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On the Stability of Stochastically Forced Parametric Oscillators

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Abstract

We present a method for determining the moment stability of stochastically forced ordinary differential equations. We provide many details omitted in [6]. We consider the case where the forcing arises from passing white noise through an nth order filter. We carry out a perturbation analysis for marginal moment tensors, based on a small parameter ε that gives the amplitude of the forcing. Our perturbation analysis is based on ladder operator approach to the vector Ornstein-Uhlenbeck process developed by the authors in [6].

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1 Introduction

This work is an elaboration on some of the techniques developed in [6]. The original goal of [6] was to develop a framework for analyzing the stability of the stochastically forced Mathieu equation:

$$\ddot{x} + \gamma\dot{x} + (\omega_0^2 + \varepsilon f(t))x = 0, \tag{1}$$

where f is a stochastic process, and the stability is determined by the boundedness of the second moment $\langle\langle x^2(t)\rangle\rangle$ [2, 9]. Here, $\langle\langle\cdot\rangle\rangle$ denotes the sample-average. We wanted to avoid heuristic methods, and consider cases where $f(t)$ is a stochastic process with a realistic power spectral density. In particular, we do not want to have to assume that f is white noise. Hence we want to analyze the case where $f(t)$ is colored noise. However, in order to rigorously derive a Fokker-Planck equation for a stochastic differential equation, the governing equation must include only white noise [2]. We can achieve both goals of rigor and realistic power spectral density by letting f be the output of a linear filter. That is,

$$\begin{aligned} \dot{\mathbf{s}} &= \mathbf{H}\mathbf{s} + \xi(t), \\ f(t) &= \langle \mathbf{a}, \mathbf{s}(t)\rangle, \end{aligned} \tag{2}$$

where $\mathbf{H}$ is an $n\times n$ real, diagonalizable matrix, whose eigenvalues have negative real part, and $\mathbf{a} \in \mathbb{R}^n$ ($\langle\cdot,\cdot\rangle$ is the standard inner product on $\mathbb{C}^n$). The noise ξ has mean zero and $\langle\langle \xi(t+\tau)\xi^T(t)\rangle\rangle = \mathbf{B}\delta(\tau)$, where $\mathbf{B}$ is symmetric positive semi-definite. We refer to the scalar process, $\langle \mathbf{a}, \mathbf{s}(t)\rangle$, as colored noise or as an nth order filter.

We were originally interested in equation (1) as a model for the response of capillary gravity waves to a time-varying gravitational field arising from random vertical motions of a container with a free surface (as in [14]). Here $f(t)$ represents the random fluctuations in acceleration. Since the Fourier transform of an acceleration should vanish at zero, along with its derivative, the power spectral density of a realistic process f should satisfy $S(0) = S'(0) = 0$. For example, we can construct a two-dimensional filter using the system (2) that has the power spectral density

$$S(\omega) = \frac{\sigma\beta^2\omega^2}{(\omega^2+\kappa^2)(\omega^2+\alpha^2)}, \tag{3}$$

for choices

$$\mathbf{H} = \begin{pmatrix} -\kappa & 0 \\ \beta & -\alpha \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{a} = \begin{pmatrix} \beta \\ -\alpha \end{pmatrix}, \quad \alpha, \kappa, \sigma > 0.$$

Given $\mathbf{H}$, $\mathbf{B}$, and $\mathbf{a}$, the formula for $S(\omega)$ in equation (3) follows from Corollary 13 in [6].

In [6] we analyzed the Fokker-Planck operator for the process $\mathbf{s}(t)$ in terms of its ladder operators. Here, we apply that analysis to develop perturbation theory for parametrically forced equations of the form

$$\dot{\mathbf{x}} = \mathbf{A}_0\mathbf{x} + \varepsilon\langle \mathbf{a}, \mathbf{s}\rangle \mathbf{A}_1\mathbf{x}, \tag{4}$$

where $\mathbf{A}_0$, $\mathbf{A}_1$ are $N\times N$ real matrices, and $\mathbf{s}$ is the solution to the stochastic equation (2).

The purpose of this paper is to present a method for determining the stability of the solution $\mathbf{x}(t)$ of (4), by which we mean the boundedness of the second moments of $\mathbf{x}(t)$. However, our method applies to the pth moment, so we will not limit our analysis to second moments only.

The Fokker-Planck equation for the combined system (2) and (4) is

$$\partial_t P = \frac{1}{2}\mathrm{div}_{\mathbf{s}}\left(\mathbf{B}\nabla_{\mathbf{s}}P\right) - \mathrm{div}_{\mathbf{s}}\left(\mathbf{H}\mathbf{s}P\right) - \mathrm{div}_{\mathbf{x}}\left((\mathbf{A}_0 + \varepsilon\langle\mathbf{a},\mathbf{s}\rangle\mathbf{A}_1)\mathbf{x}P\right) \tag{5}$$

See [8] for a derivation of this equation. The notation $\mathrm{div}_{\mathbf{s}}$ and $\nabla_{\mathbf{s}}$ refer to divergence and gradient with respect to only the s_j variables, and similarly $\mathrm{div}_{\mathbf{x}}$ is divergence in $\mathbf{x}$ variables. We note that (5) is the same in both the Itô and Stratonovich interpretations because the matrix $\mathbf{B}$ is independent of $\mathbf{s}$ and $\mathbf{x}$ (see [8]).

