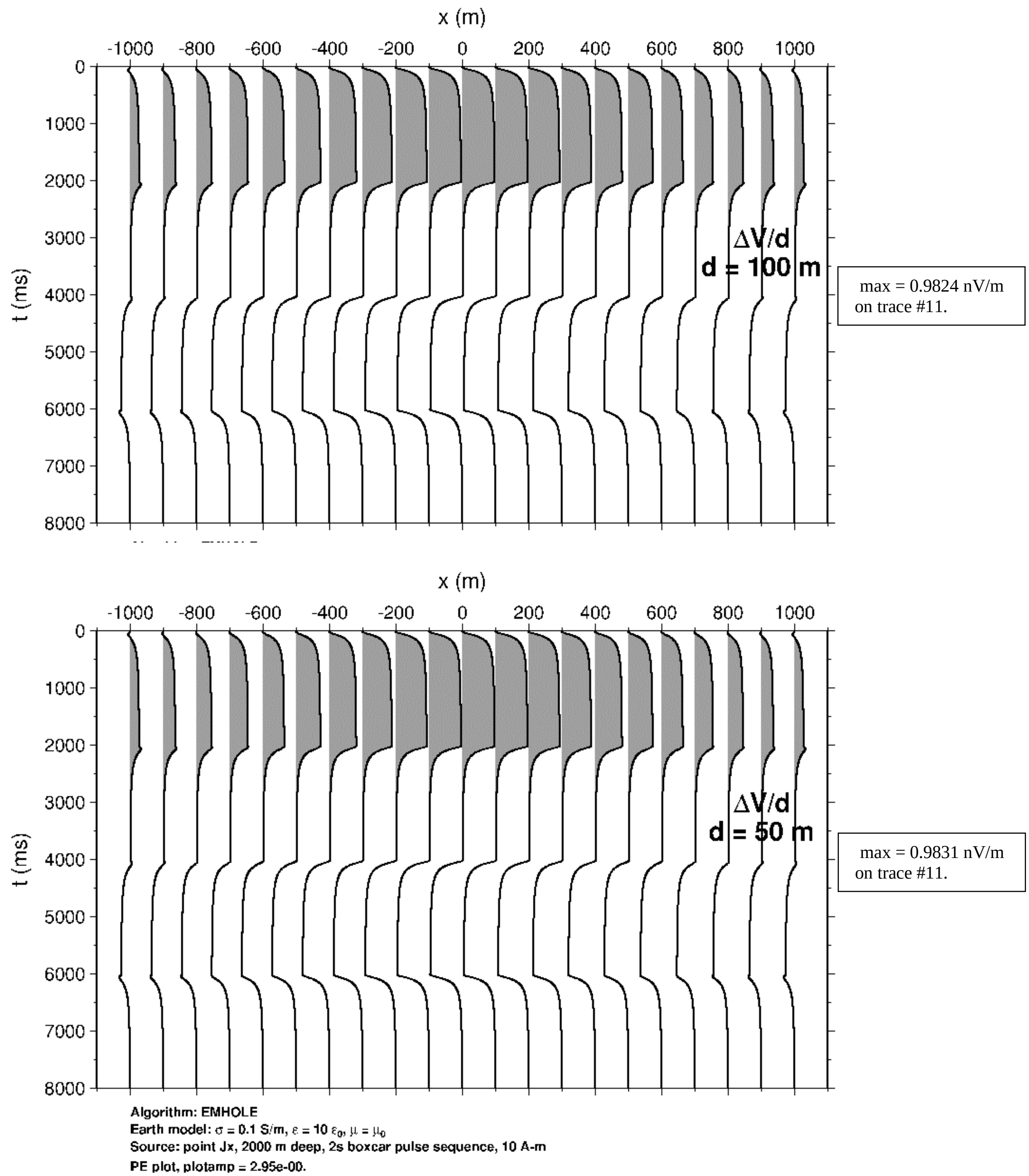

Figure 11.3. "Voltage difference traces". Each trace is divided by the horizontal separation distance *d* between two point voltage electrodes centered at the receiver station location.

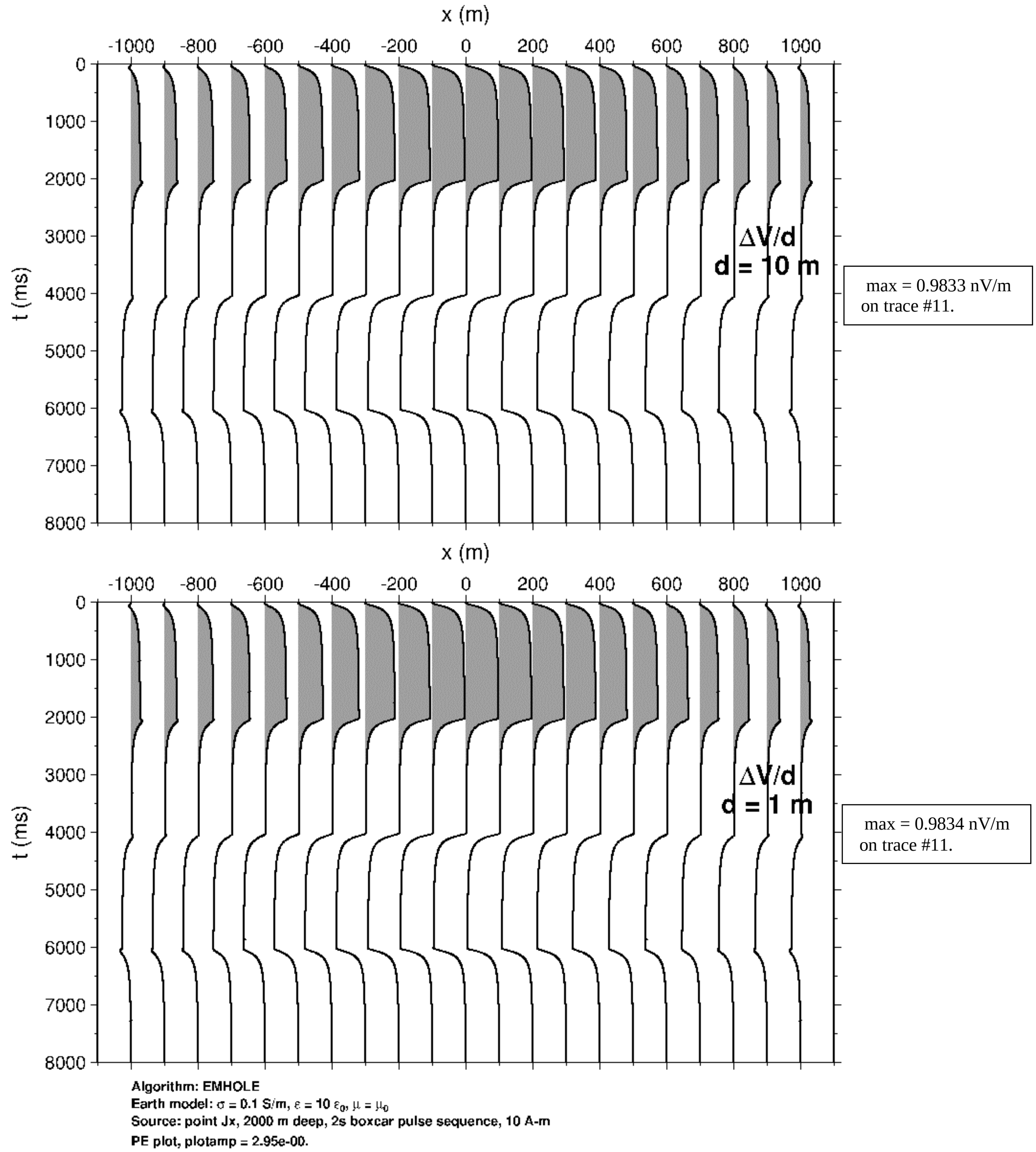

Figure 11.4. "Voltage difference traces". Each trace is divided by the horizontal separation distance d between two point voltage electrodes centered at the receiver station location. Traces in the bottom panel for $d = 1$ m closely approximate the "$\mathbf{grad}\,\phi(\mathbf{x},t)$ traces" displayed in figure 11.1.

