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Radar Antenna Pointing for Optimized Signal to Noise Ratio

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Abstract

The Signal-to-Noise Ratio (SNR) of a radar echo signal will vary across a range swath, due to spherical wavefront spreading, atmospheric attenuation, and antenna beam illumination. The antenna beam illumination will depend on antenna pointing. Calculations of geometry are complicated by the curved earth, and atmospheric refraction. This report investigates optimizing antenna pointing to maximize the minimum SNR across the range swath.

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Foreword

This report details the results of an academic study. It does not presently exemplify any modes, methodologies, or techniques employed by any operational system known to the authors.

The specific mathematics and algorithms presented herein do not bear any release restrictions or distribution limitations.

This distribution limitations of this report are in accordance with the classification guidance detailed in the memorandum “Classification Guidance Recommendations for Sandia Radar Testbed Research and Development”, DRAFT memorandum from Brett Remund (Deputy Director, RF Remote Sensing Systems, Electronic Systems Center) to Randy Bell (US Department of Energy, NA-22), February 23, 2004. Sandia has adopted this guidance where otherwise none has been given.

This report formalizes preexisting informal notes and other documentation on the subject matter herein.

1 Introduction & Background

In this report we concern ourselves with airborne radar systems intending to interrogate the earth's surface. Examples include Synthetic Aperture Radar (SAR) systems, Ground Moving Target Indicator (GMTI) radar systems, Wide Area Search (WAS) radar systems (maritime and other), and similar radar systems.

Such radar systems typically want to interrogate some interval of ranges, termed a "range swath". However, signal quality, measured in terms of Signal to Noise Ratio (SNR) is generally not uniform across any range swath. An examination of the radar equation shows that this is due to several factors, including the following[1,2]

1. Elevation antenna beam pattern losses,

2. Spherical wavefront curvature losses, often termed R^4 losses, and

3. Atmospheric attenuation losses, also generally a function of range.

These factors conspire with the geometric effects of a curved earth, along with atmospheric refraction, to substantially complicate the calculation of how radar echo power is a function of range, particularly at long ranges and shallow angles. Note that atmospheric refraction effectively makes antenna boresight calibration range-dependent. Even more complicated is calculating an antenna pointing direction that optimizes SNR across a range swath, or even maximizes such a range swath given some SNR criteria.

Many radars ignore one or more of these factors, often simply pointing to some nominal center-swath range based on simple geometry and assuming straight-line propagation. This might by tolerable for relatively short ranges and narrow swaths, but becomes particularly problematic for long-range wide-swath operation.

Some systems do allow for manual trimming of the antenna pointing to assist target detection.

Other systems seek to mitigate antenna pattern effects by creating an antenna pattern to yield constant antenna gain as a function of range. An example of this is the well-known Cosecant Squared antenna pattern.[3] However, this comes at the expense of requiring a larger antenna aperture size to facilitate the requisite beam shaping. In addition, such a pattern is typically not very 'adjustable' if the collection geometry changes.

Some radar systems employ a Sensitivity Time Control (STC, or other 'fast automatic gain control'. These features address dynamic range issues, and are not related to optimizing SNR.

In this report, we detail calculations to optimize antenna beam pointing and/or range swath given the factors described above. Optimization is with respect to SNR.

“An ounce of performance is worth pounds of promises.”
-- Mae West

2 Models and Metrics

The purpose of antenna pointing is to bring antenna gain to bear for targets at the ranges and directions we want to interrogate. Antenna pointing is necessarily an angular adjustment of the antenna orientation. Herein this report we concern ourselves with choosing an optimum antenna depression angle for the ranges we wish to interrogate.

We begin by presenting a summary of the relationship between depression angle and range for a curved earth and atmospheric refraction.

Once this relationship is established, we can investigate the combined effects of range loss and antenna beam effects on Signal to Noise Ratio (SNR) across a range swath of interest, and various optimizations of swath versus SNR reduction.

2.1 Curved Earth Geometry with Refraction

The intent here is to derive useable functions that relate depression angle to range, for a curved earth and with a refractive atmosphere. An extensive discussion of refraction is given in an earlier report.[4]

2.1.1 The Basics

We shall assume a spherical earth, with the following parameters identified as

R_e = radius of the earth, nominally 6378 km,
h_a = altitude of aircraft,
h_s = altitude of target,
R = propagation path range from aircraft to surface target,
ϕ_e = earth surface angular change,
ψ_d = depression angle at aircraft (positive below horizontal), and
ψ_g = grazing angle at target (positive above horizontal). (1)

From the earlier report we will presume that atmospheric refraction is adequately accounted for by suitably scaling the earth's radius by a factor

k = earth radius scale factor. (2)

We stipulate that the propagation path range is in fact a distance measure along the curved propagation path that is not dependent on propagation velocity. Radar range calculation errors due to non-constant velocity of propagation are beyond the scope of this report, but will be addressed in a future report.

If the target surface is at a non-zero altitude above the surface of the spherical earth, then the various angles can be calculated from range and heights as

$$\sin\psi_d = \frac{(h_a - h_s)}{R}\left(1 - \frac{(h_a - h_s)}{2(kR_e + h_a)}\right) + \frac{R}{2(kR_e + h_a)},$$

$$\sin\psi_g = \frac{(h_a - h_s)}{R}\left(1 + \frac{(h_a - h_s)}{2(kR_e + h_s)}\right) - \frac{R}{2(kR_e + h_s)}, \text{ and}$$

$$\cos(\phi_e') \approx 1 - \frac{R^2 - (h_a - h_s)^2}{2(kR_e + h_s)(kR_e + h_a)}, \tag{3}$$

where

$$\phi_e' = \psi_d - \psi_g . \tag{4}$$

Furthermore, range can in turn be calculated from various angles and heights as

$$R = (kR_e + h_a)\sin\psi_d - \sqrt{(kR_e + h_s)^2 - (kR_e + h_a)^2\cos^2\psi_d}\ , \text{ or}$$

$$R = -(kR_e + h_s)\sin\psi_g + \sqrt{(kR_e + h_s)^2\sin^2\psi_g + 2(kR_e + h_s)(h_a - h_s) + (h_a - h_s)^2}\ . \tag{5}$$

Note that for such a range to exist, we require depression angle to satisfy

$$\cos\psi_d \le \frac{(kR_e + h_s)}{(kR_e + h_a)}. \tag{6}$$

The angle for which this is an equality is the minimum depression angle that will still yield a path to the target surface. This defines the radar horizon. More on this later.