We can derive an equation for the pth marginal moments by multiplying (5) by monomials $\mathbf{x}^\alpha$ and integrating with respect to $d\mathbf{x}$, where α is a multi-index of order p. The result is an equation for $\mathbf{M}(\mathbf{s},t)$, a symmetric tensor of the pth marginal moments. In the case $p=2$ the equation for $\mathbf{M}$ is of the form

$$\partial_t\mathbf{M} = \mathscr{D}\mathbf{M} + \mathbf{A}_0\mathbf{M} + \mathbf{M}\mathbf{A}_0^T + \varepsilon\langle\mathbf{a},\mathbf{s}\rangle\left(\mathbf{A}_1\mathbf{M} + \mathbf{M}\mathbf{A}_1^T\right) \tag{6}$$

where $\mathscr{D}$ is a differential operator only with respect to the $\mathbf{s}$ variables, given by

$$\mathscr{D}\varphi = \frac{1}{2}\mathrm{div}_{\mathbf{s}}\left(\mathbf{B}\nabla_{\mathbf{s}}\varphi\right) - \mathrm{div}_{\mathbf{s}}\left(\mathbf{H}\mathbf{s}\varphi\right), \tag{7}$$

and $\mathbf{M}$ is the $N\times N$ symmetric matrix with $\mathbf{M}_{ij} = \int_{\mathbb{R}^N} x_i x_j P(\mathbf{s},\mathbf{x},t)d\mathbf{x}$. In (6), $\mathscr{D}\mathbf{M}$ indicates $\mathscr{D}$ applied to each component of $\mathbf{M}$. The case $p>2$ is addressed in Section 3.4.

The operator $\mathscr{D}$ is the same operator $\mathscr{D}$ that was characterized in terms of its associated ladder operators in [6]. Analysis of $\mathscr{D}$ was also done in [12, 13], but not in terms of the the ladder-operators developed in [6], which is what we need for carrying out perturbation analysis of (4). There are several useful properties of $\mathscr{D}$ that were shown in [6], and Section 2 we collect those properties that are used in our perturbation analysis.

As in a standard stability analysis, in order to determine the stability of (6), we look for solutions of the form $\widetilde{\mathbf{M}}(\mathbf{s},t) = e^{\lambda t}\mathbf{M}(\mathbf{s})$. Our equation for $\mathbf{M}(\mathbf{s})$ becomes

$$\lambda\mathbf{M} = \mathscr{D}\mathbf{M} + \mathbf{A}_0\mathbf{M} + \mathbf{M}\mathbf{A}_0^T + \varepsilon\langle\mathbf{a},\mathbf{s}\rangle\left(\mathbf{A}_1\mathbf{M} + \mathbf{M}\mathbf{A}_1^T\right) \tag{8}$$

That is, the equation for the second marginal moments of $\mathbf{x}(t)$ can be written as an eigenvalue problem, and stability is decided by the sign of the real part of the largest eigenvalue. The same is true for pth moments (see Section 3.4). As in [6] we assume that our eigenfunctions have bounded moments of all orders in $\mathbf{s}$. We present a perturbation method for determining this eigenvalue in Section 3. For $\varepsilon = 0$ we can analytically determine the eigenvalues of (8), and for nonzero ε we determine λ in a series expansion in the parameter ε. The coefficients of the expansion involves the *extended power spectral density* $G(z)$, which is defined for the process $\mathbf{s}(t)$ in (18), and is given explicitly in (19) for the filter $\langle\mathbf{a},\mathbf{s}(t)\rangle$. In particular, if λ_0 is the unperturbed eigenvalue, then in Section 3 we show $\lambda = \lambda_0 + \lambda_2\varepsilon^2 + \ldots$ with

$$\lambda_2 = 8 \sum_{j,k=1}^{N} \frac{C_{jkqr} C_{qrjk}}{1+\delta_{qr}} G(\sigma_q + \sigma_r - \sigma_j - \sigma_k), \tag{9}$$

where σ_k are the eigenvalues of $\mathbf{A}_0$, $\lambda_0 = \sigma_q + \sigma_r$, and the tensor C_{jklm} is given by

$$C_{jklm} = \frac{1}{4} \left(\delta_{jm} \langle \mathbf{g}_k, \mathbf{A}_1 \mathbf{h}_l \rangle + \delta_{km} \langle \mathbf{g}_j, \mathbf{A}_1 \mathbf{h}_l \rangle + \delta_{jl} \langle \mathbf{g}_k, \mathbf{A}_1 \mathbf{h}_m \rangle + \delta_{kl} \langle \mathbf{g}_j, \mathbf{A}_1 \mathbf{h}_m \rangle \right)$$

where $\mathbf{h}_k$ and $\mathbf{g}_k$ are the eigenvector and adjoint eigenvector (normalized so that $\langle \mathbf{h}_k, \mathbf{g}_j \rangle = \delta_{jk}$) of the matrix $\mathbf{A}_0$.

The generalization to higher moments is discussed in Section 3.4.

2 Preliminaries

In this section, we collect the results from [6] that we will need to develop perturbation theory for equation (8) .

2.1 Ladder Operators for $\mathscr{D}$

Lemma 1. *If* $\mathbf{H}$ *has simple eigenvalues* $-\mu_k < 0, , k = 1, n$ *and* $\mathbf{H}$ *and* $\mathbf{B}$ *form a controllable pair, then there are* $2n$ *linearly independent ladder operators,* $\mathscr{L}_k$*, associated with the operator* $\mathscr{D}$ *in equation (7). These operators satisfy*

$$[\mathscr{D}, \mathscr{L}_{\pm k}] = \pm\mu_k \mathscr{L}_{\pm k}, \quad \text{for } k = 1, n \tag{10}$$

where $[\mathscr{D}, \mathscr{L}_k] = \mathscr{D}\mathscr{L}_k - \mathscr{L}_k\mathscr{D}$*. Furthermore, we can write the operator* $\mathscr{D}$ *as*

$$\mathscr{D} = \sum_{k=1}^{n} \mu_k \mathscr{L}_{-k}\mathscr{L}_k. \tag{11}$$

The eigenfunction $\Phi_0(s)$ *of* $\mathscr{D}$ *associated with the eigenvalue with the largest real part satisfies*

$$\mathscr{L}_k \Phi_0 = 0 \quad k = 1, \ldots, n. \tag{12}$$

This eigenfunction has eigenvalue 0*, and that there is a matrix* $\blacksquare$ *such that* $\Phi_0(\mathbf{s}) = e^{-\frac{1}{2}\langle \mathbf{s}, \blacksquare \mathbf{s}\rangle}$.