There are two other types of point sources to consider. A point *displacement current source* is represented by the vector

$$\mathbf{l}_s(\mathbf{x},t) = \frac{\partial \mathbf{d}_s(\mathbf{x},t)}{\partial t} = D\mathbf{d}w'(t)\delta(\mathbf{x}-\mathbf{x}_s)\,, \tag{11.19}$$

where magnitude scalar D has SI unit C-m, and the prime signifies differentiation with respect to time t. Examination of PDEs (11.9) (for the vector potential) and (11.11) (for the scalar potential) indicates a displacement current source acts in a manner similar to a conduction current source $\mathbf{j}_s(\mathbf{x},t)$. The only difference is that the source waveform is differentiated. Hence, by comparison with equations (11.16) and (11.17a), frequency-domain expressions for the vector and scalar potentials generated by a point displacement current source are

$$\mathbf{A}(\mathbf{x},\omega) = \left(\frac{D\mu}{4\pi r}\right)\big(-i\omega W(\omega)\big)\ \exp\big(+ik(\omega)r\big)\mathbf{d}\,, \tag{11.20a}$$

and

$$\Phi(\mathbf{x},\omega) = \left(\frac{DW(\omega)}{4\pi\varepsilon r^2}\right)\left(\frac{-i\omega}{\omega_t - i\omega}\right)\big(1-ik(\omega)r\big)\ \exp\big(+ik(\omega)r\big)\big(\mathbf{d}\cdot\mathbf{e}_r\big), \tag{11.20b}$$

respectively. The final source type is the point *magnetic current source*, represented by

$$\mathbf{k}_s(\mathbf{x},t) = \frac{\partial \mathbf{b}_s(\mathbf{x},t)}{\partial t} = B\mathbf{d}w'(t)\delta(\mathbf{x}-\mathbf{x}_s)\,, \tag{11.21}$$

where magnitude scalar B has SI unit T-m^3. The solution of PDE (11.9) for the vector potential function (in the frequency-domain) is

$$\mathbf{A}(\mathbf{x},\omega) = \left(\frac{BW(\omega)}{4\pi r^2}\right)\big(1-ik(\omega)r\big)\ \exp\big(+ik(\omega)r\big)\big(\mathbf{d}\times\mathbf{e}_r\big). \tag{11.22a}$$

In this case, the vector potential function is normal to both the source direction $\mathbf{d}$ and the source-to-receiver direction $\mathbf{e}_r$. Interestingly, from PDE (11.11) for the scalar potential, a magnetic current source has no influence on $\phi(\mathbf{x},t)$. Thus we conclude

$$\Phi(\mathbf{x},\omega) = 0\,, \tag{11.22b}$$

for a magnetic current source.

11.6 Heterogeneous Medium

For a heterogeneous electromagnetic medium, PDEs (11.7) and (11.8) above for the scalar and vector potential functions remain coupled. However, the same convolutional solution approach used for homogeneous media may be pursued for the scalar potential function. Re-write equation (11.7) as

$$\frac{\partial\phi(\mathbf{x},t)}{\partial t}+\omega_t(\mathbf{x})\phi(\mathbf{x},t)=-c_\infty^2(\mathbf{x})\mathrm{div}\,\mathbf{a}(\mathbf{x},t)\,. \tag{11.23}$$

At any fixed spatial position $\mathbf{x}$, this is an inhomogeneous *ordinary* differential equation for the scalar potential $\phi(\mathbf{x},t)$. The solution is

$$\phi(\mathbf{x},t)=-c_\infty^2(\mathbf{x})\exp\left(-\omega_t(\mathbf{x})t\right)H(t)*\mathrm{div}\,\mathbf{a}(\mathbf{x},t)\,, \tag{11.24}$$

where the asterisk denotes temporal convolution, and $H(t)$ is the Heaviside unit step function. Hence, the Lorenz gauge condition implies that the scalar potential $\phi(\mathbf{x},t)$ is determined by the vector potential divergence div $\mathbf{a}(\mathbf{x},t)$, via convolution with a one-sided decaying exponential function.

11.6.1 Vector Potential Equation

We will attempt to obtain separated space-time equations for the vector potential function $\mathbf{a}(\mathbf{x},t)$ and the scalar potential function $\phi(\mathbf{x},t)$. Plugging relation (11.23) into equation (11.8) gives

$$\begin{aligned}&\nabla^2\mathbf{a}(\mathbf{x},t)-\frac{\omega_t(\mathbf{x})}{c_\infty^2(\mathbf{x})}\frac{\partial\mathbf{a}(\mathbf{x},t)}{\partial t}-\frac{1}{c_\infty^2(\mathbf{x})}\frac{\partial^2\mathbf{a}(\mathbf{x},t)}{\partial t^2}\\&\quad-\mathbf{grad}\ln\left(\kappa_\varepsilon(\mathbf{x})\kappa_\mu(\mathbf{x})\right)\mathrm{div}\,\mathbf{a}(\mathbf{x},t)+\mathbf{grad}\ln\kappa_\mu(\mathbf{x})\times\mathbf{curl}\,\mathbf{a}(\mathbf{x},t)\\&\quad-\mathbf{grad}\,\omega_t(\mathbf{x})\left[\exp\left(-\omega_t(\mathbf{x})t\right)H(t)*\mathrm{div}\,\mathbf{a}(\mathbf{x},t)\right]\\&\qquad=-\mu(\mathbf{x})\left(\mathbf{j}_s(\mathbf{x},t)+\frac{\partial\mathbf{d}_s(\mathbf{x},t)}{\partial t}\right)-\mathbf{curl}\,\mathbf{b}_s(\mathbf{x},t)+\mathbf{grad}\ln\kappa_\mu(\mathbf{x})\times\mathbf{b}_s(\mathbf{x},t)\,.\end{aligned} \tag{11.25}$$

So now we have an *integro-differential equation* for the vector potential! Once $\mathbf{a}(\mathbf{x},t)$ is determined, the scalar potential $\phi(\mathbf{x},t)$ may be obtained from the convolution (11.11). However, it is debatable whether solution of equations (11.24) and (11.25) is any easier than the solution of the coupled partial differential system (11.7) and (11.8).

