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A Method for the Analysis of Impact Systems using Numerical Continuation

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Abstract

The existence of impacts in mechanical systems inherently makes these systems nonlinear in nature. As a result, these systems can be difficult to thoroughly characterize. Furthermore, the results obtained through the analysis of these systems can be highly dependent on the models used to describe the contact during an impact event. Due to the inherent difficulties in the analysis of systems with impacts, more efficient methods are desired to carry out these analyses and compare the results obtained from different contact models. This report proposes numerical continuation as an efficient method for the analysis of impact systems and presents the results of an analysis of a single degree of freedom impact oscillator. Also, recommendations are given for future improvements to this method in order to make it practical for the analysis of complex systems.

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We would like to thank the following individuals for their contributions to this project: Brandon Deaner and Professor Matthew Allen of the University of Wisconsin, for providing the codes and papers that ultimately made this project a success, and Kurtis Ford, for his guidance throughout the summer and proofreading this report.

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1. INTRODUCTION

Countless mechanical systems contain components that experience impact behavior. This behavior, sometimes intentionally designed for, often introduces undesirable dynamics, such as wear and premature failure. In other cases, such as the case of impact dampers used to reduce vibrations in large buildings, impacts are a necessary behavior for the proper functioning of a system. The area of impact dynamics is an important field of research for any applications involving contact between bodies.

Mechanical impact is an inherently nonlinear phenomenon, which can have dramatic effects on their responses. The responses predicted by mathematical analyses can vary significantly with the methods used to model the contact experienced during impact [1]. In an effort to deal with the computational difficulties associated with the modeling of mechanical systems with impacts (*impact systems*), an efficient method is sought to simulate these systems and compare results using several different contact models. This report details the progress made during the author's summer internship period toward the implementation of such a method for the analysis of forced oscillating impact systems.

The information and quantities to be obtained through these simulations consisted of the following: 1) frequency response diagrams of position amplitude and contact velocity, 2) Lyapunov exponents to determine stability, 3) wear work rates to determine the degree of damage to a system caused by impact, and 4) phase portraits and Poincaré sections to visualize responses. Calculation of all of these items requires the discovery of periodic solutions of the equations of motion of the system being analyzed. The method used in this project to obtain these periodic solutions is a numerical continuation algorithm.

The remainder of this report is structured as follows: In Section 2, the numerical continuation method is introduced and explained. Four different numerical continuation toolboxes are considered in this study. Three open source toolboxes are detailed in Section 3, and the fourth, which is used throughout the rest of this study, is described in Section 4. This method is based on a toolbox developed at the University of Wisconsin and is specifically developed to analyze forced systems (unlike the three methods detailed in Section 3). An impact oscillator is analyzed with the toolbox in Section 5, and Section 6 presents recommendations for future work and the continuation of this project. Finally, Section 7 presents conclusions about the project.

2. NUMERICAL CONTINUATION

In order to investigate the frequency response, bifurcations, stability, and wear work of an impact system, it is necessary to obtain periodic solutions to the equations of motion over a range of forcing frequencies. From these periodic solutions, all items necessary to determine the desired quantities can be extracted.

A simple way to obtain a series of periodic solutions would be to carry out time integration for each forcing frequency in the desired range. However, this method is extremely inefficient for the following reasons: 1) Impact systems usually have long transients; therefore, a very long time span is usually needed to find a periodic response. Consequently, to integrate the equations of motion over very long time spans for a large number of forcing frequencies is computationally expensive and time consuming. 2) Unstable periodic responses can be extremely difficult to detect with time integration. This is because there may be very few sets of initial conditions that give rise to these unstable solutions, so their discovery usually occurs by chance. For these reasons, a long series of time integrations is not the most suitable method for the analysis of impact systems.

Numerical continuation is a method that provides a more efficient way to find a large set of solutions to a system's equations of motion. Given an initial solution (usually obtained via time integration), numerical continuation algorithms can quickly find entire branches of solutions, using information derived from the initial solution and equations of motion and a predictor-corrector method. Numerical continuation also has the ability to find unstable solutions.

2.1. Method

For the toolboxes used in this study, a similar method of numerical continuation is used in each case. The application of a numerical continuation method begins by first finding an initial solution. This initial solution is then perturbed as the control parameters (such as excitation frequency), and the change in system dynamics is calculated, from which the system's Jacobian about the previous solution is derived. This Jacobian is then used to predict a set of initial conditions and/or system parameters that give rise to another solution. Next, an iterative method (usually Newton-Raphson iterations) is used to correct the initial conditions and system parameters until convergence criteria are satisfied. This process is repeated, using the new solution obtained from the Jacobian analysis as an initial solution for the algorithm. For more information about numerical continuation techniques, see [2].

3. OPEN SOURCE CONTINUATION TOOLBOXES

Three open source numerical continuation toolboxes (AUTO [3], MATCONT [4], and COCO [5]) are commonly used for numerical continuation. In what follows, the suitability of these toolboxes is discussed in terms of their ability to be applied to analyze impact systems.

3.1. Toolbox Capabilities

The capabilities of the three open source numerical continuation toolboxes are summarized below.

3.1.1. AUTO

AUTO can be run by using either Python or Unix commands. It has the ability to analyze systems of ordinary differential equations of the form

$$\frac{dx}{dt} = f(x, \alpha)$$

where x is a vector of state variables and α represents all free parameters. It can compute solution curves to these systems subject to general nonlinear boundary conditions and integral constraints, and can determine folds and branch points along families of these solutions. AUTO can also compute families of periodic solutions and the Floquet multipliers that determine their stability, as well as the locations of folds, branch points, period doubling bifurcations, and bifurcations to tori along those families. Auto also has many other bifurcation detection and continuation abilities; these abilities are summarized below in Table 1. It should also be noted that AUTO has the ability to analyze algebraic systems and parabolic partial differential equations, though these features are not necessary for the analysis of impact systems.

3.1.2. MATCONT

MATCONT is written in Matlab and is controlled by a convenient graphical user interface (though a command line based version does exist), and has the ability to analyze systems of ordinary differential equations of the same form as AUTO. It can perform time integration for any set of initial conditions and free parameters to locate equilibrium points and limit cycles. Once these equilibrium points and limit cycles are located, MATCONT can continue them with respect to one or two parameters and detect many types of bifurcations. It can also compute curves of many types of bifurcation points. These and other abilities are summarized below in Table 1.