We denote by $\mathbf{u}_k$ the eigenvectors of $\mathbf{H}$ with eigenvalue $-\mu_k$, and the adjoint eigenvectors by $\mathbf{v}_k$, with normalization $\langle \mathbf{v}_k, \mathbf{u}_j \rangle = \delta_{jk}$. We assume throughout that the $\{\mathbf{u}_k\}$ are complete.

In [6] we also proved

Lemma 2. *Let* $\mathbf{H}$ *have a complete set of eigenvectors. For any vector* $\mathbf{a}$*, there exist constants* α_k *and* β_k *such that*

$$\langle \mathbf{a}, \mathbf{s} \rangle = \sum_{k=1}^{n} \alpha_k \mathscr{L}_k + \beta_k \mathscr{L}_{-k}, \tag{13}$$

In particular,

$$\alpha_k = \langle \overline{\mathbf{u}}_k, \mathbf{a} \rangle, \qquad \beta_k = -\sum_{m=1}^{n} \frac{\overline{\alpha}_m}{\overline{\mu}_m + \mu_k} \langle \mathbf{v}_k, \mathbf{B}\mathbf{v}_m \rangle, \tag{14}$$

where $\mathbf{u}_k$ *and* $\mathbf{v}_k$ *are the eigenvectors and adjoint eigenvectors of the matrix* $\mathbf{H}$*. The equation for* α_k *can be written* $\alpha = \mathbf{U}^T\mathbf{a}$ *with* $\mathbf{U} = [\mathbf{u}_1, \ldots, \mathbf{u}_n]$.

The commutator relations for the ladder operators will also be useful in the perturbation analysis.

Lemma 3. *The ladder operators $\mathscr{L}_k$ satisfy the following commutator relations.*

$$[\mathscr{L}_j, \mathscr{L}_k] = 0, \qquad [\mathscr{L}_{-j}, \mathscr{L}_k] = \delta_{jk}, \qquad \textit{for } j,k = 1,\ldots,n. \tag{15}$$

We use the notation $\Phi_k = \mathscr{L}_{-k}\Phi_0$ to denote eigenfunctions that are the result of a lowering operator applied to the "top" eigenfunction Φ_0. The fact that $\mathscr{L}_k\Phi_0 = 0$ for $k > 0$ implies (using the second equation in equation (15)) $\mathscr{L}_k\Phi_k = \mathscr{L}_k\mathscr{L}_{-k}\Phi_0 = \mathscr{L}_{-k}\mathscr{L}_k\Phi_0 - \Phi_0 = -\Phi_0$. Collecting our notation, we have the following definition and lemma.

Definition 1. The functions $\Phi_k(\mathbf{s})$ are defined as

$$\Phi_k(\mathbf{s}) = \mathscr{L}_{-k}\Phi_0(\mathbf{s}) \tag{16}$$

where $\Phi_0(\mathbf{s})$ is the eigenfunction of $\mathscr{D}$ associated with the eigenvalue with the largest real part.

Lemma 4.

$$\mathscr{L}_k\Phi_k(\mathbf{s}) = -\Phi_0(\mathbf{s}), \qquad k = 1,\ldots,n. \tag{17}$$

2.2 The Expression for the Extended PSD

The expression for the perturbed eigenvalue λ will be written in terms of the Laplace transform of the autocorrelation function of the filter $\langle \mathbf{a}, \mathbf{s}(t)\rangle$. If $\mathbf{s}(t)$ is a stochastic process that becomes stationary in the limit $t \to \infty$ with autocorrelation function $\mathbf{R}(\tau)$, then in [6] we defined the *extended power spectral density* of the vector process $\mathbf{s}(t)$ as

$$\mathbf{G}(z) = \int_0^\infty \mathbf{R}(\tau)e^{-z\tau}\,d\tau. \tag{18}$$

Lemma 5. *Let* $\mathbf{H}$ *have simple eigenvalues* $-\mu_k < 0, , k = 1, n$ *and* $\mathbf{H}$ *and* $\mathbf{B}$ *form a controllable pair. With the definition of* $\mathbf{G}$ *in (18), the scalar filter* $\langle \mathbf{a}, \mathbf{s}(t)\rangle$ *has extended power spectral density* $G(z) = \langle \mathbf{a}, \mathbf{G}(z)\mathbf{a}\rangle$. *This can be written as*

$$G(z) = -\sum_{l=1}^{n} \frac{\alpha_l \beta_l}{\mu_l + z}, \tag{19}$$

provided $Re\,[\mu_l + z] > 0$ *for* $l = 1,\ldots,n$, *and* α_l, β_l *are defined in (14).*

Note that the power spectral density, $\mathbf{S}(\omega) = \int_{\mathbb{R}} \mathbf{R}(\tau)e^{i\omega\tau}d\tau$, of an asymptotically stationary process is defined only for $\omega \in \mathbb{R}$. Evaluating $\mathbf{S}(\omega)$ at complex values is not well-defined and does not make physical sense. However, the domain of the extended power spectral density is much larger, and $\mathrm{Re}\,[\mathbf{G}(i\omega)] = \frac{1}{2}\mathbf{S}(\omega)$. The domain of $\mathbf{G}$ is $\{z \in \mathbb{C} : \mathrm{Re}\,[\mu_l + z] > 0 \text{ for } l = 1,\ldots,n\}$, and therefore contains $i\mathbb{R}$ by our assumption that the eigenvalues of $\mathbf{H}$ have negative real part. Thus, $\mathbf{G}$ is an extension of $\mathbf{S}$ to an upper half-plane $\{z \in \mathbb{C} : \mathrm{Im}[i\mu_l + z] > 0, l = 1,\ldots,n\}$, which contains $\mathbb{R}$.

3 Perturbation Method for Moment Tensors

We begin with the case of second marginal moments, in which case we will have a 2-tensor of the second marginal moments. This is a special case, because the algebra can be expressed in terms of matrix algebra.

In this section we will use the following notation.

Definition 2. We define the operation $\mathbf{A}\star\mathbf{M}$ between two matrices as

$$\mathbf{A}\star\mathbf{M} = \mathbf{A}\mathbf{M} + \mathbf{M}\mathbf{A}^T \tag{20}$$

The following lemma shows what the equation for the evolution of the second marginal moments looks like in matrix form.