Interestingly, if the medium is uniform in the transition frequency ω_t, then the temporal convolution is removed from equation (11.25). If current conductivity σ vanishes (implying the transition frequency ω_t = 0), and source electric and magnetic flux densities $\mathbf{d}_s$ and $\mathbf{b}_s$ are ignored, then (11.25) reduces to

$$\begin{aligned}&\nabla^2\mathbf{a}(\mathbf{x},t)-\frac{1}{c_\infty^2(\mathbf{x})}\frac{\partial^2\mathbf{a}(\mathbf{x},t)}{\partial t^2}-\mathbf{grad}\ln\left(\kappa_\varepsilon(\mathbf{x})\kappa_\mu(\mathbf{x})\right)\mathrm{div}\,\mathbf{a}(\mathbf{x},t)+\mathbf{grad}\ln\kappa_\mu(\mathbf{x})\times\mathbf{curl}\,\mathbf{a}(\mathbf{x},t)\\&\qquad=-\mu(\mathbf{x})\mathbf{j}_s(\mathbf{x},t)\,.\end{aligned}$$

This is *not* compatible with equation (VII) in Welch (1960). His expression is obtained only if the medium is additionally uniform in both permittivity ε and permeability μ (which implies the infinite-frequency phase speed c_∞^2 also is uniform). Our equation (11.25) for the vector potential is far more general, accommodating conducting media, heterogeneity in all medium parameters, and a greater diversity of electromagnetic body sources.

11.6.2 Scalar Potential Equation

What equation does the scalar potential function $\phi(\mathbf{x},t)$ satisfy in a *heterogeneous* medium? Equation (11.7) above indicates that it is directly coupled to the divergence of the vector potential $\mathbf{a}(\mathbf{x},t)$. Can the scalar potential be un-coupled from the vector potential?

Recall that the two remaining Maxwell equations that have not yet been utilized in the development of the potential formulation are the charge continuity equation (11.1c) and the Gauss electric law (11.1e). Introducing the constitutive relations (11.2c) for the current density vector and (11.2b) for the electric displacement vector yields the pair of expressions

$$\frac{\partial\theta(\mathbf{x},t)}{\partial t} + \mathrm{div}\left[\sigma(\mathbf{x})\mathbf{e}(\mathbf{x},t) + \mathbf{j}_s(\mathbf{x},t)\right] = 0\,, \tag{11.26a}$$

$$-\theta(\mathbf{x},t) + \mathrm{div}\left[\varepsilon(\mathbf{x})\mathbf{e}(\mathbf{x},t) + \mathbf{d}_s(\mathbf{x},t)\right] = 0\,. \tag{11.26b}$$

Eliminate the charge density $\theta(\mathbf{x},t)$ by differentiating the second equation with respect to time and adding to the first equation. We obtain

$$\mathrm{div}\left[\sigma(\mathbf{x})\mathbf{e}(\mathbf{x},t) + \varepsilon(\mathbf{x})\frac{\partial\mathbf{e}(\mathbf{x},t)}{\partial t} + \mathbf{j}_s(\mathbf{x},t) + \mathbf{l}_s(\mathbf{x},t)\right] = 0\,, \tag{11.27}$$

where $\mathbf{l}_s(\mathbf{x},t) = \partial\mathbf{d}_s(\mathbf{x},t)/\partial t$ is the displacement current body source. This expression motivates defining a generalized or *total* current density vector as

$$\mathbf{j}_{\mathrm{tot}}(\mathbf{x},t) \equiv \sigma(\mathbf{x})\mathbf{e}(\mathbf{x},t) + \varepsilon(\mathbf{x})\frac{\partial\mathbf{e}(\mathbf{x},t)}{\partial t} + \mathbf{j}_s(\mathbf{x},t) + \mathbf{l}_s(\mathbf{x},t)\,. \tag{11.28}$$

The total current density is the sum of conduction current density, displacement current density, and body sources of each type. Hence, the combined charge continuity law and Gauss electric law of Maxwell's equations is equivalent to the statement that the divergence of the total current density vector vanishes:

$$\mathrm{div}\,\mathbf{j}_{\mathrm{tot}}(\mathbf{x},t) = 0\,. \tag{11.29}$$

This may be thought of as a generalized charge continuity law.

Next, substitute from equation (11.4) for the electric vector in terms of the two potential functions. Straightforward manipulation yields

$$\nabla^2 \frac{\partial \phi(\mathbf{x},t)}{\partial t} + \omega_t(\mathbf{x})\nabla^2\phi(\mathbf{x},t) + \mathbf{grad}\ln\kappa_\varepsilon(\mathbf{x})\cdot\mathbf{grad}\frac{\partial\phi(\mathbf{x},t)}{\partial t}$$

$$+\left[\mathbf{grad}\,\omega_t(\mathbf{x}) + \omega_t(\mathbf{x})\mathbf{grad}\ln\kappa_\varepsilon(\mathbf{x})\right]\cdot\mathbf{grad}\,\phi(\mathbf{x},t)$$

$$+\frac{\partial^2 \mathrm{div}\,\mathbf{a}(\mathbf{x},t)}{\partial t^2} + \omega_t(\mathbf{x})\frac{\partial \mathrm{div}\,\mathbf{a}(\mathbf{x},t)}{\partial t} + \mathbf{grad}\ln\kappa_\varepsilon(\mathbf{x})\cdot\frac{\partial^2\mathbf{a}(\mathbf{x},t)}{\partial t^2}$$

$$+\left[\mathbf{grad}\,\omega_t(\mathbf{x}) + \omega_t(\mathbf{x})\mathbf{grad}\ln\kappa_\varepsilon(\mathbf{x})\right]\cdot\frac{\partial\mathbf{a}(\mathbf{x},t)}{\partial t} = \frac{1}{\varepsilon(\mathbf{x})}\mathrm{div}\left(\mathbf{j}_s(\mathbf{x},t) + \mathbf{l}_s(\mathbf{x},t)\right),$$

where $\omega_t(\mathbf{x}) = \sigma(\mathbf{x})/\varepsilon(\mathbf{x})$ is the spatially-variable transition frequency. Also recall $\mathrm{div}\,\mathbf{grad}\,\phi \equiv \nabla^2\phi$.

Not really knowing how to effectively proceed from this point, we substitute for div $\mathbf{a}(\mathbf{x},t)$ from the Lorenz gauge condition (11.7) and group terms to obtain

$$\frac{\partial}{\partial t}\left\{\nabla^2\phi(\mathbf{x},t) - \frac{\omega_t(\mathbf{x})}{c_\infty^2(\mathbf{x})}\frac{\partial\phi(\mathbf{x},t)}{\partial t} - \frac{1}{c_\infty^2(\mathbf{x})}\frac{\partial^2\phi(\mathbf{x},t)}{\partial t^2}\right\}$$
$$+\omega_t(\mathbf{x})\left\{\nabla^2\phi(\mathbf{x},t) - \frac{\omega_t(\mathbf{x})}{c_\infty^2(\mathbf{x})}\frac{\partial\phi(\mathbf{x},t)}{\partial t} - \frac{1}{c_\infty^2(\mathbf{x})}\frac{\partial^2\phi(\mathbf{x},t)}{\partial t^2}\right\}$$

$$+\mathbf{grad}\ln\kappa_\varepsilon(\mathbf{x})\cdot\left[\frac{\partial}{\partial t}\left(\mathbf{grad}\,\phi(\mathbf{x},t) + \frac{\partial\mathbf{a}(\mathbf{x},t)}{\partial t}\right) + \omega_t(\mathbf{x})\left(\mathbf{grad}\,\phi(\mathbf{x},t) + \frac{\partial\mathbf{a}(\mathbf{x},t)}{\partial t}\right)\right]$$
$$+\mathbf{grad}\,\omega_t(\mathbf{x})\cdot\left(\mathbf{grad}\,\phi(\mathbf{x},t) + \frac{\partial\mathbf{a}(\mathbf{x},t)}{\partial t}\right)$$

$$= \frac{1}{\varepsilon(\mathbf{x})}\mathrm{div}\left(\mathbf{j}_s(\mathbf{x},t) + \mathbf{l}_s(\mathbf{x},t)\right). \tag{11.30}$$