Table 1. Summary of Abilities of AUTO and MATCONT

Toolbox Capability	AUTO	MATCONT
Time Integration		+
Computation of solutions of boundary value problems	+	
Poincaré maps		+
Monitoring user functions along curves computed by continuation	+	+
Continuation of equilibria	+	+
Detection of branch points and codimension-1 bifurcations of equilibria	+	+
Computation of normal forms for codimension-1 bifurcations of equilibria		+
Continuation of codimension-1 bifurcations of equilibria	+	+
Detection of codimenion-2 bifurcations of equilibria		+
Computation of normal forms for codimension-2 bifurcations of equilibria		+
Continuation of limit cycles	+	+
Computation of phase response curves and their derivatives		+
Detection of branch points and codimension-1 bifurcations of cycles	+	+
Continuation of codimension-1 bifurcations of cycles	+	+
Branch switching at equilibrium and cycle bifurcations	+	+
Continuation of branch points of equilibria of cycles		+
Computation of normal forms for codimension-1 bifurcations of cycles		+
Detection of codimension-2 bifurcations of cycles		+
Continuation of orbits homoclinic to equilibria	+	+
Locate extrema of an integral objective functional along a family of periodic solutions and continue such extrema in other parameters	+	

3.1.3. COCO

COCO is written in Matlab and appears to be in early stages of development. Very little documentation detailing its use is provided. While COCO appears to be a potentially powerful continuation toolbox, it would not be practical to attempt to learn all of its functionality in the time frame of a summer internship. It is possible that COCO will become a very powerful and useful toolbox at some point in the future.

3.2. Results and Conclusions

AUTO is not compatible with Matlab, and the contact force models currently exist only as Matlab functions. More importantly, there is no evidence that AUTO can perform continuation on forced response systems [3]. Therefore, while AUTO is a very powerful toolbox, it is not suitable for use in this project.

As mentioned above, COCO has very little documentation detailing its capabilities or method of use. For this reason, COCO is not suitable for use in this project, especially given the time constraints in place.

While MATCONT is Matlab-based, easy to use, and has a wealth of documentation available, there is no evidence that it can perform continuation of forced systems, with or without impact [3]. After much investigation, it is concluded that MATCONT is unsuitable for this project.

Because none of the three open source tool boxes tested are deemed suitable for the continuation of forced response systems with impacts, another toolbox with the necessary capabilities is necessary.

4. FORCED SYSTEM CONTINUATION CODE

A set of codes (termed WisCont) with the ability to perform continuation on forced systems is provided by the University of Wisconsin with the permission of Professor Matthew Allen. These codes are not written for use with impact systems, so some modifications are necessary. Information about the mathematical theory behind this code is given in [3].

4.1. Modifications to the Continuation Code

Modifications made to WisCont to enable the continuation of impact systems are described below.

4.1.1. Variable Response Period

The first change made to WisCont is to allow the code to search for periodic responses that are integer multiples of the forcing frequency. The original version of WisCont uses the assumption that after the transients decay, the period of the system response is equal to the period of the forcing function. While this is true for most periodically forced systems, it is not necessarily true for those with impacts. In impact systems, the response period is an integer multiple of the forcing period.

4.1.2. Frequency Response Plotting

Response parameters of interest that are particular to impact systems include impact velocities and impact times. The new version of the code now includes functions that are designed to locate both the local extrema and contact velocities encountered during one period of the system response. Additional code also is used to plot these values as functions of forcing frequency. Therefore, frequency response diagrams for both position amplitude and contact velocity can be plotted (the user may choose one or both frequency responses in the initialization parameters).

4.1.3. Additional Storage for Post-Processing

In order to provide more information about the system dynamics and response, the continuation script now generates more output data (local extrema, contact velocities, response periods, and full time histories for each solution) to be stored for the calculation of wear work rates and Lyapunov exponents and the re-plotting of the frequency response(s) with color coding during post-processing.

4.1.4. Wear Work Rates and Lyapunov Exponents

Wear work rates and finite time Lyapunov exponents, quantities specifically developed for impact systems, are now calculated by the code during the post-processing phase. More information about the computation of wear work rates can be found in [1]. The Lyapunov exponents are calculated using the finite time method [4], though other, Jacobian based methods exist.

4.1.5. Color-Coding of Frequency Response(s)

The frequency response plots are color-coded in order to denote stable responses and unstable responses based off of marker color. The stabilities are determined by the signs of the Lyapunov exponents corresponding to each response (stable responses have non-positive Lyapunov exponents, and unstable responses have positive Lyapunov exponents).

4.2. Method of Use

This section documents how the modified version of WisCont is used to analyze an impact system.

4.2.1. Main File

Most of the variables needed for the analysis of impact systems are generated or stored by a main file, which performs the computations outlined below:

- **Initialization**

  First, the main file selects a contact model and calls scripts that initialize the parameters needed for that model. Parameters for the continuation (such as predictor and Jacobian step sizes and error tolerances for the integrators and correction steps) are also initialized.

  In order to produce a full frequency response diagram with as few gaps between points as possible, the continuation must be run multiple times, starting with an initial solution corresponding to a different forcing frequency each time. For this reason, a vector of starting forcing frequencies (in Hertz) is specified.

  Finally, a function containing the equations of motion is identified.

- **Initial Solution**

  A function is called that finds a set of initial conditions leading to an immediately periodic solution (one with virtually no transient response) corresponding to the first forcing frequency specified in the vector mentioned above. The function accomplishes this by integrating the equations of motion over many periods of the forcing function (this number of periods may be changed, but is usually around 50), and extracting a state vector at a time at or near the end of the time history corresponding to a zero of the forcing function. A time history is then calculated and plotted using the new initial conditions.

- **Continuation and Frequency Response**

  A loop is initiated that performs the continuation process, using the equations of motion, initial conditions, initial forcing period, and continuation parameters as inputs for the continuation code. During each instance of the loop, the continuation code calculates and plots a specified number of new solutions and outputs the following quantities: local extrema of the positions of the responses, contact velocities of the responses, initial state vector for each solution, periods of the forcing function and response for each solution, and a full time history for each solution. These quantities are all stored in a structure by the main file for post-processing (however, the reciprocals of the forcing periods are stored instead of the periods themselves to obtain a set of forcing frequencies).

  For every instance of the loop except the first, a new set of initial conditions leading to an immediately periodic solution is calculated using the next starting forcing frequency in the vector specified during initialization.

- **Wear Work Rates**

  After the continuation loop is completed, post-processing begins. The stored time histories are used (along with a function that calculates contact force, normally used in the equations of motion), to calculate the average wear work rate over one period of forcing for each solution. These wear work rates are plotted as a function of forcing frequency.