Lemma 6. *Let* $\mathbf{M}$ *be the* 2*-tensor of marginal second moments*

$$M_{jk} = \int_{\mathbb{R}^N} x_j x_k P(\mathbf{x},\mathbf{s},t)\,d\mathbf{x}. \tag{21}$$

$\mathbf{M}$ *can also be viewed as an* $N\times N$ *matrix that satisfies the evolution equation*

$$\partial_t\mathbf{M} = \mathscr{D}\mathbf{M} + (\mathbf{A}_0 + \varepsilon\langle\mathbf{a},\mathbf{s}\rangle\mathbf{A}_1)\star\mathbf{M}. \tag{22}$$

Proof. If we multiply (5) by $x_j x_k$ and integrate over all $\mathbf{x}$, we arrive at the equation (after integrating the $x_j x_k \mathrm{div}_{\mathbf{x}}(A\mathbf{x}P)$ term by parts)

$$\partial_t M_{jk} = \mathscr{D}M_{jk} + \int_{\mathbb{R}^N} \nabla(x_j x_k)\cdot((\mathbf{A}_0 + \varepsilon\langle\mathbf{a},\mathbf{s}\rangle\mathbf{A}_1)\mathbf{x}P)\,d\mathbf{x}. \tag{23}$$

If we let $\mathbf{A} = \mathbf{A}_0 + \varepsilon(\mathbf{a}\cdot\mathbf{s})\mathbf{A}_1$, the integral term in (23) has the form

$$\begin{aligned}\nabla(x_j x_k)\cdot\mathbf{A}\mathbf{x} &= \sum_{l,m=1}^{N} \partial_{x_m}(x_j x_k)A_{ml}x_l = \sum_{l,m=1}^{N}(x_j\delta_{mk} + x_k\delta_{mj})A_{ml}x_l \\ &= \sum_{l=1}^{N} A_{kl}x_l x_j + A_{jl}x_l x_k = (\mathbf{A}\star\mathbf{X})_{jk}\end{aligned} \tag{24}$$

where $\mathbf{X}$ is the matrix whose (j,k) component is $x_j x_k$. If we integrate this last expression over all values of $\mathbf{x}$, we arrive at the result of the lemma. □

This interpretation works only in the $p=2$ case. The case for general p is addressed in Section 3.4.

Looking for solutions of the form $\widetilde{\mathbf{M}}(\mathbf{s},t) = e^{\lambda t}\mathbf{M}(\mathbf{s})$, yields the eigenvalue problem for $\mathbf{M}(\mathbf{s})$

$$\lambda\mathbf{M} = \mathscr{D}\mathbf{M} + \mathbf{A}_0\star\mathbf{M} + \varepsilon\langle\mathbf{a},\mathbf{s}\rangle\mathbf{A}_1\star\mathbf{M}. \tag{25}$$

The marginal moment tensor $\mathbf{M}$ is symmetric ($M_{jk} = M_{kj}$), so we will use a basis of symmetric tensors to express $\mathbf{M}$.

Definition 3. We define the matrices $\mathbf{E}_{jk}$ by

$$\mathbf{E}_{jk} = \frac{1}{2}\left(\mathbf{h}_j\mathbf{h}_k^T + \mathbf{h}_k\mathbf{h}_j^T\right). \tag{26}$$

where $\mathbf{h}_j$ are eigenvectors of $\mathbf{A}_0$ with eigenvalues σ_j, We also define

$$\mathbf{F}_{jk} = \frac{1}{2}\left(\mathbf{g}_j\mathbf{g}_k^T + \mathbf{g}_k\mathbf{g}_j^T\right) \tag{27}$$

where $\mathbf{g}_k$ are the normalized adjoint eigenvectors of $\mathbf{A}_0$ (that is, $\langle \mathbf{g}_j, \mathbf{h}_k\rangle = \delta_{jk}$).

We will assume that the eigenvectors $\mathbf{h}_j$ of $\mathbf{A}_0$ form a complete set.

Lemma 7. *The matrices $\mathbf{E}_{jk}$ are eigenmatrices of $\mathbf{A}_0$ in the sense that*

$$\mathbf{A}_0 \star \mathbf{E}_{jk} = \mathbf{A}_0\mathbf{E}_{jk} + \mathbf{E}_{jk}\mathbf{A}_0^T = (\sigma_j + \sigma_k)\mathbf{E}_{jk}. \tag{28}$$

Assuming $\{\mathbf{h}_j\}$, the eigenvectors of $\mathbf{A}_0$, form a complete set, any symmetric matrix (with the same dimension as $\mathbf{A}_0$) can be written as a linear combination of the matrices $\mathbf{E}_{jk}$.

Proof. The first part of the lemma can be proved by direct substitution. The second part can be shown by showing that these matrices are linearly independent. We leave this as an exercise for the reader. □

Definition 4. We define the inner product of two 2-tensors as

$$\langle \mathbf{P}, \mathbf{Q}\rangle_2 = \operatorname{tr}\left(\mathbf{P}^T\mathbf{Q}\right) = \sum_{j,k=1}^{N} P_{jk}Q_{jk}, \tag{29}$$

This is the same as the standard matrix inner product for real $N \times N$ matrices.