Unfortunately, we have not been able to completely eliminate the vector potential $\mathbf{a}(\mathbf{x},t)$. Our only consolation is that, for a medium that is uniform in both electric permittivity ε and current conductivity σ, the above equation reduces to

$$\frac{\partial}{\partial t}\left\{\nabla^2\phi(\mathbf{x},t) - \frac{\omega_t}{c_\infty^2(\mathbf{x})}\frac{\partial\phi(\mathbf{x},t)}{\partial t} - \frac{1}{c_\infty^2(\mathbf{x})}\frac{\partial^2\phi(\mathbf{x},t)}{\partial t^2}\right\}$$
$$+\omega_t(\mathbf{x})\left\{\nabla^2\phi(\mathbf{x},t) - \frac{\omega_t}{c_\infty^2(\mathbf{x})}\frac{\partial\phi(\mathbf{x},t)}{\partial t} - \frac{1}{c_\infty^2(\mathbf{x})}\frac{\partial^2\phi(\mathbf{x},t)}{\partial t^2}\right\}$$

$$= \frac{1}{\varepsilon}\mathrm{div}\left(\mathbf{j}_s(\mathbf{x},t) + \mathbf{l}_s(\mathbf{x},t)\right),$$

with convolutional solution

$$\nabla^2\phi(\mathbf{x},t) - \frac{\omega_t}{c_\infty^2(\mathbf{x})}\frac{\partial\phi(\mathbf{x},t)}{\partial t} - \frac{1}{c_\infty^2(\mathbf{x})}\frac{\partial^2\phi(\mathbf{x},t)}{\partial t^2} = \frac{1}{\varepsilon}H(t)\exp\left(-\omega_t t\right) * \mathrm{div}\left(\mathbf{j}_s(\mathbf{x},t) + \mathbf{l}_s(\mathbf{x},t)\right).$$

This is consistent with the previous deduction of equation (11.11). The slight difference is that the present expression allows the magnetic permeability $\mu(\mathbf{x})$ to be spatially variable, leading to a variable infinite-frequency phase speed $c_\infty^2(\mathbf{x}) = 1/\varepsilon\mu(\mathbf{x})$.

As a check, we note that the Fourier transform of equation (11.30) above is compatible with equation (5) in Weiss (2013) after the simplifying assumption $\mu(\mathbf{x}) = \mu_0$ is made. However, we caution that (11.30) does not imply div $\mathbf{j}(\mathbf{x},t) = 0$ (where $\mathbf{j}(\mathbf{x},t)$ is the conduction current density vector) but rather div $\mathbf{j}_{\mathrm{tot}}(\mathbf{x},t) = 0$ (where $\mathbf{j}_{\mathrm{tot}}(\mathbf{x},t)$ is the total current density vector defined above). The distinction between these two current density vectors is not clear in Weiss (2013).

In the static limit where the potentials are independent of time (equivalent to the DC limit in the frequency-domain) equation (11.30) reduces to

$$\omega_t(\mathbf{x})\nabla^2\phi(\mathbf{x}) + \left[\mathbf{grad}\,\omega_t(\mathbf{x}) + \omega_t(\mathbf{x})\,\mathbf{grad}\ln\kappa_\varepsilon(\mathbf{x})\right]\cdot\mathbf{grad}\,\phi(\mathbf{x}) = \frac{1}{\varepsilon(\mathbf{x})}\mathrm{div}\,\mathbf{j}_s(\mathbf{x}) \;,$$

which is equivalent to the well-known expression

$$\mathrm{div}\left[\sigma(\mathbf{x})\,\mathbf{grad}\,\phi(\mathbf{x})\right] = \mathrm{div}\,\mathbf{j}_s(\mathbf{x}), \tag{11.31}$$

for a static scalar electric potential function $\phi(\mathbf{x})$. Note that spatially-variable permittivity $\varepsilon(\mathbf{x})$ or permeability $\mu(\mathbf{x})$ do not influence the static scalar potential. The analogous static (or DC) limit of the vector potential equation (11.25) is

$$\nabla^2\mathbf{a}(\mathbf{x}) - \mathbf{grad}\ln\left(\kappa_\sigma(\mathbf{x})\kappa_\mu(\mathbf{x})\right)\mathrm{div}\,\mathbf{a}(\mathbf{x}) + \mathbf{grad}\ln\kappa_\mu(\mathbf{x})\times\mathbf{curl}\,\mathbf{a}(\mathbf{x}) =$$

$$-\mu(\mathbf{x})\mathbf{j}_s(\mathbf{x}) - \mathbf{curl}\,\mathbf{b}_s(\mathbf{x}) + \mathbf{grad}\ln\kappa_\mu(\mathbf{x})\times\mathbf{b}_s(\mathbf{x}), \tag{11.32}$$

where $\kappa_\sigma(\mathbf{x}) = \sigma(\mathbf{x})/\sigma_{ref}$ is a dimensionless relative current conductivity. Permittivity $\varepsilon(\mathbf{x})$ does not influence the static vector potential, although permeability $\mu(\mathbf{x})$ does. Thus, in the static limit, separated PDEs for the two potential functions are obtained. The static electric and magnetic fields are obtained from the potentials via the spatial differentiations

$$\mathbf{e}(\mathbf{x}) = -\mathbf{grad}\,\phi(\mathbf{x}), \qquad\qquad \mathbf{b}(\mathbf{x}) = \mathbf{curl}\,\mathbf{a}(\mathbf{x}),$$

respectively. So, in the DC limit, ϕ determines **e** and **a** determines **b**.

With the simplifying assumptions $\mu(\mathbf{x}) = \mu_0$ and $\mathbf{b}_s(\mathbf{x}) = \mathbf{0}$, the *separated* equations (11.31) and (11.32) above for the static potentials are consistent with the *coupled* equations (4) and (6) in Weiss (2014) for the DC potentials (i.e., in the limit of vanishing frequency there, and with proper regard for the different definition of the scalar potential function).

11.7 Reciprocity of Potentials

Recall that the ultimate goal of this Appendix is to develop reciprocity theorems involving the vector potential function $\mathbf{a}(\mathbf{x},t)$ and the scalar potential function $\phi(\mathbf{x},t)$. Although Welch (1960) and Bojarski (1983) derive reciprocity relations of this type, their developments are restricted to an electromagnetic medium that is homogeneous and isotropic, *and* has vanishing conductivity (σ = 0 S/m). In this final section of Appendix C, we derive reciprocity theorems for the potentials that are appropriate for heterogeneous media characterized by general linear, time-invariant, and local constitutive relations. Moreover, the medium may possess internal interfaces (i.e., discontinuity surfaces in material parameters).