- **Lyapunov Exponents**

  The stored initial state vectors and forcing frequencies are used as inputs to a function that computes the Lyapunov exponent for each solution. The Lyapunov exponents are also plotted as a function of forcing frequency.

- **Color-Coded Frequency Response**

  The last step is a loop that re-plots each point on the frequency response diagram(s), using the Lyapunov exponents calculated earlier to determine stability (different colors are used for stable and unstable responses).

4.2.2. Phase Portraits and Poincaré Sections

After the main file is run and frequency response plots are examined, it may be useful to plot phase portraits and Poincaré sections to verify results and explore them further (for example, it may be desirable to look at the response behavior before, during, and after bifurcation points). Phase portraits and Poincaré sections can be generated by a script separate from the main file. The computations performed by this script are outlined as follows.

- **Initialization**

  The contact parameters and integrator options must be initialized in the same way as in the main file. The integrator options exist in the same file as the rest of the continuation parameters, so it is sufficient to simply initialize them as in the main file. Also, the forcing frequency of interest must be specified.

- **Initial Point**

  The same function that found the initial conditions leading to an immediately periodic solution for the primary analysis is used to find an initial point. By finding this specific initial point, the clutter in the phase portrait that would be caused by a transient is eliminated, and only a single periodic orbit will appear (in the case of unstable solutions, the orbit might not be truly periodic, but the initial transient will still be eliminated).

- **Time Integration**

  A time integration of the equations of motion is performed for a time span specified by the user (usually at least five forcing periods).

- **Plotting**

  A loop is initiated that plots each point in the generated time history in the phase plane. The Poincaré section for the specified frequency is generated by plotting specially marked points at time intervals equal to the period of the forcing function (this is accomplished by plotting special points for the times that are sufficiently near zeroes of a sinusoidal curve whose period is twice the forcing period).

In the next Section, this new code is applied to an impact system in order to further illustrate its capabilities.

5. RESULTS

An analysis of a simple impact oscillator consisting of a mass and a linear spring, with contact occurring at the equilibrium position of the spring (the contact force is applied in the negative direction any time the position is positive), is presented in what follows. A sinusoidal forcing function is applied to the mass. The mass and stiffness are chosen such that the natural frequency of the oscillator is 10 Hz. Details about this system, including equations of motion, are shown in Appendix A.

5.1. Frequency Response of Position Amplitude

The frequency response of the position amplitude is shown in Figure 1. The plot shows the absolute values of all local minima (because the mass is out of contact for all points where the position is negative) of the position vs. time curve present in one period of the response. This plot is color coded to show stability, with the local minima of stable solutions shown in blue and the local minima of unstable solutions shown in red.

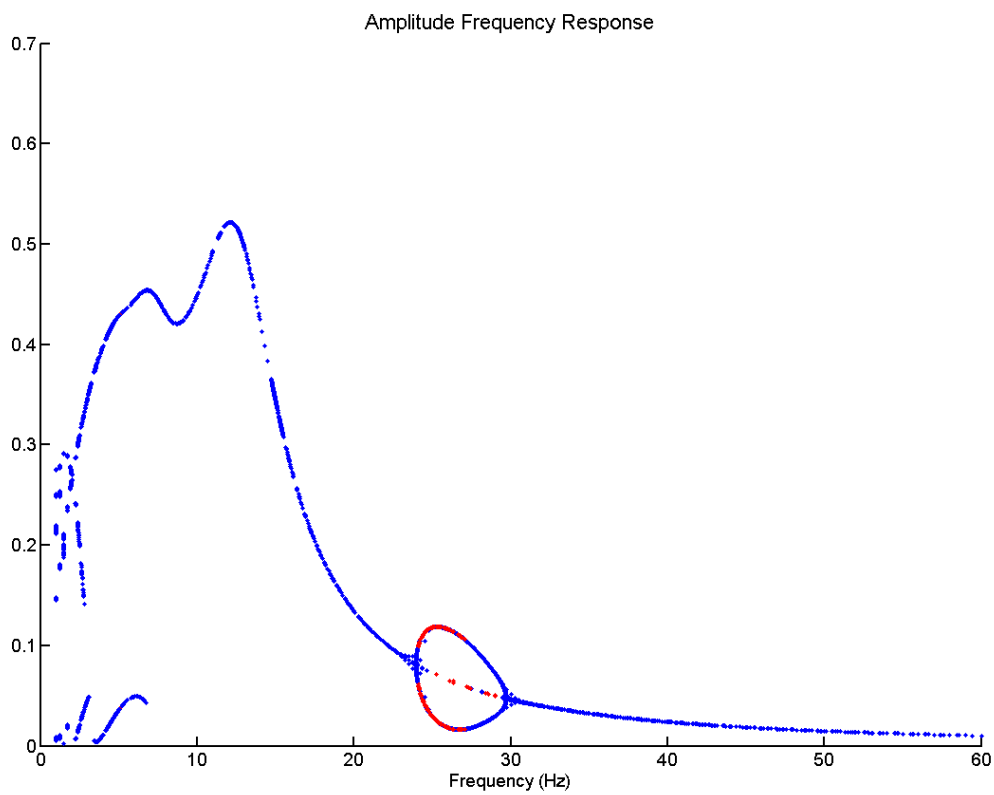


Figure 1. Position Amplitude Frequency Response

Figure 1. Position Amplitude Frequency Response shows very high amplitudes near the natural frequency of 10 Hz, and very small amplitudes at high frequencies. At very low frequencies, the response develops several local minima for each forcing frequency. Two bifurcations occur

between 20 and 30 Hz; between these bifurcations, the system has multiple attractors: one with two apparent local minima that changes stability at about 27 Hz, and one with one apparent local minimum that is unstable everywhere between the two bifurcations. In this case, unstable does not mean an unbounded response, but rather one that is likely to shift to a different equilibrium provided a small perturbation. Unstable, thus, indicates a local maximum in the system's potential energy function while stable indicates a local minimum.

5.2. Frequency Response of Contact Velocity

The frequency response of the contact velocity is shown in Figure 2. The plot shows the velocities at which the mass enters contact during one period of each solution. Like the position amplitude frequency response plot, this plot is also color coded for stability.