With this definition, we have the following lemma

Lemma 8. *We have*

$$\langle \mathbf{F}_{lm}, \mathbf{E}_{jk}\rangle_2 = \frac{1}{2}\left(\delta_{km}\delta_{jl} + \delta_{kl}\delta_{jm}\right), \tag{30}$$

Proof. This follows from $\langle \mathbf{g}_j, \mathbf{h}_k\rangle = \delta_{jk}$. □

3.1 Zeroth Order

We assume an expansion of the form

$$\lambda = \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \ldots, \qquad \mathbf{M} = \mathbf{M}_0 + \varepsilon\mathbf{M}_1 + \varepsilon^2\mathbf{M}_2 + \ldots. \tag{31}$$

If we collect the terms at order zero we get

$$\lambda_0 \mathbf{M}_0 - \mathscr{D}\mathbf{M}_0 - (\mathbf{A}_0\mathbf{M}_0 + \mathbf{M}_0\mathbf{A}_0^T) = 0. \tag{32}$$

We expand about the eigenvalue of the operator $\mathscr{D}\mathbf{M} + \mathbf{A}_0 \star \mathbf{M}$ with the largest real part. The eigenfunctions of $\mathscr{D}$ are scalar-valued, and the eigenmatrices of $\mathbf{A}_0 \star \mathbf{M}$ are constant matrices. Assuming that both the eigenfunctions of $\mathscr{D}$ and the eigenmatrices of $\mathbf{A}_0 \star \mathbf{M}$ are complete, then the most general solution $\mathbf{M}_0$ to (32) will be a product of an eigenfunction of $\mathscr{D}$ with an eigenmatrix of $\mathbf{A}_0 \star \mathbf{M}$, and λ_0 will be the sum of the eigenvalues of $\mathscr{D}$ and $\mathbf{A}_0 \star \mathbf{M}$.

Since the eigenvalues of $\mathbf{A}_0 \star \mathbf{M}$ are of the form $\sigma_j + \sigma_k$, we see that for some values of q and r the eigenvalue $\mathbf{A}_0 \star \mathbf{M}$ with the largest real part can be written as $\sigma = \sigma_q + \sigma_r$. Since 0 is the largest eigenvalue of $\mathscr{D}$, (see [12, 13, 6]), the eigenvalue of $\mathscr{D}\mathbf{M} + \mathbf{A}_0 \star \mathbf{M}$ with the largest real part can be written as

$$\lambda_0 = \sigma_q + \sigma_r \tag{33}$$

Note that there may be several choices for σ_q and σ_r. For example, if the eigenvalue of $\mathbf{A}_0$ with the smallest real part is complex, we could take $\sigma_q = \sigma_r$, or $\sigma_q = \overline{\sigma}_r$. For now we will assume that we have chosen a particular q and r to begin the perturbation expansion.

The solution to (32) is

$$\mathbf{M}_0(\mathbf{s}) = \mathbf{E}_{qr}\Phi_0(\mathbf{s}), \qquad \lambda_0 = \sigma_q + \sigma_r. \tag{34}$$

We have the following lemma.

Lemma 9. *Let* $\mathbf{H}$ *have simple eigenvalues* $-\mu_k < 0, , k = 1, n$ *and* $\mathbf{H}$ *and* $\mathbf{B}$ *form a controllable pair. Assume the eigenvectors* $\{\mathbf{h}_j\}_{j=1}^N$ *of* $\mathbf{A}_0$ *are complete and have eigenvalues* σ_j*. Let* $\mathbf{P}$ *be a symmetric* $N \times N$ *matrix,* $\Phi(s)$ *be an eigenfunction of* $\mathscr{D}$ *with eigenvalue* $-\mu \neq 0$*, and* $\lambda_0 = \sigma_q + \sigma_r$*. If* $\sigma_q + \sigma_r + \mu \neq \sigma_j + \sigma_k$ *for each* j, k*, then the equation*

$$\lambda_0 \mathbf{Z} - \mathscr{D}\mathbf{Z} - (\mathbf{A}_0\mathbf{Z} + \mathbf{Z}\mathbf{A}_0^T) = \Phi(s)\mathbf{P}$$

has a solution $\mathbf{Z}$*, where* $\mathbf{Z} = \mathbf{Z}^T$ *and is explicitly given by*

$$\mathbf{Z} = \sum_{j,k=1}^{N} \frac{\langle \mathbf{F}_{jk}, \mathbf{P} \rangle_2}{\sigma_q + \sigma_r - \sigma_j - \sigma_k + \mu} \mathbf{E}_{jk}\Phi(s). \tag{35}$$

If $\sigma_q + \sigma_r \neq \sigma_j + \sigma_k$ *for* $q, r \neq j, k$*, then the equation*

$$\lambda_0 \mathbf{Z} - \mathscr{D}\mathbf{Z} - (\mathbf{A}_0\mathbf{Z} + \mathbf{Z}\mathbf{A}_0^T) = \Phi_0(s)\mathbf{P}$$

has a solution if and only if $\langle \mathbf{F}_{qr}, \mathbf{P} \rangle_2 = 0$*. In this case, the solution is given by*

$$\mathbf{Z} = \kappa \mathbf{E}_{qr}\Phi_0(s) + \sum_{j,k=1}^{N} \frac{\langle \mathbf{F}_{jk}, \mathbf{P} \rangle_2}{\sigma_q + \sigma_r - \sigma_j - \sigma_k} \mathbf{E}_{jk}\Phi_0(s), \tag{36}$$

where the $\mathbf{E}_{qr}$ *and* $\mathbf{E}_{rq}$ *terms are omitted from the sum, and* κ *is an arbitrary constant.*

The constant κ can be used to choose a normalization for $\mathbf{Z}$. We do not need to choose a specific normalization for $\mathbf{Z}$, so we set $\kappa = 0$ because it is convenient.

Proof. If $\mu \neq 0$, then when we write $\mathbf{P}$ in the $\mathbf{E}_{jk}$ basis, which is possible because $\mathbf{P}$ is symmetric and $\{\mathbf{h}_j\}_{j=1}^N$ are complete, and make the ansatz $\mathbf{Z}(\mathbf{s}) = \Phi(\mathbf{s})\mathbf{C}$, where $\mathbf{C}$ is a constant matrix, we arrive at the expression for $\mathbf{m}$ in equation (35). If $\mu = 0$, and hence $\Phi(\mathbf{s}) = \Phi_0(\mathbf{s})$, then we cannot solve this equation if $\mathbf{P}$ has any component in the direction of $\mathbf{E}_{qr}$. This gives the compatibility condition $\langle \mathbf{F}_{qr}, \mathbf{P}\rangle_2 = 0$. Assuming this holds, the solution is given by equation (36). It may appear that there is over-counting in formulas (35) and (36) because $\mathbf{E}_{jk} = \mathbf{E}_{kj}$, but the normalization in (30) compensates for this. □

We will now describe the outline of the perturbation analysis. In order to help us describe the perturbation analysis we will use the definition.