In their derivations, Welch (1960) and Bojarski (1983) work with Green function representations for the electromagnetic potentials in a homogeneous and isotropic medium. In contrast, we start with the previously-derived reciprocity theorem for the EM vector wavefields $\mathbf{e}(\mathbf{x},t)$ and $\mathbf{h}(\mathbf{x},t)$. Both time-convolution and time-correlation reciprocity relations are developed. The derivation does not make recourse to the frequency-domain. However, in order to facilitate comparisons with Welch (1960) and Bojarski (1983), we consider only current density body sources.

11.7.1 Time-Convolution Reciprocity

The point of departure for the analysis is the time-convolution global reciprocity theorem (3.20). As indicated, this theorem applies to the case where media A and B are identical and satisfy adjoint conditions. We make the additional simplifying assumption that EM wavefield states A and B are activated only via body electric current sources. That is, there are no body magnetic currents, body displacement currents, or internal surface currents. Finally, we assume that the outer bounding surface S surrounding volume V is removed to infinity, where radiation conditions apply. Reciprocity theorem (3.20) simplifies dramatically to

$$\int_V \left\{ e_i^A(\mathbf{x},t) * j_i^{B-s}(\mathbf{x},t) - e_i^B(\mathbf{x},t) * j_i^{A-s}(\mathbf{x},t) \right\} dV(\mathbf{x}) = 0 . \tag{11.33}$$

Recall that the asterisks denote temporal convolution, and summation over repeated subscripts is implied. Substituting from equation (11.4) for the two electric vectors in terms of vector and scalar potential functions yields

$$\int_V \left\{ \left[\frac{\partial a_i^A(\mathbf{x},t)}{\partial t} + \frac{\partial \phi^A(\mathbf{x},t)}{\partial x_i} \right] * j_i^{B-s}(\mathbf{x},t) - \left[\frac{\partial a_i^B(\mathbf{x},t)}{\partial t} + \frac{\partial \phi^B(\mathbf{x},t)}{\partial x_i} \right] * j_i^{A-s}(\mathbf{x},t) \right\} dV(\mathbf{x}) = 0 .$$

Shifting the time differentiations from the vector potentials to the current density sources gives the equivalent form

$$\int_V \left\{ a_i^A(\mathbf{x},t) * \frac{\partial j_i^{B-s}(\mathbf{x},t)}{\partial t} - a_i^B(\mathbf{x},t) * \frac{\partial j_i^{A-s}(\mathbf{x},t)}{\partial t} \right\} dV(\mathbf{x})$$

$$+ \int_V \left\{ \frac{\partial \phi^A(\mathbf{x},t)}{\partial x_i} * j_i^{B-s}(\mathbf{x},t) - \frac{\partial \phi^B(\mathbf{x},t)}{\partial x_i} * j_i^{A-s}(\mathbf{x},t) \right\} dV(\mathbf{x}) = 0 .$$

From the product rule of differentiation, it is easy to establish the relation

$$\mathbf{grad}\left(f(\mathbf{x},t)*g(\mathbf{x},t)\right)=\mathbf{grad}\ f(\mathbf{x},t)*g(\mathbf{x},t)+f(\mathbf{x},t)*\mathbf{grad}\ g(\mathbf{x},t)\,,$$

where $f(\mathbf{x},t)$ and $g(\mathbf{x},t)$ are two scalar functions of position and time. Hence, the above is re-written as

$$\int_V \left\{ a_i^A(\mathbf{x},t)*\frac{\partial j_i^{B-s}(\mathbf{x},t)}{\partial t} - a_i^B(\mathbf{x},t)*\frac{\partial j_i^{A-s}(\mathbf{x},t)}{\partial t} \right\} dV(\mathbf{x})$$

$$-\int_V \left\{ \phi^A(\mathbf{x},t)*\frac{\partial j_i^{B-s}(\mathbf{x},t)}{\partial x_i} - \phi^B(\mathbf{x},t)*\frac{\partial j_i^{A-s}(\mathbf{x},t)}{\partial x_i} \right\} dV(\mathbf{x})$$

$$+\int_V \frac{\partial}{\partial x_i}\left\{ \phi^A(\mathbf{x},t)*j_i^{B-s}(\mathbf{x},t) - \phi^B(\mathbf{x},t)*j_i^{A-s}(\mathbf{x},t) \right\} dV(\mathbf{x}) = 0\,.$$

Applying the generalized divergence theorem (2.7) to the third volume integral above gives

$$\int_V \frac{\partial}{\partial x_i}\left\{ \phi^A(\mathbf{x},t)*j_i^{B-s}(\mathbf{x},t) - \phi^B(\mathbf{x},t)*j_i^{A-s}(\mathbf{x},t) \right\} dV(\mathbf{x}) =$$

$$\int_S \left\{ \phi^A(\mathbf{x},t)*j_i^{B-s}(\mathbf{x},t) - \phi^B(\mathbf{x},t)*j_i^{A-s}(\mathbf{x},t) \right\} m_i(\mathbf{x})\, dS(\mathbf{x})$$
$$-\sum_{n=1}^{N}\int_{S_n} \left[\phi^A(\mathbf{x},t)*j_i^{B-s}(\mathbf{x},t) - \phi^B(\mathbf{x},t)*j_i^{A-s}(\mathbf{x},t) \right] n_i(\mathbf{x})\, dS(\mathbf{x}) =$$

$$-\sum_{n=1}^{N}\int_{S_n} \left[\phi^A(\mathbf{x},t)*j_i^{B-s}(\mathbf{x},t) - \phi^B(\mathbf{x},t)*j_i^{A-s}(\mathbf{x},t) \right] n_i(\mathbf{x})\, dS(\mathbf{x})\,.$$

The integral over the outer bounding surface S (with unit normal $\mathbf{m}(\mathbf{x})$) vanishes due to the assumed radiation conditions. Next, utilizing the rule for the jump discontinuity of a convolution (developed in chapter 3.0) gives

$$\int_V \frac{\partial}{\partial x_i}\left\{ \phi^A(\mathbf{x},t)*j_i^{B-s}(\mathbf{x},t) - \phi^B(\mathbf{x},t)*j_i^{A-s}(\mathbf{x},t) \right\} dV(\mathbf{x}) =$$

$$-\sum_{n=1}^{N}\int_{S_n} \Big\{ \left[\phi^A(\mathbf{x},t)\right]*\left\langle j_i^{B-s}(\mathbf{x},t)\right\rangle n_i(\mathbf{x}) + \left\langle \phi^A(\mathbf{x},t)\right\rangle*\left[j_i^{B-s}(\mathbf{x},t)\right] n_i(\mathbf{x})$$
$$-\left[\phi^B(\mathbf{x},t)\right]*\left\langle j_i^{A-s}(\mathbf{x},t)\right\rangle n_i(\mathbf{x}) - \left\langle \phi^B(\mathbf{x},t)\right\rangle*\left[j_i^{A-s}(\mathbf{x},t)\right] n_i(\mathbf{x}) \Big\}\, dS(\mathbf{x})\,.$$

Recall that $[\mathrm{Q}]=\mathrm{Q}^+-\mathrm{Q}^-$ is the jump discontinuity of quantity Q and $\langle \mathrm{Q}\rangle = \left(\mathrm{Q}^+ + \mathrm{Q}^-\right)/2$ is the average value of Q at an interface.