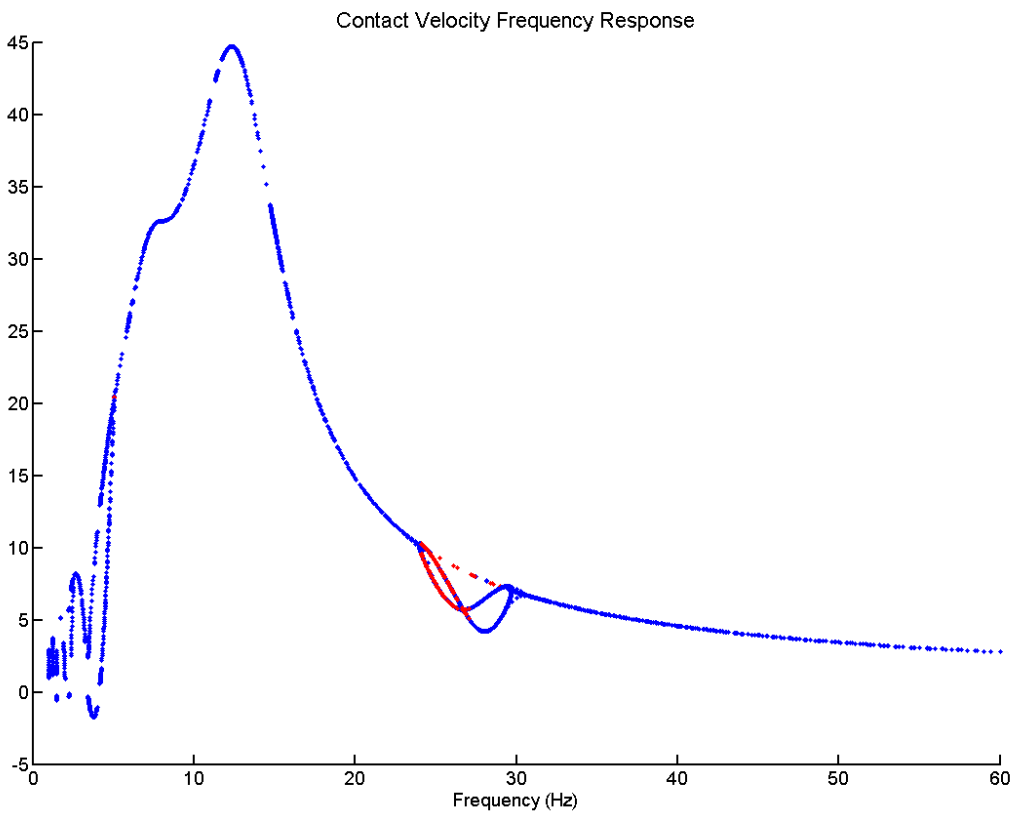


Figure 2. Contact Velocity Frequency Response

This frequency response exhibits very similar characteristics to that of the position amplitude; the response is large near the natural frequency and small at high frequencies. Also, in a manner much like the previous plot, there are several contact velocities at low frequencies. The two bifurcations between 20 and 30 Hz are also shown clearly. Note that over the region between the bifurcation points, the magnitude of the impact velocity is less than the period one solutions for the same region.

5.3. Wear Work Rates

A plot of wear work rates (averaged per forcing period) as a function of forcing frequency is shown in Figure 3.

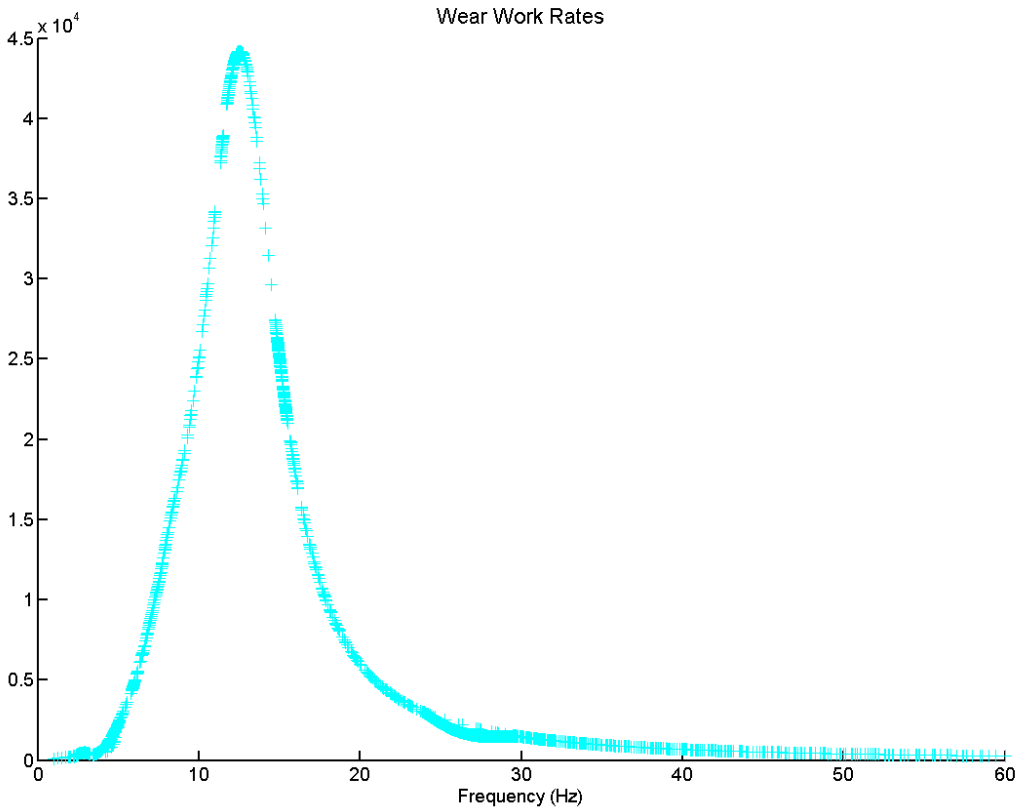


Figure 3. Wear Work Rates

The results of this plot are consistent with the frequency response plots. The highest wear work rates occur at the same frequencies as the highest contact velocities; this is expected because the wear work is computed as a time integral of the absolute value of the product of the forces and velocities experienced during contact. Consequently, the higher period responses exhibit lower wear work rates than adjacent period one responses.

5.4. Lyapunov Exponents

A plot of Lyapunov Exponents for each response is shown in Figure 4.