The arc length along the earth's surface between nadir and the target is still given by

$$d = (R_e + h_s)\phi_e \approx (kR_e + h_s)\phi_e' . \tag{7}$$

Various models for k were given in the earlier report,[4] but we shall use an average value for k calculated based on an average radius of curvature, specifically calculated as

$$k_{avg} = \frac{1}{1 - \left(\frac{R_e}{\rho_{avg}}\right)} \approx \frac{1}{1 - \left(\frac{10^{-6} N_s \cos\psi_g R_e}{H_b}\right)\left(\frac{\frac{(h_a - h_s)}{H_b}}{e^{\frac{(h_a - h_s)}{H_b}} - 1}\right)}. \tag{8}$$

where

$$H_b = \frac{h_b - h_s}{\ln\left(\frac{N_s}{N_b}\right)}. \tag{9}$$

This model may be tailored to trade accuracy versus altitude. A reasonable set of parameters for 0 to 50 kft altitude is

$$h_b = 40\,\text{kft} = 12192\,\text{m}\text{, and}$$
$$N_b = 66.65 \text{ N-units.} \tag{10}$$

In the absence of prior knowledge of these factors, we will assume a value for surface refractivity that is average for the continental United States, namely

$$N_s = 313 \text{ m.} \tag{11}$$

Note that $k = 1$ yields the spherical earth model without any atmospheric refraction, and a flat earth is essentially the case where $k \to \infty$.

2.1.2 Radar Horizon

We identify the radar horizon as the range at which the grazing angle goes to zero. We calculate the radar horizon propagation path range as

$$R_{horizon} = \sqrt{2(kR_e + h_s)(h_a - h_s) + (h_a - h_s)^2}\ . \tag{12}$$

The target surface is not visible to the radar beyond this range. The corresponding depression angle is

$$\psi_{d,horizon} = \text{acos}\left(\frac{kR_e + h_s}{kR_e + h_a}\right). \tag{13}$$

We do note that this calculation for radar horizon does rely on a model for the average atmosphere. We also know that 'average' means exactly that, average. Any given atmosphere may depart from this model, and it may be possible to at times see farther than the nominal radar horizon. Consequently, it might be prudent when selecting radar parameter constraints to allow operation beyond the nominal radar horizon somewhat. That is, we may wish to allow ranges such that

$$R \le (1 + \beta_{horizon})R_{horizon}\ , \tag{14}$$

where

$$\beta_{horizon} = \text{radar horizon margin factor.} \tag{15}$$

A number like $\beta_{horizon}$ = 0.10 seems reasonable in absence of any other information.

Furthermore, if we wish to use k_{avg} for the value for *k*, then it depends on depression/grazing angle. However, since the radar horizon is expected to be at very shallow angles, we may often assume for the calculation of k_{avg} that the depression angle is effectively zero.

2.1.3 Useable Functions

Our task here is to define functions that relate depression angle to range.

Accordingly we define the range-to-angle function as

$$\psi_d = \Psi(R, h_a, h_s) = \operatorname{asin}\left(\frac{(h_a - h_s)}{R} \left(1 - \frac{(h_a - h_s)}{2(kR_e + h_a)} \right) + \frac{R}{2(kR_e + h_a)} \right). \tag{16}$$

We then also define the angle-to-range function as

$$R = \Psi^{-1}(\psi_d, h_a, h_s) = (kR_e + h_a)\sin\psi_d - \sqrt{(kR_e + h_s)^2 - (kR_e + h_a)^2 \cos^2\psi_d}\ , \tag{17}$$

subject to the constraint

$$\psi_d \geq \psi_{d,horizon}\,. \tag{18}$$

Hidden in these equations is the fact that if we employ $k = k_{avg}$, then there is an implicit dependence of *k* on depression angle ψ_d. While this works fine for calculating $\Psi^{-1}(\psi_d, h_a, h_s)$, it is somewhat more problematic for $\Psi(R, h_a, h_s)$, where ψ_d appears on both sides of the equal sign. Numerical techniques are likely required for best accuracy.

One way to address this is iteratively, noting that k_{avg} is relatively insensitive to ψ_d. Our procedure is then the following.

<u>*Step 1.*</u> Initialize $\psi_d = 0$,

<u>*Step 2.*</u> Use ψ_d to calculate k_{avg}.

<u>*Step 3.*</u> Use k_{avg} to calculate a new $\psi_d = \Psi(R, h_a, h_s)$.

Step 4. If convergence is achieved (the change in ψ_d is sufficiently small),
then exit this procedure,
else go back to Step 2.

In practice, a single iteration is often suitably accurate.

2.1.4 Depression Angle Relative Offset

Typical radar systems point their antennas based on geometry along, often accounting for earth curvature, but ignoring atmospheric refraction. A useful quantity is then the angular offset or bias required to be applied the antenna boresight depression angle from the boresight direction otherwise assumed by the radar. We then define the depression bias angle as the difference between the 'best' depression angle and the 'default' depression angle with whatever assumptions the radar otherwise makes. That is,

$$\Delta\psi_d = \text{depression bias angle.} \tag{19}$$

If the radar normally compensates for a curved earth, then the bias angle is calculated as

$$\Delta\psi_d = \Psi(R, h_a, h_s) - \Psi(R, h_a, h_s)\big|_{k=1} = \text{depression bias angle.} \tag{20}$$

If the radar normally compensates for a flat earth, then the bias angle is calculated as

$$\Delta\psi_d = \Psi(R, h_a, h_s) - \Psi(R, h_a, h_s)\big|_{k\to\infty} = \text{depression bias angle.} \tag{21}$$

2.1.5 Comments

We note that to some degree we might be able to perhaps calibrate the refraction by using an elevation IFSAR. For example, by observing the direction of arrival information we might discern where the antenna beam center (e.g. monopulse null) falls onto the target surface, and calculate refraction from this. In turn, we might be able to then say something about atmospheric characteristics such as humidity, and ultimately other characteristics like propagation velocity which might help in range calibration, and perhaps an expected atmospheric loss. This is beyond the scope of this report.