Definition 5. We say a function $\mathbf{f}(\mathbf{s})$ is in $\mathscr{V}_k$ if it can be written as the sum of eigenfunctions of $\mathscr{D}$ times constant vectors, where each of the eigenfunctions is the product of k or less ladder operators $\mathscr{L}_{-j}, j = 1, \ldots, n$ times the eigenfunction $\Phi_0(\mathbf{s})$.

The perturbation analysis proceeds as follows. We have a zeroth-order solution $\mathbf{M}_0 = \mathbf{E}_{qr}\Phi_0$, which is clearly in $\mathscr{V}_0$. We will see by induction , that the function $\mathbf{M}_k(\mathbf{s})$ will be in $\mathscr{V}_k$.

The equation at each higher order will be of the form

$$\lambda_0 \mathbf{M}_k - \mathscr{D}\mathbf{M}_k - \mathbf{A}_0 \star \mathbf{M}_k = -\lambda_k \mathbf{M}_0 + \mathbf{r}_{k-1}(s) \tag{37}$$

where $\mathbf{r}_{k-1}(s)$ is function that can be computed using the $\mathbf{M}_j$ and λ_j for $j < k$. In particular, we have

$$\mathbf{r}_{k-1}(\mathbf{s}) = -\sum_{j=1}^{k-1} \lambda_j \mathbf{M}_{k-j} + \langle \mathbf{a}, \mathbf{s}\rangle \mathbf{A}_1 \star \mathbf{M}_{k-1} \tag{38}$$

Assuming that for $j < k$ the functions $\mathbf{M}_j(\mathbf{s})$ are in $\mathscr{V}_j$, then using equation (13) we can see that the term $\mathbf{r}_{k-1}(\mathbf{s})$ will be in $\mathscr{V}_k$. We can write

$$\mathbf{r}_{k-1}(\mathbf{s}) = \Phi_0(\mathbf{s})\mathbf{b}_{k-1} + \hat{\mathbf{r}}_{k-1}(\mathbf{s}) \tag{39}$$

where the term $\hat{\mathbf{r}}_{k-1}(\mathbf{s})$ can be written as a sum of eigenfunctions of $\mathscr{D}$ times constant tensors, where none of the eigenfunctions is $\Phi_0(\mathbf{s})$. With this in mind we see that we will be able to solve equation (37) if and only if $\lambda_k \langle \mathbf{F}_{qr}, \mathbf{E}_{qr}\rangle_2 = \langle \mathbf{F}_{qr}, \mathbf{b}_{k-1}\rangle_2$, and therefore by (30)

$$\lambda_k = \frac{2}{1+\delta_{qr}} \langle \mathbf{F}_{qr}, \mathbf{b}_{k-1}\rangle_2. \tag{40}$$

Once we have chosen λ_k in this way, we can solve for $\mathbf{m}_k$, and it will clearly be in $\mathscr{V}_k$, thus allowing us to continue the process to the next value of k by induction.

Terms in $\mathbf{r}_{k-1}(\mathbf{s})$ proportional to Φ_0 can only arise at even steps in the process (i.e. equations for $\lambda_{2j}, \mathbf{M}_{2j}$) because $\mathscr{L}_k \Phi_k = -\Phi_0$ (see Lemma 17). These terms proportional to Φ_0 must satisfy the compatibility condition $\langle \mathbf{F}_{qr}, \mathbf{P} \rangle = 0$ as in Lemma 9.

3.2 First Order

Substituting the expansions in (31) into equation (25) and collecting terms of order ε, we arrive at the equation for $\mathbf{M}_1$

$$\lambda_0 \mathbf{M}_1 - \mathscr{D}\mathbf{M}_1 - \mathbf{A}_0 \star \mathbf{M}_1 = -\lambda_1 \mathbf{M}_0 + \left(\sum_{\ell=1}^{n} \alpha_\ell \mathscr{L}_\ell + \beta_\ell \mathscr{L}_{-\ell} \right) \mathbf{A}_1 \star \mathbf{M}_0. \tag{41}$$

Lemma 10. *Under the same assumptions as Lemma 9, we have $\lambda_1 = 0$ and*

$$\mathbf{M}_1 = \sum_{\ell=1}^{n} \Phi_\ell(s) \mathbf{C}_\ell \tag{42}$$

where

$$\mathbf{C}_\ell = \sum_{j,k=1}^{N} 2\beta_\ell \frac{\langle \mathbf{F}_{jk}, \mathbf{A}_1 \mathbf{E}_{qr} \rangle_2}{\sigma_q + \sigma_r - \sigma_j - \sigma_k + \mu_\ell} \mathbf{E}_{jk}, \tag{43}$$

and $\mathbf{E}_{jk}$ are defined in (26) and Φ_ℓ defined in Definition 1.

Proof. The only term proportional to $\Phi_0(s)$ is $-\lambda_1 \mathbf{M}_0$. But $\langle \mathbf{F}_{qr}, \mathbf{M}_0 \rangle_2 \neq 0$ because $\mathbf{M}_0 = \mathbf{E}_{qr} \Phi_0(s)$, so $\lambda_1 = 0$.