It is not immediately obvious that the jump discontinuities in the above expression vanish. Hence, we include this surface integral term in the derived reciprocity theorem, written in the form

$$\int_V \left\{ a_i^A(\mathbf{x},t) * \frac{\partial j_i^{B-s}(\mathbf{x},t)}{\partial t} - a_i^B(\mathbf{x},t) * \frac{\partial j_i^{A-s}(\mathbf{x},t)}{\partial t} \right\} dV(\mathbf{x})$$

$$- \int_V \left\{ \phi^A(\mathbf{x},t) * \frac{\partial j_i^{B-s}(\mathbf{x},t)}{\partial x_i} - \phi^B(\mathbf{x},t) * \frac{\partial j_i^{A-s}(\mathbf{x},t)}{\partial x_i} \right\} dV(\mathbf{x})$$

$$= \sum_{n=1}^{N} \int_{S_n} \Big\{ \big[\phi^A(\mathbf{x},t)\big] * \big\langle j_i^{B-s}(\mathbf{x},t)\big\rangle n_i(\mathbf{x}) + \big\langle \phi^A(\mathbf{x},t)\big\rangle * \big[j_i^{B-s}(\mathbf{x},t)\big] n_i(\mathbf{x})$$
$$- \big[\phi^B(\mathbf{x},t)\big] * \big\langle j_i^{A-s}(\mathbf{x},t)\big\rangle n_i(\mathbf{x}) - \big\langle \phi^B(\mathbf{x},t)\big\rangle * \big[j_i^{A-s}(\mathbf{x},t)\big] n_i(\mathbf{x}) \Big\} \, dS(\mathbf{x}) \,. \qquad (11.34)$$

If the jump discontinuities in the scalar potential functions and the *normal* components of the body current sources vanish, then the right-hand-side equals zero.

The above time-convolution reciprocity theorem involving the two potential functions is reasonably general. In particular, it applies to spatially-heterogeneous media characterized by the linear, time-invariant, and locally-reacting constitutive relations (2.2) or (2.3) (although body sources of magnetic induction and electric displacement are assumed to vanish). Interfaces (i.e., discontinuity surfaces in material parameters) may be present within the medium. Remarkably, the derivation does not pre-suppose any particular gauge condition linking the vector and scalar potential functions. From the prior equation (11.4), the form of the two integrands on the left-hand-side are perhaps intuitively obvious: the *vector potential* convolves with the *time-derivative* of the current density source, and the *scalar potential* convolves with the *divergence* of the current density source.

For comparison with the analogous reciprocity theorem for potentials developed by Bojarski (1983), we Fourier transform expression (11.34) to the frequency-domain to obtain

$$(-i\omega)\int_V \left\{ A_i^A(\mathbf{x},\omega) J_i^{B-s}(\mathbf{x},\omega) - A_i^B(\mathbf{x},\omega) J_i^{A-s}(\mathbf{x},\omega) \right\} dV(\mathbf{x})$$

$$- \int_V \left\{ \Phi^A(\mathbf{x},\omega) \frac{\partial J_i^{B-s}(\mathbf{x},\omega)}{\partial x_i} - \Phi^B(\mathbf{x},\omega) \frac{\partial J_i^{A-s}(\mathbf{x},\omega)}{\partial x_i} \right\} dV(\mathbf{x})$$

$$= \sum_{n=1}^{N} \int_{S_n} \Big\{ \big[\Phi^A(\mathbf{x},\omega)\big] \big\langle J_i^{B-s}(\mathbf{x},\omega)\big\rangle n_i(\mathbf{x}) + \big\langle \Phi^A(\mathbf{x},\omega)\big\rangle \big[J_i^{B-s}(\mathbf{x},\omega)\big] n_i(\mathbf{x})$$
$$- \big[\phi^B(\mathbf{x},t)\big] * \big\langle j_i^{A-s}(\mathbf{x},t)\big\rangle n_i(\mathbf{x}) - \big\langle \phi^B(\mathbf{x},t)\big\rangle * \big[j_i^{A-s}(\mathbf{x},t)\big] n_i(\mathbf{x}) \Big\} \, dS(\mathbf{x}) \,. \qquad (11.35)$$

For a uniform medium (in both conductivity and permittivity), it is straightforward to demonstrate (from the charge continuity law (11.26a) and the Gauss electric law (11.26b) above) that

$$\frac{\partial \theta(\mathbf{x},t)}{\partial t} + \omega_t \theta(\mathbf{x},t) = -\text{div } \mathbf{j}_s(\mathbf{x},t) \, ,$$

which has the convolutional solution $\theta(\mathbf{x},t) = -H(t)\exp(-\omega_t t) * \text{div } \mathbf{j}_s(\mathbf{x},t)$. Fourier transforming to the frequency-domain gives $\text{div } \mathbf{J}_s(\mathbf{x},\omega) = -(\omega_t - i\omega)\Theta(\mathbf{x},\omega)$. Introducing this into equation (11.35) yields

$$(-i\omega)\int_V \left\{ A_i^A(\mathbf{x},\omega) J_i^{B-s}(\mathbf{x},\omega) - A_i^B(\mathbf{x},\omega) J_i^{A-s}(\mathbf{x},\omega) \right\} dV(\mathbf{x})$$

$$+ (\omega_t - i\omega)\int_V \left\{ \Phi^A(\mathbf{x},\omega)\Theta^B(\mathbf{x},\omega) - \Phi^B(\mathbf{x},\omega)\Theta^A(\mathbf{x},\omega) \right\} dV(\mathbf{x}) = 0 \, ,$$

where the right-hand-side vanishes because there are no interfaces in a uniform (i.e., homogeneous) wholespace. This constitutes a slight generalization of Bojarski's (1983) reciprocity theorem to a medium with non-zero current conductivity. For $\sigma = 0$ (implying the transition frequency ω_t vanishes) it appears to reduce to Bojarski's equation (27) (that is, if the four-vector product in his equation (26) is interpreted as a "four-vector dot product"). With $\sigma = 0$, the inverse Fourier transform of the above expression is

$$\int_V \left\{ a_i^A(\mathbf{x},t) * j_i^{B-s}(\mathbf{x},t) - a_i^B(\mathbf{x},t) * j_i^{A-s}(\mathbf{x},t) \right\} dV(\mathbf{x})$$

$$+ \int_V \left\{ \phi^A(\mathbf{x},t) * \theta^B(\mathbf{x},t) - \phi^B(\mathbf{x},t) * \theta^A(\mathbf{x},t) \right\} dV(\mathbf{x}) = 0 \, ,$$

which agrees with Bojarski's time-domain equation (66) (again, if a "four-vector dot product" evaluation rule is employed).

11.7.2 Time-Correlation Reciprocity

Welch (1960) also develops a reciprocity theorem for the potential functions (his "Theorem I"), again under the restricting assumption of a homogeneous and isotropic wholespace with vanishing conductivity $\sigma(\mathbf{x}) = 0$. The Lorenz gauge condition (11.7) (with vanishing transition frequency $\omega_t(\mathbf{x}) = 0$) is apparently imposed, although it is not clear if this is actually required for the derivation. Our impression is that Welch's reciprocity theorem (his equation (3)) is of the "time-correlation type". This is bolstered by remarks in Bojarski (1983) and de Hoop (1987). Hence, we develop a more general reciprocity theorem of the same type in this sub-section, and compare with Welch's (1960) result.