2.2 Radar Signal Quality Dependencies

The principal geometric parameters that influence the SNR of the scene being imaged include

1. The loss due to range variations.
2. The loss due to atmospheric attenuation
3. The loss due to antenna illumination function.

We address these in turn.

2.2.1 Range Loss

In general, range affects illumination by two mechanisms. The first is spherical spreading of the wavefront with range, and the second is via atmospheric attenuation. Of these, the spherical spreading is normally dominant. We identify a normalized gain function due to spherical spreading as

$$G_{range}(R) = \text{two-way range gain.} \tag{22}$$

For a monostatic radar, this can be calculated as[1]

$$G_{range}(R) = \left(\frac{R_{ref}}{R}\right)^4 . \tag{23}$$

We assume and identify crude properties as

$$\begin{aligned} &G_{range}(R) = 1, \text{ when } R = R_{ref} \text{ , and} \\ &G_{range}(R) \text{ increases as range decreases.} \end{aligned} \tag{24}$$

Note that this has the effect of a spatial lowpass filter (with respect to range). If we calculate a specific range loss

$$g = G_{range}(R), \tag{25}$$

then we define the inverse function as

$$R = G_{range}^{-1}(g) = \frac{R_{ref}}{\sqrt[4]{g}} , \tag{26}$$

with the stipulation that R is real and non-negative.

2.2.2 Atmospheric Loss

While not insignificant at long ranges, atmospheric attenuation is more difficult to predict. Atmospheric loss models may be found in the literature, including in papers by Doerry.[5,6]

We define here a function that is a relative loss function, namely

$$G_{atm}(R) = 10^{\frac{-\alpha(R-R_{ref})}{10}} \text{ = two-way atmospheric loss,} \tag{27}$$

where

$$\alpha \text{ = two-way atmospheric loss rate in dB per unit distance.} \tag{28}$$

This loss rate is a positive number and depends on atmospheric conditions as well as geometry factors. If we calculate a specific atmospheric loss

$$g = G_{atm}(R), \tag{29}$$

then we define the inverse function as

$$R = G_{atm}^{-1}(g) = R_{ref} - \frac{10\log_{10}(g)}{\alpha}. \tag{30}$$

This of course assumes that $\alpha > 0$, meaning that there is indeed atmospheric loss.

We note that a representative value for α in 50% relative humidity for Ku-band at 10 kft might be in the neighborhood of

$$\alpha \text{ = 0.05 dB/km.} \tag{31}$$

Rain will increase this substantially. Higher altitude will decrease this loss rate.

2.2.3 Antenna Beam Loss

Antenna illumination functions are unique to the electro-mechanical design of the antenna. We identify a generic illumination function with normalized elevation-direction dependence as

$$G_{el,ant}(\phi) \text{ = one-way antenna elevation power-gain pattern,} \tag{32}$$

where

$$\phi \text{ = angle from antenna boresight (positive towards nadir).} \tag{33}$$

We assume and identify crude properties as

$$G_{el,ant}(0) = 1,$$
$$G_{el,ant}(\phi_2) = G_{el,ant}(\phi_1) = \frac{1}{2} \text{ for some } \phi_2 \neq \phi_1 \text{, such that } |\phi_2 - \phi_1| = 1 \text{, and}$$
$$G_{el,ant}(\phi) \text{ is real and non-negative.} \tag{34}$$

This simply states that $G_{el,ant}(\phi)$ has unity maximum power gain, and unity –3 dB width, but need not be symmetrical in shape.

As a practical matter, $G_{el,ant}(\phi)$ is merely a model for the real behavior of the antenna. As such, the model will usually be chosen to be accurate over some limited interval around the boresight of the antenna, and then usually limited to the mainlobe. Accordingly, we define the limits of this model with two parameters,

$$\phi_{a,limit} = \text{angle above antenna boresight at limit of antenna model, and}$$
$$\phi_{b,limit} = \text{angle below antenna boresight at limit of antenna model.} \tag{35}$$

Both of these are normalized to a unit beamwidth.

For the subsequent discussion we reference Figure 1. The principal relevant geometric parameters are identified as

$$h_a = \text{radar height,}$$
$$h_s = \text{target surface height,}$$
$$\psi_{d,ant} = \text{antenna boresight depression angle,}$$
$$\psi_{d,n0} = \text{near range depression angle,}$$
$$\psi_{d,f0} = \text{far range depression angle,}$$
$$\psi_{g,ant} = \text{antenna boresight grazing angle,}$$
$$\psi_{g,n0} = \text{near range grazing angle,}$$
$$\psi_{g,f0} = \text{far range grazing angle,}$$
$$r_{c0,ant} = \text{reference slant range in direction of antenna boresight,}$$
$$r_{f0} = \text{maximum slant range of interest, and}$$
$$r_{n0} = \text{minimum slant range of interest,}$$
$$\theta_{el,nom} = \text{nominal elevation beamwidth (–3 dB) of the antenna.} \tag{36}$$

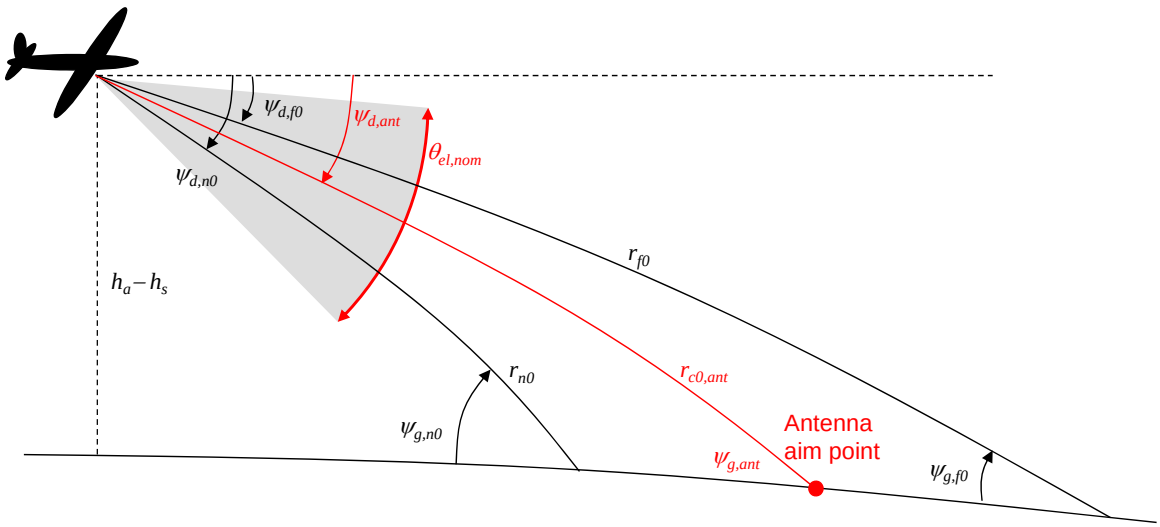

Figure 1. Geometry definitions for usable antenna beam illumination.