From Lemma 9, we have that

$$\begin{aligned} \mathbf{M}_1 &= \sum_{\ell=1}^{n} \sum_{j,k=1}^{N} \beta_\ell \frac{\langle \mathbf{F}_{jk}, \mathbf{A}_1 \mathbf{E}_{qr} + \mathbf{E}_{qr} \mathbf{A}_1^T \rangle_2}{\sigma_q + \sigma_r - \sigma_j - \sigma_k + \mu_\ell} \mathbf{E}_{jk} \Phi_\ell(s) \\ &= \sum_{\ell=1}^{n} \sum_{j,k=1}^{N} \beta_\ell \frac{2\langle \mathbf{F}_{jk}, \mathbf{A}_1 \mathbf{E}_{qr} \rangle_2}{\sigma_q + \sigma_r - \sigma_j - \sigma_k + \mu_\ell} \mathbf{E}_{jk} \Phi_\ell(s). \end{aligned} \tag{44}$$

We used the fact that $\mathbf{F}_{jk}$ and $\mathbf{E}_{qr}$ are symmetric to write $\langle \mathbf{F}_{jk}, \mathbf{A}_1 \mathbf{E}_{qr} + \mathbf{E}_{qr} \mathbf{A}_1^T \rangle_2 = 2\langle \mathbf{F}_{jk}, \mathbf{A}_1 \mathbf{E}_{qr} \rangle_2$ □

3.3 Second Order

If λ and $\mathbf{M}$ solve (25), and have the form as in (31), we can collect terms of order ε^2 to get the equation for $\mathbf{M}_2$

$$\lambda_0 \mathbf{M}_2 - \mathscr{D}\mathbf{M}_2 - \mathbf{A}_0 \star \mathbf{M}_2 = -\lambda_2 \mathbf{M}_0 + \left(\sum_{\ell=1}^{n} \alpha_\ell \mathscr{L}_\ell + \beta_\ell \mathscr{L}_{-\ell} \right) \mathbf{A}_1 \star \mathbf{M}_1. \tag{45}$$

Lemma 11. *Under the same assumptions as Lemma 9, we have*

$$\lambda_2 = -\frac{2}{\langle \mathbf{F}_{qr}, \mathbf{E}_{qr}\rangle_2} \sum_{\ell=1}^{n} \alpha_\ell \langle \mathbf{F}_{qr}, \mathbf{A}_1 \mathbf{C}_\ell \rangle_2 \tag{46}$$

Proof. With the definition of $\mathbf{C}_\ell$ in (43) we have $\mathbf{M}_1 = \sum_{\ell=1}^{n} \mathbf{C}_\ell \Phi_\ell(\mathbf{s})$. Using (13), (15), and (17) when expanding $\langle \mathbf{a}, \mathbf{s}\rangle \mathbf{A}_1 \star \mathbf{M}_1$, the only term proportional to $\Phi_0(\mathbf{s})$ is

$$-\lambda_2 \mathbf{E}_{qr} \Phi_0(\mathbf{s}) - \sum_{\ell=1}^{n} \alpha_\ell \mathbf{A}_1 \star \mathbf{C}_\ell \Phi_0(\mathbf{s}). \tag{47}$$

The compatibility condition from Lemma 9 gives

$$\lambda_2 \langle \mathbf{F}_{qr}, \mathbf{E}_{qr}\rangle_2 = -\sum_{k}^{n} \alpha_\ell \langle \mathbf{F}_{qr}, \mathbf{A}_1 \star \mathbf{C}_\ell \rangle_2. = -\sum_{\ell=1}^{n} 2\alpha_\ell \langle \mathbf{F}_{qr}, \mathbf{A}_1 \mathbf{C}_\ell \rangle_2$$

□

Theorem 1. *Assume the conditions of Lemma 9. For q, r fixed, if we have Re $\left[\sigma_q + \sigma_r - \sigma_j - \sigma_k + \mu_\ell\right] > 0$ for each $j, k = 1, \ldots, N$ and $\ell = 1, \ldots, n$, then*

$$\lambda_2 = 8 \sum_{j,k=1}^{N} \frac{C_{jkqr} C_{qrjk}}{1+\delta_{qr}} G(\sigma_q + \sigma_r - \sigma_j - \sigma_k), \tag{48}$$

where

$$C_{jklm} = \frac{1}{4}\left(\delta_{jm}\langle \mathbf{g}_k, \mathbf{A}_1 \mathbf{h}_\ell\rangle + \delta_{km}\langle \mathbf{g}_j, \mathbf{A}_1 \mathbf{h}_\ell\rangle + \delta_{jl}\langle \mathbf{g}_k, \mathbf{A}_1 \mathbf{h}_m\rangle + \delta_{kl}\langle \mathbf{g}_j, \mathbf{A}_1 \mathbf{h}_m\rangle\right)$$

Proof. From Lemma 10 we have

$$\begin{aligned}
\lambda_2 &= -\frac{1}{\langle \mathbf{F}_{qr}, \mathbf{E}_{qr}\rangle_2} \sum_{\ell=1}^{n} 2\alpha_\ell \sum_{j,k=1}^{N} 2\beta_\ell \frac{\langle \mathbf{F}_{jk}, \mathbf{A}_1 \mathbf{E}_{qr}\rangle_2 \langle \mathbf{F}_{qr}, \mathbf{A}_1 \mathbf{E}_{jk}\rangle_2}{\sigma_q + \sigma_r - \sigma_j - \sigma_k + \mu_\ell} \\
&= 4 \sum_{j,k=1}^{N} \frac{\langle \mathbf{F}_{jk}, \mathbf{A}_1 \mathbf{E}_{qr}\rangle_2 \langle \mathbf{F}_{qr}, \mathbf{A}_1 \mathbf{E}_{jk}\rangle_2}{\langle \mathbf{F}_{qr}, \mathbf{E}_{qr}\rangle_2} \sum_{\ell=1}^{n} \frac{-\alpha_\ell \beta_\ell}{\sigma_q + \sigma_r - \sigma_j - \sigma_k + \mu_\ell} \\
&= 4 \sum_{j,k=1}^{N} \frac{\langle \mathbf{F}_{jk}, \mathbf{A}_1 \mathbf{E}_{qr}\rangle_2 \langle \mathbf{F}_{qr}, \mathbf{A}_1 \mathbf{E}_{jk}\rangle_2}{\frac{1}{2}(1+\delta_{qr})} G(\sigma_q + \sigma_r - \sigma_j - \sigma_k).
\end{aligned} \tag{49}$$