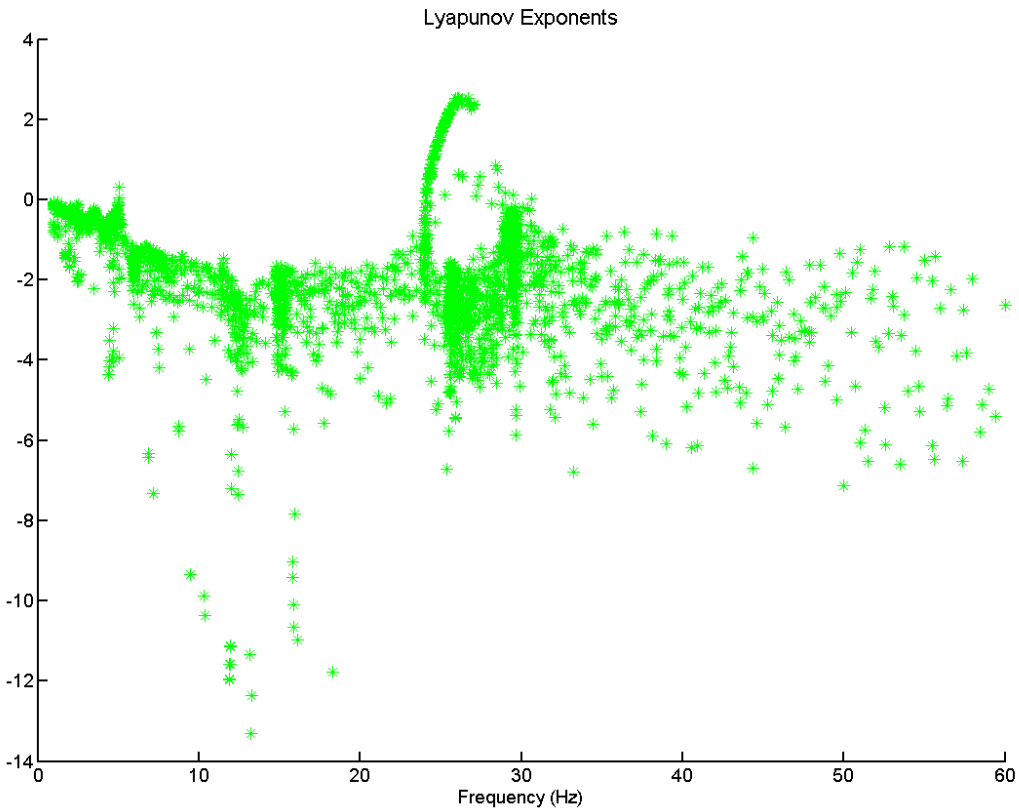


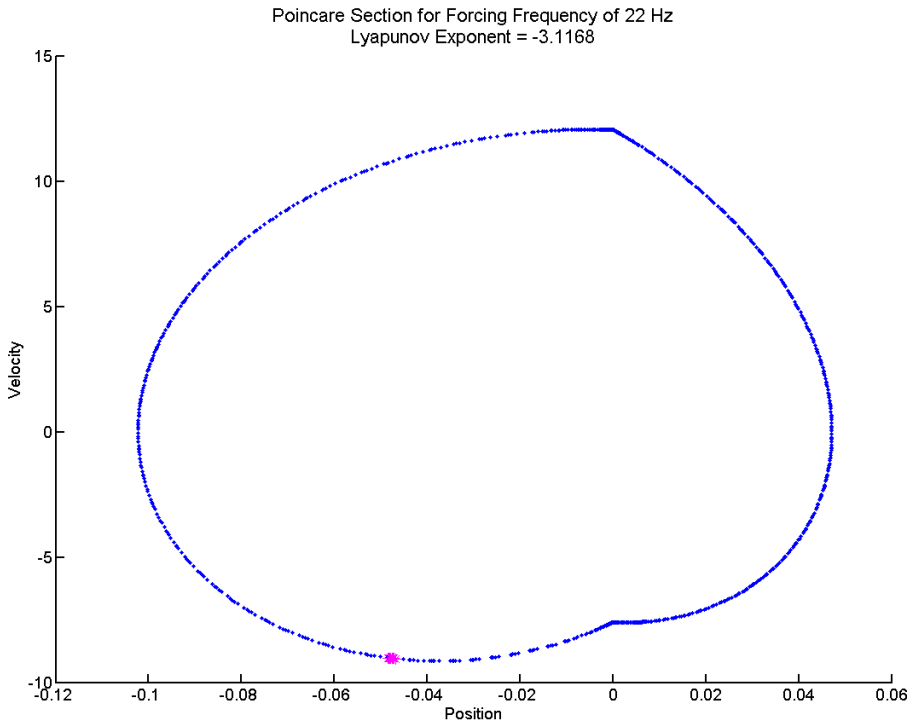
Figure 4. Lyapunov Exponents

As expected, the Lyapunov exponents are negative for nearly all solutions outside the region between the two bifurcation points. Inside that region, there are some positive Lyapunov exponents corresponding to the unstable responses shown in the frequency response plots, as well as negative ones corresponding to the stable responses; it is clear that multiple attractors are present. The positive values of Lyapunov exponents, in this case, confirm that small perturbations of the solution will cause it to move away from the unstable manifold. This is not indicative of a chaotic response (characterized by a broad band of responses in the frequency response plot), but rather the presence of an unstable manifold It should be noted that the values of the Lyapunov exponents are dependent on the manner in which they are calculated, so one method may produce a different plot than another. However, the qualitative behaviors should be similar.

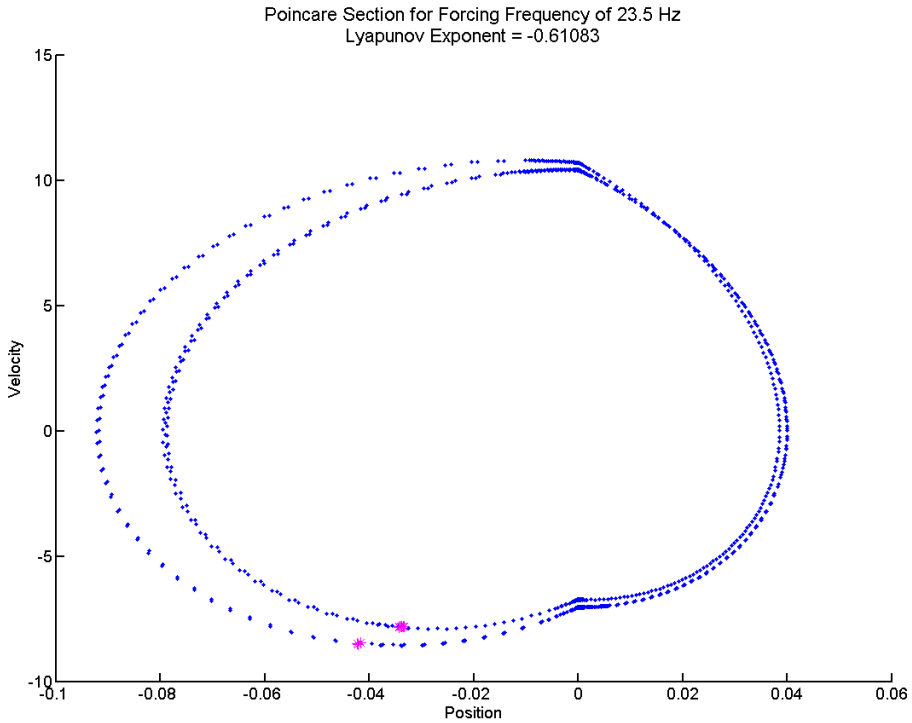
5.5. Phase Portraits and Poincaré Sections

Phase portraits and Poincaré sections are particularly useful for the analysis and visualization of the system's response at or near bifurcation points especially for assessing the unstable responses (unstable equilibriums, aperiodic solutions, or chaotic responses). Figure 5 shows phase portraits (blue dots) and Poincaré sections at zero phase (magenta asterisks) for the impact oscillator system at a series of frequencies around and between the bifurcations.