We define positive depression angles as below the horizontal (towards nadir), and positive grazing angles as above the horizontal (towards zenith).

Generally, the relationship between range and depression/grazing angles is dependent on the altitude of the radar and the topography of the scene, and includes effects of earth curvature and atmospheric refraction. That is, range R is a function of depression angle ψ_d as modeled in the previous sections.

With respect to our geometry, the antenna gain function with respect to depression angle ψ_d is

$$G_{el,ant}\left(\frac{\psi_d - \psi_{d,ant}}{\theta_{el,nom}}\right) = \text{one-way antenna elevation power-gain pattern.} \tag{37}$$

For monostatic radar systems, the data from a single pulse will exhibit the square of this gain, due to combined effects of transmission and reception.

Note that this has the effect of a spatial bandpass filter.

We also observe that other things equal, any particular gain value can typically be achieved with at least two different depression angles, one above boresight, and one below boresight. This complicates slightly the calculation of an inverse function.

If we calculate a specific antenna pattern loss factor as

$$g = G_{el,ant}(\phi), \tag{38}$$

then we define the inverse function as having two results, namely

$\phi_a = {}^a G_{el,ant}^{-1}(g)$ = the angle above boresight, and

$\phi_b = {}^b G_{el,ant}^{-1}(g)$ = the angle below boresight. (39)

Note that ϕ_a is negative, whereas ϕ_b is positive. Implicit are that these angles are within the interval where the antenna gain model is valid, that is, both angles are within $\lfloor \phi_{a,limit}, \phi_{b,limit} \rfloor$.

It is quite likely that a closed form expression is unobtainable for all but the most simple antenna gain models. Consequently, numerical techniques such as iterative techniques should be considered. Using Appendix C, an iterative technique to find $\phi_a = {}^a G_{el,ant}^{-1}(g)$ might be as follows.

<u>*Step 1.*</u> Initialize the angle estimate

$$\hat{\phi}_a = (0.95)\, \phi_{a,limit} \,. \tag{40}$$

<u>*Step 2.*</u> Select sample angles for estimating the derivative

$$\phi_1 = \hat{\phi}_a \text{, and}$$
$$\phi_2 = \phi_1 + 0.01. \tag{41}$$

<u>*Step 3.*</u> Calculate the angle step

$$\Delta\phi_1 = \mu \frac{\varepsilon(\phi_1)}{m(\phi_1)} = \operatorname{sgn}\left(\mu \frac{\varepsilon(\phi_1)}{m(\phi_1)} \right) \min\left(\left| \mu \frac{\varepsilon(\phi_1)}{m(\phi_1)} \right|, \Delta\phi_{1,\max} \right), \tag{42}$$

where the constituent values are

$$\varepsilon(\phi_1) = G_{el,ant}(\phi_1) - g \,,$$
$$m(\phi_1) = \frac{G_{el,ant}(\phi_2) - G_{el,ant}(\phi_1)}{\phi_2 - \phi_1} \text{, and}$$
$$\mu = 1.0 \,. \tag{43}$$

<u>*Step 4.*</u> Calculate the updated angle estimate $\hat{\phi}_a = \phi_1 - \Delta\phi_1$.

$$\hat{\phi}_a = \phi_1 - \Delta\phi_1 \,. \tag{44}$$

Step 5. If convergence criteria are met,
then exit iteration loop assuming $\phi_a = \hat{\phi}_a$,
else return to step 2.

The iterative technique for finding $\phi_b = {}^b G_{el,ant}^{-1}(g)$ can be similarly derived.

2.2.3.1 Simple Antenna Model

For a uniformly illuminated antenna aperture in elevation, we model the elevation beam pattern as

$$G_{el,ant}(\phi) = \text{sinc}^2(0.884\,\phi), \tag{45}$$

where

$$\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}. \tag{46}$$

Limits for this model might be the first null on either side of the mainlobe, that is

$$\begin{aligned} \phi_{a,limit} &= -1/0.884 \approx -1.13, \text{ and} \\ \phi_{b,limit} &= 1/0.884 \approx 1.13. \end{aligned} \tag{47}$$

2.2.3.2 Simpler Antenna Model

A somewhat simpler antenna elevation beam pattern is a simple quadratic given by

$$G_{el,ant}(\phi) = \left(1 - 2\phi^2\right)\text{rect}\left(\frac{\phi}{\sqrt{2}}\right), \tag{48}$$

where

$$\text{rect}(z) = \begin{cases} 1 & |z| \le 0.5 \\ 0 & else \end{cases}, \tag{49}$$

Limits for this model might be where the gain goes to zero, that is

$$\begin{aligned} \phi_{a,limit} &= -\sqrt{2}, \text{ and} \\ \phi_{b,limit} &= \sqrt{2}. \end{aligned} \tag{50}$$

2.2.3.3 Polynomial Antenna Model

Some radar systems employ a polynomial model for the antenna gain. For example, the model might be

$$G_{el,ant}(\phi) = \sum_{m=0}^{M} c_m \phi^m \,, \tag{51}$$

where

c_m = the polynomial coefficients,
m = polynomial index, and
M = the order of the polynomial. (52)

Often, $M = 4$ is quite adequate. Coefficients will be based on a fit to the actual antenna pattern.

Limits for this model will also need to be based on how well the model fits the actual antenna pattern. We simply state here

$\phi_{a,limit}$ = model limit above boresight, and
$\phi_{b,limit}$ = model limit below boresight (towards nadir). (53)

2.2.3.4 Electronically Steered Array

We note that for an Electronically Steered Array (ESA) antenna, whether an Active ESA (AESA) or a Passive ESA (PESA), the beam pattern depends on depression angle, and perhaps even squint angle. This of course means that the gain function is not adequately a simple function of angle off boresight alone, and therefore complicates calculation of both $G_{el,ant}(\phi)$ and its inverse(s). Dealing with this complication is beyond the scope of this report.