Neglecting magnetic current sources and displacement current sources in the global time-correlation reciprocity theorem (11.18) yields the simplified form

$$\int_V \left\{ e_i^A(\mathbf{x},t) * \bar{j}_i^{B-s}(\mathbf{x},t) + \bar{e}_i^B(\mathbf{x},t) * j_i^{A-s}(\mathbf{x},t) \right\} dV(\mathbf{x}) = 0 \, . \tag{11.36}$$

Following exactly the same derivational procedure as in the previous sub-section for a time-convolution reciprocity theorem for the potential functions, we arrive at the result

$$\int_V \left\{ a_i^A(\mathbf{x},t) * \frac{\partial \bar{j}_i^{B-s}(\mathbf{x},t)}{\partial t} - \bar{a}_i^B(\mathbf{x},t) * \frac{\partial j_i^{A-s}(\mathbf{x},t)}{\partial t} \right\} dV(\mathbf{x})$$

$$-\int_V \left\{ \phi^A(\mathbf{x},t) * \frac{\partial \bar{j}_i^{B-s}(\mathbf{x},t)}{\partial x_i} + \bar{\phi}^B(\mathbf{x},t) * \frac{\partial j_i^{A-s}(\mathbf{x},t)}{\partial x_i} \right\} dV(\mathbf{x})$$

$$= \sum_{n=1}^{N} \int_{S_n} \Big\{ \left[\phi^A(\mathbf{x},t)\right] * \left\langle \bar{j}_i^{B-s}(\mathbf{x},t) \right\rangle n_i(\mathbf{x}) + \left\langle \phi^A(\mathbf{x},t) \right\rangle * \left[\bar{j}_i^{B-s}(\mathbf{x},t)\right] n_i(\mathbf{x})$$
$$+ \left[\bar{\phi}^B(\mathbf{x},t)\right] * \left\langle j_i^{A-s}(\mathbf{x},t) \right\rangle n_i(\mathbf{x}) + \left\langle \bar{\phi}^B(\mathbf{x},t) \right\rangle * \left[j_i^{A-s}(\mathbf{x},t)\right] n_i(\mathbf{x}) \Big\} \, dS(\mathbf{x}) . \qquad (11.37)$$

As with the prior equation (11.34), this reciprocity theorem applies to heterogeneous media with general linear, time-invariant, and local constitutive relations. The conditions under which the right-hand-side vanishes are not investigated here.

Can our general time-correlation reciprocity theorem (11.37) be reduced to Welch's (1960) form, under the assumption of vanishing conductivity? Consider the charge continuity law combined with the current density constitutive relation:

$$\frac{\partial \theta(\mathbf{x},t)}{\partial t} + \sigma(\mathbf{x}) \text{div } \mathbf{e}(\mathbf{x},t) + \mathbf{grad}\, \sigma(\mathbf{x}) \cdot \mathbf{e}(\mathbf{x},t) = -\text{div } \mathbf{j}_s(\mathbf{x},t) .$$

For Welch's assumption $\sigma(\mathbf{x}) = 0$ this reduces to $\partial \theta(\mathbf{x},t)/\partial t = -\text{div } \mathbf{j}_s(\mathbf{x},t)$. Inserting this into our reciprocity theorem (11.37) gives

$$\int_V \left\{ a_i^A(\mathbf{x},t) * \frac{\partial \bar{j}_i^{B-s}(\mathbf{x},t)}{\partial t} - \bar{a}_i^B(\mathbf{x},t) * \frac{\partial j_i^{A-s}(\mathbf{x},t)}{\partial t} \right\} dV(\mathbf{x})$$

$$-\int_V \left\{ \phi^A(\mathbf{x},t) * \frac{\partial \bar{\theta}^B(\mathbf{x},t)}{\partial t} - \bar{\phi}^B(\mathbf{x},t) * \frac{\partial \theta^A(\mathbf{x},t)}{\partial t} \right\} dV(\mathbf{x}) = 0 ,$$

where the right-hand-side vanishes because there are no internal interfaces within a homogeneous wholespace. Next, integrating with respect to time (equivalent to convolving with the Heaviside step function $H(t)$) gives

$$\int_V \left\{ a_i^A(\mathbf{x},t) * \bar{j}_i^{B-s}(\mathbf{x},t) - \bar{a}_i^B(\mathbf{x},t) * j_i^{A-s}(\mathbf{x},t) \right\} dV(\mathbf{x})$$

$$-\int_V \left\{ \phi^A(\mathbf{x},t) * \bar{\theta}^B(\mathbf{x},t) - \bar{\phi}^B(\mathbf{x},t) * \theta^A(\mathbf{x},t) \right\} dV(\mathbf{x}) = 0 .$$

Writing out the convolutions as integrals over time yields the equivalent form

$$\int_V \left\{ \int_{-\infty}^{+\infty} a_i^A(\mathbf{x},\tau)\bar{j}_i^{B-s}(\mathbf{x},t-\tau)\,d\tau - \int_{-\infty}^{+\infty} \bar{a}_i^B(\mathbf{x},t-\tau) j_i^{A-s}(\mathbf{x},\tau)\,d\tau \right\} dV(\mathbf{x})$$

$$-\int_V \left\{ \int_{-\infty}^{+\infty} \phi^A(\mathbf{x},\tau)\bar{\theta}^B(\mathbf{x},t-\tau)\,d\tau - \int_{-\infty}^{+\infty} \bar{\phi}^B(\mathbf{x},t-\tau)\theta^A(\mathbf{x},\tau)\,d\tau \right\} dV(\mathbf{x}) = 0 .$$

Our reciprocity theorem is a function of time t, whereas Welch's theorem pertains to a single fixed value, independent of time. So, evaluating the above expression at t = 0 gives

$$\int_V \left\{ \int_{-\infty}^{+\infty} a_i^A(\mathbf{x},\tau)\bar{j}_i^{B-s}(\mathbf{x},-\tau)\,d\tau - \int_{-\infty}^{+\infty} \bar{a}_i^B(\mathbf{x},-\tau) j_i^{A-s}(\mathbf{x},\tau)\,d\tau \right\} dV(\mathbf{x})$$

$$-\int_V \left\{ \int_{-\infty}^{+\infty} \phi^A(\mathbf{x},\tau)\bar{\theta}^B(\mathbf{x},-\tau)\,d\tau - \int_{-\infty}^{+\infty} \bar{\phi}^B(\mathbf{x},-\tau)\theta^A(\mathbf{x},\tau)\,d\tau \right\} dV(\mathbf{x}) = 0 .$$

However, a doubly time-reversed function equals the original function, as per $\bar{\bar{x}}(t) = x(t)$. Thus, the above expression is finally re-written as:

$$\int_V \left\{ \int_{-\infty}^{+\infty} a_i^A(\mathbf{x},t) j_i^{B-s}(\mathbf{x},t)\,dt - \int_{-\infty}^{+\infty} a_i^B(\mathbf{x},t) j_i^{A-s}(\mathbf{x},t)\,dt \right\} dV(\mathbf{x})$$

$$-\int_V \left\{ \int_{-\infty}^{+\infty} \phi^A(\mathbf{x},t)\theta^B(\mathbf{x},t)\,dt - \int_{-\infty}^{+\infty} \phi^B(\mathbf{x},t)\theta^A(\mathbf{x},t)\,dt \right\} dV(\mathbf{x}) = 0 ,$$

in which the dummy time integration variable τ is replaced by t.