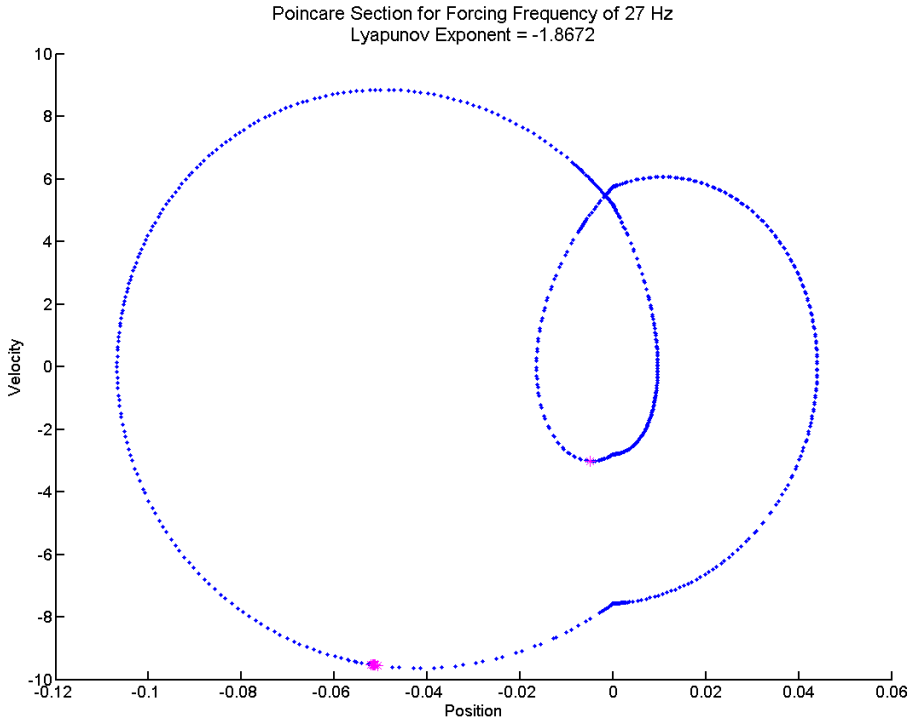
(a)


Poincare Section for Forcing Frequency of 22 Hz
Lyapunov Exponent = -3.1168
Velocity
Position
15
10
5
0
-5
-10
-0.12
-0.1
-0.08
-0.06
-0.04
-0.02
0
0.02
0.04
0.06

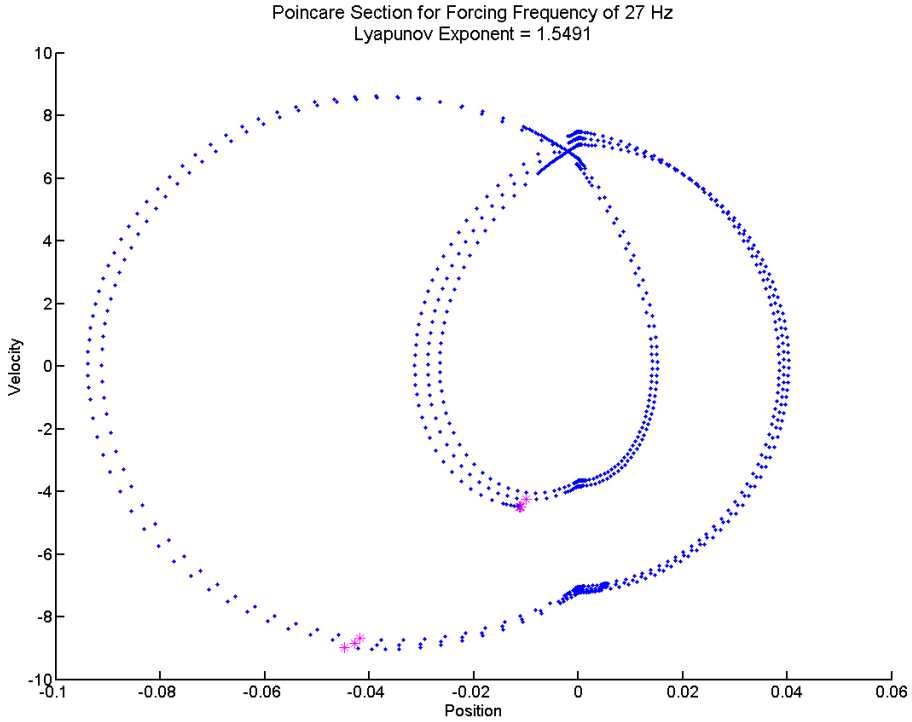
(b)


Poincare Section for Forcing Frequency of 23.5 Hz
Lyapunov Exponent = -0.61083
Velocity
Position
15
10
5
0
-5
-10
-0.1
-0.08
-0.06
-0.04
-0.02
0
0.02
0.04
0.06