Hereafter in this report we will assume a simpler mechanically-steered antenna, and defer analysis of ESA systems for a future rainy day.

2.2.4 Combined Radar Signal Effects Model

The combined gain due to antenna illumination and range variations is given by

$$G_{combined}(\psi_{\mathrm{d}}, \psi_{d,ant}) = \left[G_{el,ant}\left(\frac{\psi_d - \psi_{d,ant}}{\theta_{el,nom}} \right) \right]^2 G_{range}(R)\, G_{atm}(R). \tag{54}$$

We make several important observations as follows.

- The depression angle ψ_d and range R are related by geometry that takes into account earth curvature and refraction, as discussed in an earlier section.

- Antenna pointing $\psi_{d,ant}$ clearly influences gain.

- The maximum gain for any specific range *R* is in the 'bent' boresight direction where a refracted ray of length *R* intersects the target surface. This is the case where $\psi_{d,ant} = \psi_d$.

- For a specific antenna boresight depression angle $\psi_{d,ant}$, the maximum combined gain across a swath is 'not' necessarily in the geometric boresight direction of where a ray of length *R* intersects the target surface, nor is it necessarily in the 'bent' boresight direction where a refracted ray of length *R* intersects the target surface.

- There is nothing magical about the –3 dB antenna beamwidth. Useable data can be collected from outside this region, as long as the minimum required gain is provided.

For later convenience, we have written the combined gain function as specifically a function of depression angle ψ_d, but could have just as easily made it a function of slant range *R*. An example of the combined gain as well as its constituents is shown in Figure 2.

If we calculate a specific combined loss factor as

$$g = G_{combined}\left(\psi_d, \psi_{d,ant}\right), \tag{55}$$

then we define the inverse function as having two results, namely

$$\psi_{d,a} = {}^{a}G^{-1}_{combined}\left(g, \psi_{d,ant}\right) = \text{the depression angle above boresight, and}$$
$$\psi_{d,b} = {}^{b}G^{-1}_{combined}\left(g, \psi_{d,ant}\right) = \text{the depression angle below boresight.} \tag{56}$$

These are rather difficult to calculate. The inverse function depends on antenna pointing and both grazing angle and range, which are of course related in a fairly complicated way. Essentially, we wish to solve for either $\psi_{d,a}$ or $\psi_{d,b}$ in the respective equations

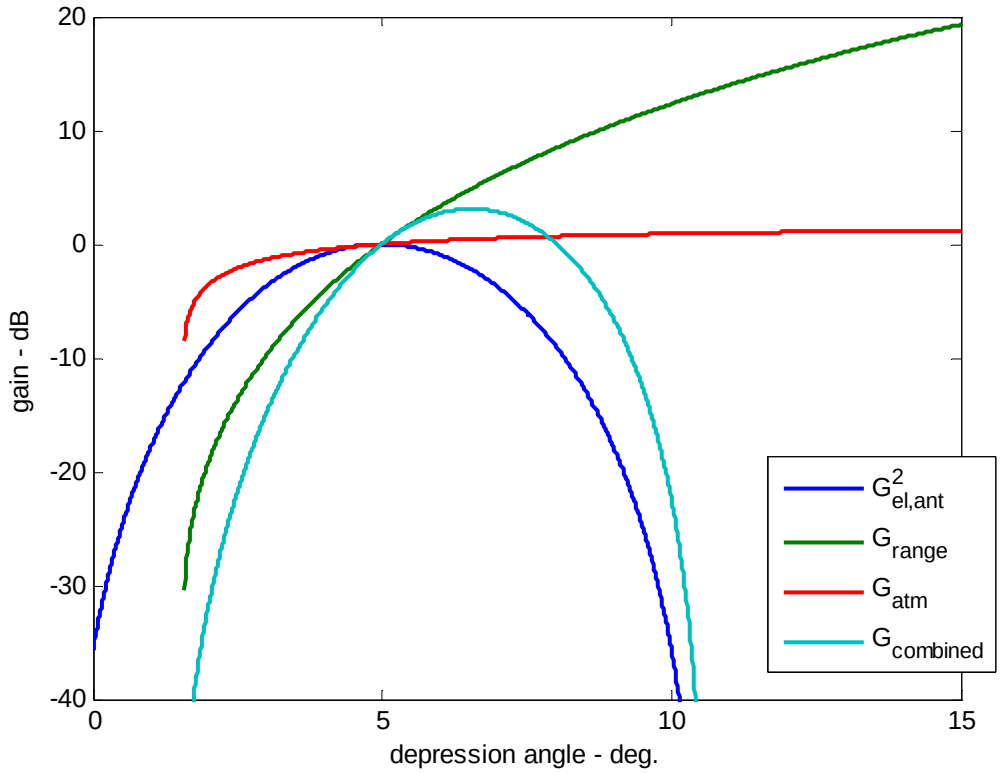

Figure 2. Example of combined gain and constituent values for h_a=10 kft, h_s = 0, α = 0.05 dB/km, $\theta_{el,nom}$ = 5 deg, $\psi_{d,ant}$ = 5 deg, and reference range at beam center. Atmospheric refraction assumes average values for the continental United States. Note that positive angles are towards nadir.

$$\left[G_{el,ant}\left(\frac{\psi_{d,a}-\psi_{d,ant}}{\theta_{el,nom}}\right)\right]^2 G_{range}(R)\,G_{atm}(R)=g\text{ , or}$$

$$\left[G_{el,ant}\left(\frac{\psi_{d,b}-\psi_{d,ant}}{\theta_{el,nom}}\right)\right]^2 G_{range}(R)\,G_{atm}(R)=g\,. \tag{57}$$

It is expected that numerical techniques will be required to solve these. Solving these with an iterative technique is discussed in Appendix A.

The remainder of this report seeks to optimize the relationship between antenna pointing $\psi_{d,ant}$ and the swath of ranges that the radar desires to process, that is, some set of ranges defined to be between r_{n0} and r_{f0}.