This compares favorably with Welch's (1960) equation (3), although he superposes a tilde "~" on the electromagnetic wavefields (but *not* the body sources) of state B. As indicated previously, Welch (1960) uses the tilde to signify "time-advanced" wavefields. However, our derivation (and subsequent reduction to Welch's particular form) does not require any such time-advancement. Curiously, the above expression does *not* agree with Bojarski's (1983) equation (76), even though he claims that equation represents "Welsh's (*sic*) reciprocity theorem for the electromagnetic vector and scalar potentials". The sign of the second integral is "+" rather than the "**–**" here.

11.7.3 Point Source Reciprocity

Several reciprocal relations involving various combinations of two point electromagnetic body sources were derived in chapter 4; these expressions contain the wavefields $\mathbf{e}(\mathbf{x},t)$ and $\mathbf{h}(\mathbf{x},t)$. It is straightforward to develop the analogous relations involving the vector potential $\mathbf{a}(\mathbf{x},t)$ and scalar potential $\phi(\mathbf{x},t)$.

We assume EM wavefield states A and B are activated by two point current density sources located at the different positions $\mathbf{x}_s^A$ and $\mathbf{x}_s^B$ within volume V, respectively. Equations (4.3) give these body sources as

$$\mathbf{j}_s^A(\mathbf{x},t) = J^A \mathbf{d}^A w^A(t)\delta\left(\mathbf{x}-\mathbf{x}_s^A\right), \qquad \mathbf{j}_s^B(\mathbf{x},t) = J^B \mathbf{d}^B w^B(t)\delta\left(\mathbf{x}-\mathbf{x}_s^B\right). \qquad \text{(4.3a,b again)}$$

Superscripts distinguish the magnitudes, orientations, waveforms, and positions of each point source. Substituting these into the time-convolution reciprocity theorem (11.34) and performing the volume integrals yields

$$J^A w^A(t) * \mathbf{d}^A \cdot \left\{ \frac{\partial \mathbf{a}^B(\mathbf{x}_s^A,t)}{\partial t} + \mathbf{grad}\,\phi^B(\mathbf{x}_s^A,t) \right\} = J^B w^B(t) * \mathbf{d}^B \cdot \left\{ \frac{\partial \mathbf{a}^A(\mathbf{x}_s^B,t)}{\partial t} + \mathbf{grad}\,\phi^A(\mathbf{x}_s^B,t) \right\}. \tag{11.38}$$

[Recall that $\int f(x)\delta'(x)dx = -f'(0)$ (with the negative sign!) as per Bracewell, 1965, page 82.] For simplicity, we assume that neither point current source is located on an internal interface S_n in V. Thus, the right-hand-side of equation (12.34) vanishes.

Equation (11.38) is a reciprocal relationship between two point current density sources located in a general (i.e., three-dimensional, heterogeneous, with LTL (= linear, time-invariant, local) constitutive relations) electromagnetic medium. Clearly, exactly the same expression is obtained (and with much less effort!) by substituting the electric vector expression $\mathbf{e}(\mathbf{x},t) = -\partial\mathbf{a}(\mathbf{x},t)/\partial t - \mathbf{grad}\,\phi(\mathbf{x},t)$ into the previously-derived reciprocity relation (4.4a). The derivational approach used above serves as a useful check on the mathematics leading to reciprocity theorem (11.34). Finally, we state that it is straightforward to verify equation (11.38) for the case of a homogeneous and isotropic EM wholespace. Transforming to the frequency-domain yields

$$J^A W^A(\omega)\mathbf{d}^A \cdot \left\{(-i\omega)\mathbf{A}^B(\mathbf{x}_s^A,\omega) - \mathbf{grad}\,\Phi^B(\mathbf{x}_s^A,\omega)\right\}$$
$$= J^B W^B(\omega)\mathbf{d}^B \cdot \left\{(-i\omega)\mathbf{A}^A(\mathbf{x}_s^B,\omega) - \mathbf{grad}\,\Phi^A(\mathbf{x}_s^B,\omega)\right\}.$$

Substituting the point source expressions for $\mathbf{A}(\mathbf{x},\omega)$ and $\Phi(\mathbf{x},\omega)$ indicates that this relation is exactly satisfied.

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Figure 5.6. Validation of point source/point receiver reciprocity for finite-difference electromagnetic modeling algorithm FDEM. **Red** and **blue** traces correspond to states A and B, respectively. **Blue** trace overplots **red** trace, confirming that reciprocity holds for this case of *current density sources* and *electric field receivers* in both states. This test validates reciprocity relation (4.4a) for the case where source waveforms of states A and B are identical. The situation is depicted in figure 4.1. Temporal/spatial FD operator orders are $O(M,N) = O(2,2)$.

Figure 5.7. Same as figure 5.6, except the temporal/spatial FD operator orders in algorithm FDEM are $O(M,N) = O(2,4)$. These EM responses are virtually identical to those displayed in figure 5.6.

Figure 5.8. Validation of point source/point receiver reciprocity for finite-difference electromagnetic modeling algorithm FDEM. **Red** and **blue** traces correspond to states A and B, respectively. **Blue** trace overplots **red** trace, confirming that reciprocity holds for this case of *dissimilar source types* activated by *different waveforms* in the two states.

Figure 6.1. Setup of the Fréchet derivative problem. An isotropic electromagnetic medium occupies volume V bounded by surface S. The medium is characterized by the three scalar material properties electric permittivity $\varepsilon(\mathbf{x})$, magnetic permeability $\mu(\mathbf{x})$, and current conductivity $\sigma(\mathbf{x})$. All parameters are functions of position $\mathbf{x}$ within V and on S, and are independent of time t. A point current density source is located at $\mathbf{x}^S$ and a point receiver (of either the electric or magnetic field) is located at $\mathbf{x}^R$. Within V, the electromagnetic wavefield satisfies the EH system of coupled first-order partial differential equations. On S, boundary conditions are imposed.

Figure 11.1. "$\partial \mathbf{a}(\mathbf{x},t)/\partial t$ traces" (top panel) and "$\mathbf{grad}\,\phi(\mathbf{x},t)$ traces" (bottom panel) recorded on a horizontal line array of receivers situated in a homogeneous and isotropic electromagnetic wholespace. The EM body source is a horizontally-directed point current density source located 2000 m below the center point of the receiver array at $x = 0$ m. Source activation waveform is an alternating-polarity boxcar pulse sequence (2 s positive, 2 s zero, 2 s negative, 2 s zero) of amplitude 10 A-m. Trace length is 8 s, with positive lobes shaded gray. The maximum absolute amplitude within each panel (occurring on the center trace at $x = 0$ m) is plotted at one trace spacing.

Figure 11.2. Top panel displays the sum of the two sets of traces depicted in figure 11.1. Bottom panel displays the x-component of the electric vector $e_x(\mathbf{x},t)$ calculated by numerical algorithm EMHOLE. The maximum absolute amplitude within each panel (occurring on the center trace at $x = 0$ m) is plotted at one trace spacing. Close agreement between the two panels is obviously obtained.

Figure 11.3. "Voltage difference traces". Each trace is divided by the horizontal separation distance d between two point voltage electrodes centered at the receiver station location.

Figure 11.4. "Voltage difference traces". Each trace is divided by the horizontal separation distance d between two point voltage electrodes centered at the receiver station location. Traces in the bottom panel for $d = 1$ m closely approximate the "$\mathbf{grad}\,\phi(\mathbf{x},t)$ traces" displayed in figure 11.1.

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