We used $\langle \mathbf{F}_{qr}, \mathbf{E}_{qr}\rangle_2 = \frac{1}{2}(1+\delta_{qr})$, which follows from (30). To prove (48) we only need to show that $\langle \mathbf{F}_{jk}, \mathbf{A}_1 \mathbf{E}_{\ell m}\rangle_2 = C_{jk\ell m}$. But writing out $\langle \mathbf{F}_{jk}, \mathbf{A}_1 \mathbf{E}_{\ell m}\rangle_2$ we get a sum of four terms, each of the form

$$\mathrm{tr}\left(\overline{\mathbf{g}}_j \overline{\mathbf{g}}_k^T \mathbf{A}_1 \mathbf{h}_\ell \mathbf{h}_m^T\right) = \mathrm{tr}\left(\mathbf{A}_1 \mathbf{h}_\ell \mathbf{h}_m^T \overline{\mathbf{g}}_j \overline{\mathbf{g}}_k^T\right) = \delta_{jm} \mathrm{tr}\left(\mathbf{A}_1 \mathbf{h}_\ell \overline{\mathbf{g}}_k^T\right) = \delta_{jm}\langle \mathbf{g}_k^T, \mathbf{A}_1 \mathbf{h}_\ell\rangle.$$

This, together with (49), yields (48). □

3.4 Higher Moments

For pth marginal moments, we define the p-tensor $\mathbf{M}$ by

$$M_{k_1 k_2 \ldots k_p} = \int_{\mathbb{R}^N} x_{k_1} x_{k_2} \ldots x_{k_p} P(\mathbf{s}, \mathbf{x}, t) d\mathbf{x}. \tag{50}$$

We define the multiplication $\mathbf{A} \star \mathbf{M}$ between the 2-tensor $\mathbf{A}$ (which is an $N \times N$ matrix) and the p-tensor $\mathbf{M}$ as the p-tensor given by

$$(\mathbf{A} \star \mathbf{M})_{k_1 k_2 \ldots k_p} = \sum_{j=1}^{J} \sum_{i=1}^{N} A_{k_j, i} M_{k_1, \ldots, k_{j-1}, i, k_{j+1}, \ldots, k_p}. \tag{51}$$

If we multiply (5) by $x_{k_1} x_{k_2} \ldots x_{k_p}$ and integrate we obtain

$$\partial_t M_{k_1 \ldots k_p} = \mathscr{D} M_{k_1 \ldots k_p} + \int_{\mathbb{R}^N} \nabla (x_{k_1} x_{k_2} \ldots x_{k_p}) \cdot (\mathbf{A}_0 + \varepsilon \langle \mathbf{a}, \mathbf{s} \rangle \mathbf{A}_1) \mathbf{x} P d\mathbf{x} \tag{52}$$

The integral term is of the form

$$\begin{aligned} \nabla (x_{k_1} x_{k_2} \ldots x_{k_p}) \cdot \mathbf{A}\mathbf{x} &= \sum_{m,\ell=1}^{N} (\delta_{m k_1} x_{k_2} \ldots x_{k_p} + \delta_{m k_2} x_{k_1} x_{k_3} \ldots x_{k_p} + \\ &\qquad \ldots + \delta_{m k_p} x_{k_1} \ldots x_{k_{p-1}}) A_{m\ell} x_\ell \\ &= \sum_{j=1}^{J} \sum_{i=1}^{N} A_{k_j, i} x_{k_1} \ldots x_{k_{j-1}} x_i, x_{k_{j+1}}, \ldots x_{k_p} \end{aligned} \tag{53}$$

Therefore, with the definition of $\mathbf{A} \star \mathbf{M}$ given in (51), can write (52) as

$$\partial_t \mathbf{M} = \mathscr{D} \mathbf{M} + \mathbf{A}_0 \star \mathbf{M} + \varepsilon \langle \mathbf{a}, \mathbf{s} \rangle \mathbf{A}_1 \star \mathbf{M}, \tag{54}$$

As mentioned before, if $p = 1$, then $\mathbf{A} \star \mathbf{M}$ is just matrix multiplication, and if $p = 2$, then $\mathbf{A} \star \mathbf{M} = \mathbf{A}\mathbf{M} + \mathbf{M}\mathbf{A}^T$. Looking for solutions of the form $\widetilde{\mathbf{M}}(\mathbf{s}, t) = e^{\lambda t} \mathbf{M}(\mathbf{s})$, yields the equation for $\mathbf{M}(\mathbf{s})$

$$\lambda \mathbf{M} = \mathscr{D} \mathbf{M} + \mathbf{A}_0 \star \mathbf{M} + \varepsilon \langle \mathbf{a}, \mathbf{s} \rangle \mathbf{A}_1 \star \mathbf{M}. \tag{55}$$

To carry out the analysis as in Section 3, we need a basis for symmetric tensors (symmetric under permutation of the indices) and built from the $\{\mathbf{h}_j\}$. We can use tensors of the form

$$\mathbf{E}_{j_1 \ldots j_p} = \mathrm{Sym}\left(\mathbf{h}_{j_1} \otimes \mathbf{h}_{j_2} \cdots \otimes \mathbf{h}_{j_p}\right), \tag{56}$$

where the notation $\mathrm{Sym}\left(\mathbf{h}_{j_1} \otimes \mathbf{h}_{j_2} \cdots \otimes \mathbf{h}_{j_p}\right)$, means to take the average of all permutations of the tensor product of $\mathbf{h}_{j_1}, \mathbf{h}_{j_2}, \ldots, \mathbf{h}_{j_p}$. The adjoint basis is $\mathbf{F}_{j_1 \ldots j_p} = \mathrm{Sym}\left(\mathbf{g}_{j_1} \otimes \mathbf{g}_{j_2} \cdots \otimes \mathbf{g}_{j_p}\right)$, and the inner product is

$$\langle \mathbf{P}, \mathbf{Q} \rangle_p = \sum_{k_1=1}^{N} \sum_{k_2=1}^{N} \cdots \sum_{k_p=1}^{N} P_{k_1, k_2, \ldots, k_p} Q_{k_1, k_2, \ldots, k_p}. \tag{57}$$

There are $n_{p,N} = \begin{pmatrix} N+p-1 \\ p \end{pmatrix}$ distinct pth order monomials in N variables, therefore there are $n_{p,N}$ basis tensors. There is an analogous result to Lemma 9 for solving the perturbation equations, and the rest of the analysis follows accordingly.

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