2.2.5 Hardware and Software Constraints

As a practical matter, antenna pointing will be limited to within some set of boundary depression angles. This may be due to hardware gimbal stops in the case of a mechanically steered antenna, or due to some signal fidelity requirements in the case of an AESA antenna. We define these limits as

$$\psi_{d,\min} \le \psi_{d,ant} \le \psi_{d,\max}, \tag{58}$$

where

$\psi_{d,\min}$ = the minimum (most shallow) depression angle limit, and

$\psi_{d,\max}$ = the maximum (most steep) depression angle limit. (59)

Later, we will also desire to limit the allowable gain reduction across some range swath with respect to the gain at some reference range. We accordingly define

$G_{\min}$ = the minimum acceptable combined gain factor. (60)

This will ultimately place limits on range swath due to antenna characteristics and pointing parameters. That is, we desire generally that

$$G_{combined}\left(\psi_{\mathrm{d}}, \psi_{d,ant}\right) \ge G_{\min} \qquad \text{for } \psi_{d,f0} \le \psi_{d,ant} \le \psi_{d,n0}. \tag{61}$$

The ranges that correspond to $\psi_{d,f0}$ and $\psi_{d,n0}$ define the allowable swath.

As a final note, we observe that using iterative techniques to solve equations means that even with convergence we get answers that are very close, and often good enough, but are not exact. This means that precision is limited. Consequently, practical calculations may need to consider this, and allow for degraded precision, especially where boundary conditions apply (e.g. radar horizon, etc.). We need to be able to deal with “close enough”.

“It is much more difficult to measure nonperformance than performance.”
-- Harold S. Geneen

3 Optimization Strategies

There are several different strategies that can be employed in selecting an optimum range swath. These differ in the constraints imposed onto the search, or calculations. We present several in the following sections.

3.1 Best Swath for Fixed Far Range

For this modality, the range swath is not fixed, and we desire to aim the antenna in elevation such that the minimum gain across the swath is some specified value, but the swath is maximized accordingly. This will occur when the combined gain at near range equals the combined gain at far range, and both are acceptable.

Our inputs will include

h_a = altitude of aircraft,
h_s = altitude of target, and
$r_{f0,desired}$ = maximum slant range of interest. (62)

Our constraints are

$G_{\min}$ = the minimum acceptable combined gain factor,
$\psi_{d,\min}$ = the minimum (most shallow) depression angle limit,
$\psi_{d,\max}$ = the maximum (most steep) depression angle limit,
$\beta_{horizon}$ = radar horizon margin factor. (63)

Our assumptions (in the absence of further information) are

N_s = 313, and
α = 0.05 dB/km (Ku-band at 10 kft, 50% RH). (64)

In addition, we have a model for our antenna beam shape that is valid over the interval defined by the normalized angles

$\phi_{a,limit}$ = angle above antenna boresight at limit of antenna model, and
$\phi_{b,limit}$ = angle below antenna boresight at limit of antenna model, (65)

with

$\theta_{el,nom}$ = nominal beamwidth of the antenna. (66)

Our outputs are the achievable near and far ranges that define the swath, r_{n0} and r_{f0}. An ancillary output is the antenna depression angle $\psi_{d,ant}$.

Although we have stated a desire for the maximum range to be $r_{f0,desired}$, this may not be feasible due to other factors. Consequently, we need to identify the actual achievable far range subject to the constraints

$$\begin{aligned} &r_{f0} \le r_{f0,desired}, \\ &r_{f0} \le (1+\beta_{horizon})R_{horizon}, \text{ and} \\ &r_{f0} \le \Psi^{-1}\left(\left[\psi_{d,\min} + \theta_{el,nom}\, {}^{a}G_{el,ant}^{-1}\left(\sqrt{G_{\min}}\right)\right], h_a, h_s\right), \end{aligned} \tag{67}$$

where angle arguments are limited to positive depression angles, and the ranges exist. In addition, we will also want to force a minimum range such that

$$r_{f0} \ge \Psi^{-1}\left(\left[\psi_{d,\max} + \theta_{el,nom}\, {}^{a}G_{el,ant}^{-1}\left(\sqrt{G_{\min}}\right)\right], h_a, h_s\right). \tag{68}$$

In the unlikely event that the minimum exceeds the maximum constraint, we will choose the minimum constraint.

With r_{f0} now chosen, we may calculate the actual depression angle to this range to be

$$\psi_{d,f0} = \Psi\left(r_{f0}, h_a, h_s\right). \tag{69}$$

In addition, we now assume that the reference range is

$$R_{ref} = r_{f0}. \tag{70}$$

This implies that the only element of the combined gain we need to worry about for selecting antenna boresight depression angle is due to the antenna beam pattern itself.

Actual antenna boresight depression angle is then calculated as

$$\psi_{d,ant} = \psi_{d,f0} - \theta_{el,nom}\, {}^{a}G_{el,ant}^{-1}\left(\sqrt{G_{\min}}\right). \tag{71}$$

Recall that the sign of ${}^{a}G_{el,ant}^{-1}\left(\sqrt{G_{\min}}\right)$ is negative. This antenna boresight depression angle should be within the limits previously identified.

With antenna boresight depression angle calculated, we now wish to find $\psi_{d,n0}$ such that

$$\psi_{d,n0} = {}^{b}G_{combined}^{-1}\left(G_{\min}, \psi_{d,ant}\right). \tag{72}$$

This is the angle below boresight that yields the minimum acceptable combined gain, which is the same as for the far range. As discussed in earlier sections, a closed form solution is generally not at all easy to calculate. Appendix A discusses an iterative technique for calculating this.

With the near edge angle calculated, we may then calculate the near range itself as

$$r_{n0} = \Psi^{-1}\left(\psi_{d,n0}, h_a, h_s\right). \tag{73}$$

With r_{n0} and r_{f0} now calculated, as well as $\psi_{d,ant}$, we are finished.

<u>Example</u>

We illustrate these calculations with an example.