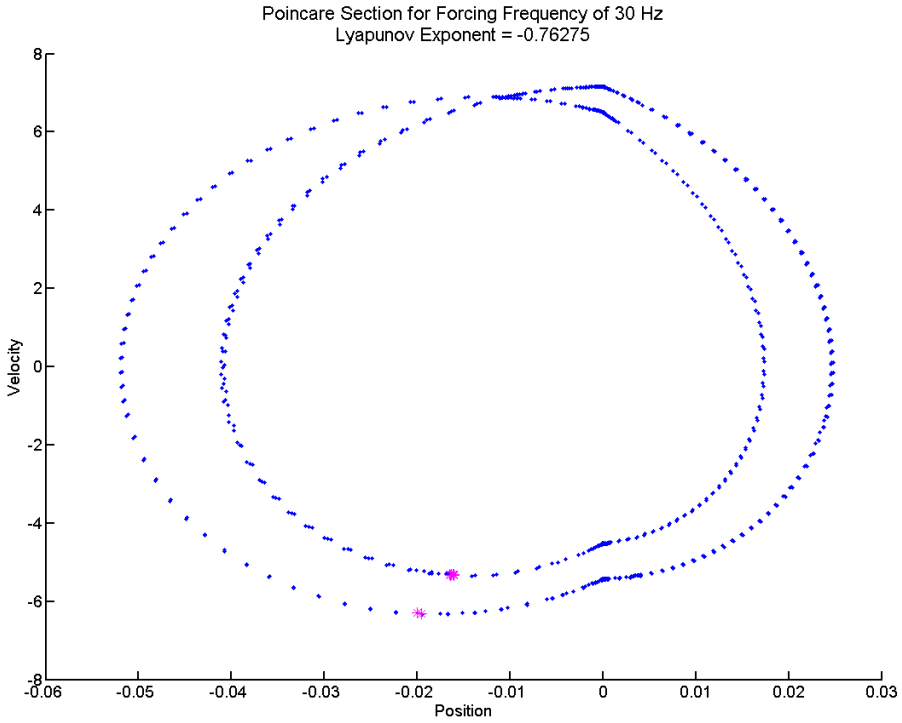
(c)


Poincare Section for Forcing Frequency of 27 Hz
Lyapunov Exponent = -1.8672
Velocity
Position
10
8
6
4
2
0
-2
-4
-6
-8
-10
-0.12
-0.1
-0.08
-0.06
-0.04
-0.02
0
0.02
0.04
0.06

(d)


Poincare Section for Forcing Frequency of 27 Hz
Lyapunov Exponent = 1.5491
Velocity
Position
10
8
6
4
2
0
-2
-4
-6
-8
-10
-0.1
-0.08
-0.06
-0.04
-0.02
0
0.02
0.04
0.06

(e)



(f)

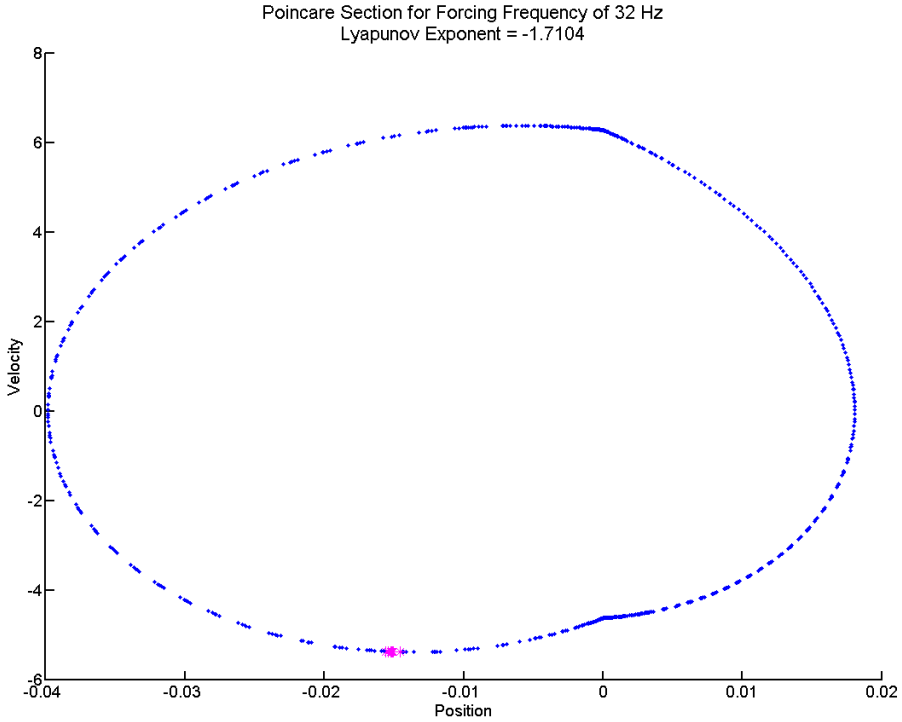


Figure 5. Phase Portraits and Poincaré Sections near Bifurcations

Figure 5 presents the evolution of the system's response across the bifurcated region: In Figure 5 (a), the phase portrait shows one contact initiation point (the sharp corner point near the top of the curve), and the Poincaré section only shows one distinct point, indicating that there is one period of forcing per period of the response. As the frequency is increased slightly from 22 Hz to 23.5 Hz in Figure 5 (b), a period doubling bifurcation occurs and there are two contact initiation points and two distinct points in the Poincaré section; there are now two periods of forcing per period of the response. The two coexisting solutions at 27 Hz are shown in Figure 5 (c)-(d). First the stable solution is presented in Figure 5 (c); note that the two loops appearing in the response are now quite different in size, as this response is approximately half way between the two bifurcation points. Second, Figure 5 (d) shows an unstable response at a forcing frequency of 27 Hz; the defining characteristic of this response is that the state vector never returns to its exact initial state. Finally, Figure 5 (e)-(f) show the transition during the second bifurcation point (between 30 and 32 Hz) to a response similar to that of the first bifurcation point.

It should be noted that other attractors may exist between the bifurcation points. Due to the difficulty and inconsistencies in calculating the Lyapunov exponents, any other attractors that exist might be difficult to distinguish and independently characterize.

6. FUTURE CONSIDERATIONS

There are several areas of future work for this project. Recommendations for improvement of the codes in order to make them of greater use are presented in what follows.

6.1. Improve Computation Time

While the code can perform analyses using the piecewise linear contact force model within a reasonable timespan, the use of more complicated contact force models can be prohibitively slow. This is due to that fact that these models require a very high degree of accuracy in time integration in order for the corrector steps in the continuation code to converge to a solution. When using one of Matlab's built-in integrators (i.e. *ode 45*, *ode113*, etc.), the relative and absolute error tolerances need to be set to a value of approximately 10^{-12} for the continuation code to find solutions.

In order to solve this problem, it may be worthwhile to develop an integrator with more speed at high accuracies than the ones built in to Matlab. This might be accomplished by using an implicit or implicit-explicit (IMEX) method, or an improved time step selection method.

However, care must be taken because some of the computations performed in the continuation code rely on Matlab's *deval* function, which acts on a structure and is produced by Matlab's integrators, but usually not by others. If a custom integrator could not produce a structure compatible with the *deval* function, extra modifications would need to be made to the continuation code.