Our inputs and constraints will be

$$\begin{aligned} &h_a = 20 \text{ kft}, \\ &h_s = 0, \\ &r_{f0,desired} = 100 \text{ km}, \\ &G_{\min} = -6 \text{ dB, and} \\ &\beta_{horizon} = 0.1. \end{aligned} \tag{74}$$

Our antenna parameters and constraints will be consistent with a sinc() function pattern, with

$$\begin{aligned} &\psi_{d,\min} = 5 \text{ deg.}, \\ &\psi_{d,\max} = 60 \text{ deg.}, \\ &\phi_{a,limit} = -1.1312, \\ &\phi_{b,limit} = 1.1312, \text{ and} \\ &\theta_{el,nom} = 7 \text{ degrees.} \end{aligned} \tag{75}$$

Our outputs are then calculated to be

$$\begin{aligned} &r_{f0} = 100 \text{ km}, \\ &r_{n0} = 24.8 \text{ km, and} \\ &\psi_{d,ant} = 7.37 \text{ deg.} \end{aligned} \tag{76}$$

3.2 Best Pointing for Fixed Swath

For this modality, the range swath is fixed, and we desire to aim the antenna in elevation such that the minimum combined gain over the entire swath is maximized. We then wish to adjust the antenna boresight depression angle to make the combined gain at the near and far ranges equal and maximum.

Our inputs will include

h_a = altitude of aircraft,
h_s = altitude of target,
$r_{f0,desired}$ far range of swath, and
$r_{n0,desired}$ = near range of swath. (77)

Our constraints are

$G_{\min}$ = the minimum acceptable combined gain factor,
$\psi_{d,\min}$ = the minimum (most shallow) depression angle limit,
$\psi_{d,\max}$ = the maximum (most steep) depression angle limit,
$\beta_{horizon}$ = radar horizon margin factor. (78)

Our assumptions (in the absence of further information) are

N_s = 313, and
α = 0.05 dB/km (Ku-band at 10 kft, 50% RH). (79)

In addition, we have a model for our antenna beam shape that is valid over the interval defined by the normalized angles

$\phi_{a,limit}$ = angle above antenna boresight at limit of antenna model, and
$\phi_{b,limit}$ = angle below antenna boresight at limit of antenna model, (80)

with

$\theta_{el,nom}$ = nominal beamwidth of the antenna. (81)

Our outputs are the achievable near and far ranges that define the swath, r_{n0} and r_{f0}. An additional output is the actual antenna depression angle $\psi_{d,ant}$.

Although we have specified near and far ranges, we might ensure that they fall within the ranges allowed by antenna pointing limits. Consequently, we require

$$
\begin{aligned}
& r_{f0} \le r_{f0,desired}, \\
& r_{f0} \le \left(1+\beta_{horizon}\right) R_{horizon}, \text{ and} \\
& r_{f0} \le \Psi^{-1}\left(\left[\psi_{d,\min} + \theta_{el,nom}\, {}^{a}G_{el,ant}^{-1}\left(\sqrt{G_{\min}}\right)\right], h_a, h_s\right),
\end{aligned}
\tag{82}
$$

where angle arguments are limited to positive depression angles, and the ranges exist. In addition, we will also want to force a minimum range such that

$$
r_{f0} \ge \Psi^{-1}\left(\left[\psi_{d,\max} + \theta_{el,nom}\, {}^{a}G_{el,ant}^{-1}\left(\sqrt{G_{\min}}\right)\right], h_a, h_s\right). \tag{83}
$$

In the unlikely event that the minimum exceeds the maximum constraint, we will choose the minimum constraint.

We might additionally require that the limits on near range satisfy

$$
\begin{aligned}
& r_{n0} \ge r_{n0,desired}, \text{ and} \\
& r_{n0} \ge \Psi^{-1}\left(\psi_{d,n0,\min}, h_a, h_s\right),
\end{aligned}
\tag{84}
$$

where the nearest range is limited by the steepest allowable antenna depression angle given as

$$
\psi_{d,n0,\min} = {}^{b}G_{combined}^{-1}\left(G_{\min}, \psi_{d,\max}\right), \tag{85}
$$

and the reference range for this particular angle calculation is also at the steepest antenna depression angle, namely

$$
R_{ref} = \Psi^{-1}\left(\psi_{d,\max}, h_a, h_s\right). \tag{86}
$$

With final edges of the swath now chosen, we may calculate the actual depression angles to these edges to be

$$
\begin{aligned}
& \psi_{d,f0} = \Psi\left(r_{f0}, h_a, h_s\right), \text{ and} \\
& \psi_{d,n0} = \Psi\left(r_{n0}, h_a, h_s\right).
\end{aligned}
\tag{87}
$$

In addition, we now also assume that for subsequent calculations, the reference range is the far edge of the swath, namely

$$
R_{ref} = r_{f0}. \tag{88}
$$

The task at hand is to find the antenna boresight angle $\psi_{d,ant}$ that allows the following equation to be true, namely

$$G_{combined}\left(\psi_{d,f0},\psi_{d,ant}\right)=G_{combined}\left(\psi_{d,n0},\psi_{d,ant}\right), \tag{89}$$

where both are greater than $G_{\min}$, which we will presume is guaranteed during the prerequisite swath selection process.

An iterative technique is presented in Appendix B.

With r_{n0} and r_{f0} now specified or calculated, as well as $\psi_{d,ant}$, we are finished.

<u>Example</u>

We continue with the example of the previous section, and now refine our swath to the desired limits

$$\begin{aligned} r_{f0,desired} &= 100 \text{ km, and} \\ r_{n0,desired} &= 40 \text{ km,} \end{aligned} \tag{90}$$

with other parameters the same.

Our outputs of the procedure in this section are then calculated to be

$$\begin{aligned} r_{f0} &= 100 \text{ km,} \\ r_{n0} &= 40 \text{ km, and} \\ \psi_{d,ant} &= 5.0 \text{ deg.} \end{aligned} \tag{91}$$

Note that the antenna is pointed at its minimum depression angle. We observe that the combined gains at the range swath edges are then

$$\begin{aligned} G_{combined}\left(\psi_{d,f0},\psi_{d,ant}\right) &= -0.6 \text{ dB, and} \\ G_{combined}\left(\psi_{d,n0},\psi_{d,ant}\right) &= 11.3 \text{ dB.} \end{aligned} \tag{92}$$

Note further that because the antenna depression angle is against a limit, the gains at the swath edges are not equal, but both are nevertheless greater than the –6 dB minimum.

Were it not for the hard limit on antenna depression angle, the optimum $\psi_{d,ant}$ for these conditions would be 3.85 degrees (pointed at the far range), providing 0.6 dB gain improvement at the far range. This is about 0.1 degrees above the geometric angle if atmospheric refraction were ignored.