6.2. Improve Coefficient of Restitution Model Analysis

The coefficient of restitution model is unique in that it requires instantaneous changes in velocity to be made during time integration. This requires that special time integration techniques be used. Currently, little success has been seen using this model, but this is likely because the integration scheme used had 1st order error and was therefore probably not accurate enough for the continuation code's corrector steps to converge to new solutions.

The use of a more accurate integration scheme with the ability to produce instantaneous velocity changes could solve this problem.

6.3. Improve Period Finding Code

The section of code that finds the integer ratio between the response period and the forcing period sometimes returns an integer that is itself an integer multiple of the true ratio. For example, a plot may indicate a response period of three times the forcing period, but the code may return a ratio of six, which is twice the ratio desired. While this issue is unlikely to inhibit the continuation code from finding new periodic solutions, it slows computation. This is because

the continuation code performs many time integrations over the response periods determined by the period ratios (a shorter period makes for shorter time integration).

6.4. Generalize Code for More Degrees of Freedom

While the analysis presented here was performed on a single degree-of-freedom system, it will later be necessary to allow for more degrees of freedom in order to analyze more complicated systems, such as impact dampers. Implementation of this change would not be difficult; the continuation code already supports multiple degrees of freedom, and most of the other codes could be modified through fairly simple changes.

6.5. Reorganize and Repackage Code

After working out computational issues, it would be worthwhile to package the codes in a more professional way (similar to the open source toolboxes), making them more amenable to use by others. Another possible change might be the addition of a graphical user interface (GUI); use of a GUI would dramatically decrease the time needed for new users to learn how to perform analyses.

7. CONCLUSIONS

This project developed an efficient method for the analysis of impact systems with forced oscillations. Numerical continuation is used as a means to efficiently obtain periodic solutions of a system's equations of motion. The three major open source numerical continuation toolboxes are assessed, and it is found that none of the three toolboxes are suitable for use in this project. Instead, a set of codes with the ability to perform continuation on forced systems developed by the University of Wisconsin is used. Several modifications are present in the final code that allow for the continuation of systems with impacts.

The capabilities of the current set of codes are the following:

- Perform numerical continuation to obtain a set of periodic solutions to the equations of motion of an impact system with one degree of freedom.
- Detect and plot the local extrema of the responses as a function of forcing frequency.
- Detect and plot the contact velocities experienced in the response as a function of forcing frequency.
- Compute and plot the average wear work rates associated with each response experienced during one forcing period.
- Compute and plot the Lyapunov exponents associated with each response.
- Re-plot frequency response curves of amplitude and contact velocity using color coding to indicate stability based on the signs of the Lyapunov exponents.
- Plot phase portraits and Poincaré sections to visualize the responses at any chosen forcing frequencies.

This code is applied in Section 5 to analyze a single degree-of-freedom impact oscillator. Finally, a set of suggestions for the continuation of this project and improvement of the codes are made:

- Improve computational efficiency by implementing a new temporal integrator.
- Improve the analysis of systems with a coefficient of restitution contact model by implementing a new integrator with the ability to account for instantaneous changes in velocity.
- Decrease integration time by increasing the accuracy of the sections of code that find the ratio of the response period to the forcing period.
- Generalize the codes to allow for the analysis of systems with more than one degree of freedom.
- Make the code more amenable to use by others by reorganizing it and repackaging it in a more logical and more professional way.

8. REFERENCES

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APPENDIX A - IMPACT OSCILLATOR SYSTEM

A.1 System Definition

The single degree of freedom system used in the analysis consists of a spring and mass connected in series. A sinusoidal forcing function acts on the mass. Contact is initiated whenever the position x is greater than the offset x_0. A diagram of the system is shown in Figure 6 Impact Oscillator System Diagram below.

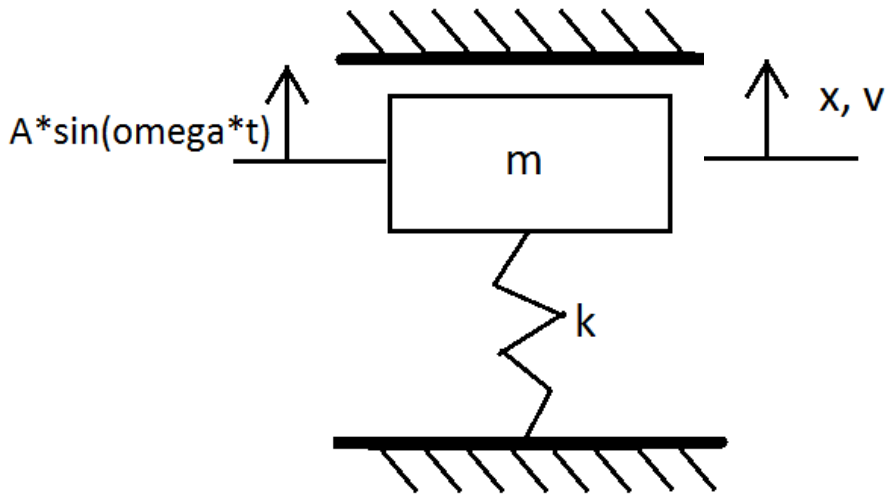

Figure 6 Impact Oscillator System Diagram

A.2 Equations of Motion

The equations of motion for the system in state space form are

$$\frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{cases} \begin{bmatrix} 0 & 1 \\ -k/m & 0 \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} * A\sin(\omega t), & x \le 0 \\ \begin{bmatrix} 0 & 1 \\ -k/m & 0 \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} * A\sin(\omega t) - \begin{bmatrix} 0 \\ 1/m \end{bmatrix} F_C(x, v), & x > 0 \end{cases}$$

In general, the contact force F_c is a function of the position and velocity of the mass *relative to the point of contact*. For this system, the point of contact is defined to be at $x = 0$.

The contact force is modeled as a piecewise linear constitutive relationship for this analysis; that is, the introduction of the new force during contact is equivalent to introducing a Kelvin-Voight

type element in parallel with the system stiffness. In general, any arbitrary contact force model may be used in place of this piecewise linear model, as long as it is only a function of relative position and velocity.

A.3 Parameters Used

The parameters that are used in the analysis in arbitrary (but consistent) units, are given in **Error! Reference source not found.**.

Table 2 System Parameters

Parameter	Value
Mass, m	1
System Stiffness, k	3947.84
Forcing Amplitude, A	1000
Piecewise Linear Stiffness	1 X 10^4
Piecewise Linear Damping	100

Note that the natural frequency of the system, $\omega_n = \sqrt{k/m}$, is equal to 62.832 radians/sec, or approximately 10 Hz.

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