3.3 Best Swath for Fixed Center-Beam Range

For this modality, the range swath is not fixed, and we desire to aim the antenna in elevation such that the combined gain is maximum at some specified range. We then wish to maximize swath around this range, but keeping this range centered, and satisfying some minimum allowable combined gain reduction.

Our inputs will include

h_a = altitude of aircraft,
h_s = altitude of target, and
$r_{c0,desired}$ = desired swath-center slant range of interest. (93)

Our constraints are

$G_{\min}$ = the minimum acceptable combined gain factor,
$\psi_{d,\min}$ = the minimum (most shallow) depression angle limit,
$\psi_{d,\max}$ = the maximum (most steep) depression angle limit,
$\beta_{horizon}$ = radar horizon margin factor. (94)

Our assumptions (in the absence of further information) are

N_s = 313, and
α = 0.05 dB/km (Ku-band at 10 kft, 50% RH). (95)

In addition, we have a model for our antenna beam shape that is valid over the interval defined by the normalized angles

$\phi_{a,limit}$ = angle above antenna boresight at limit of antenna model, and
$\phi_{b,limit}$ = angle below antenna boresight at limit of antenna model, (96)

with

$\theta_{el,nom}$ = nominal beamwidth of the antenna. (97)

Our outputs are the achievable near and far ranges that define the swath, r_{n0} and r_{f0}. An additional output is the actual antenna depression angle $\psi_{d,ant}$.

Although we have stated a desire for the reference range to be $r_{c0,desired}$, this may not be feasible due to other factors. Consequently, we need to identify the actual achievable center range subject to the constraints

$$r_{c0} \le r_{c0,desired},$$
$$r_{c0} \le (1+\beta_{horizon})R_{horizon}, \text{ and}$$
$$r_{c0} \le \Psi^{-1}(\psi_{d,\min}, h_a, h_s), \tag{98}$$

where angle arguments are limited to positive depression angles below the horizon, and the ranges exist. In addition, we will also want to force a minimum range such that

$$r_{c0} \ge \Psi^{-1}(\psi_{d,\max}, h_a, h_s). \tag{99}$$

In the unlikely event that the minimum exceeds the maximum constraint, we will choose the minimum constraint.

With r_{c0} now chosen, we may calculate the actual antenna boresight depression angle to this range to be

$$\psi_{d,ant} = \Psi(r_{c0}, h_a, h_s). \tag{100}$$

In addition, we now assume that

$$R_{ref} = r_{c0}. \tag{101}$$

With antenna boresight depression angle calculated, we now wish to find the limits of useable angles as

$$\psi_{d,f0,temp} = {}^{a}G^{-1}_{combined}(G_{\min}, \psi_{d,ant}), \text{ and}$$
$$\psi_{d,n0,temp} = {}^{b}G^{-1}_{combined}(G_{\min}, \psi_{d,ant}), \tag{102}$$

where these angles are limited to positive depression angles below the horizon. These are the angles above and below boresight that yields the minimum acceptable combined gain. However these are not yet final selections. As discussed in earlier sections, these angles are generally not at all easy to calculate. An iterative technique for doing so is given in Appendix A.

In the event that convergence is not achieved, or the inverse is otherwise not calculable, then we might use one or more of the following default calculations

$$\psi_{d,f0,temp} = \psi_{d,f0,default} = \psi_{d,ant} - \theta_{el,nom}/2, \text{ and}$$
$$\psi_{d,n0,temp} = \psi_{d,n0,default} = \psi_{d,ant} + \theta_{el,nom}/2, \tag{103}$$

but again limited by the radar horizon. With appropriate temporary depression angles calculated, we may then calculate the corresponding temporary ranges as

$$r_{f0,temp} = \Psi^{-1}\left(\psi_{d,f0,temp}, h_a, h_s\right), \text{ and}$$
$$r_{n0,temp} = \Psi^{-1}\left(\psi_{d,n0,temp}, h_a, h_s\right). \tag{104}$$

An issue now results from the observation that the reference range r_{c0} is not necessarily centered between these two temporary ranges. This means that quite possibly, in fact very likely

$$r_{f0,temp} - r_{c0} \neq r_{c0} - r_{n0,temp}. \tag{105}$$

With this observation, we now ask "What should be the final near and far ranges?" We offer three reasonable choices as answers to this, as follows.

1. Offset Swath with Minimum Gain Degradation
2. Center Beam at Center Swath – Inscribed Swath
3. Center Beam at Center Swath – Circumscribed Swath

We examine these in turn. Whichever we choose, in all cases, with r_{n0} and r_{f0} therewith calculated, as well as $\psi_{d,ant}$, we are finished.

3.3.1 Offset Swath with Minimum Gain Degradation

In this option we allow the swath to not be centered at r_{c0}. Consequently, we choose

$$r_{n0} = r_{n0,temp}, \text{ and}$$
$$r_{f0} = r_{f0,temp}. \tag{106}$$

3.3.2 Center Beam at Center Swath – Inscribed Swath

In this option, we insist that the swath is centered at r_{c0}, and all ranges within the swath meet the minimum gain requirement. Consequently, we choose

$$\begin{aligned} &\text{If} \quad r_{f0,temp} - r_{c0} \geq r_{c0} - r_{n0,temp}, \\ &\text{then} \quad r_{n0} = r_{n0,temp}, \text{ and } r_{f0} = 2r_{c0} - r_{n0}, \\ &\text{else} \quad r_{f0} = r_{f0,temp}, \text{ and } r_{n0} = 2r_{c0} - r_{f0}. \end{aligned} \tag{107}$$

3.3.3 Center Beam at Center Swath – Circumscribed Swath

In this option, we insist that the swath is centered at r_{c0}, and all ranges that meet the minimum gain requirement are within the swath. Consequently, we choose

If $r_{f0,temp} - r_{c0} \geq r_{c0} - r_{n0,temp}$,

then $r_{f0} = r_{f0,temp}$, and $r_{n0} = 2r_{c0} - r_{f0}$,

else $r_{n0} = r_{n0,temp}$, and $r_{f0} = 2r_{c0} - r_{n0}$. (108)

<u>Example</u>

We illustrate these calculations with an example.

Our inputs and constraints will be

h_a = 20 kft,
h_s = 0,
$r_{c0,desired}$ = 70 km,
$G_{\min}$ = –6 dB, and
$\beta_{horizon}$ = 0.1. (109)

Our antenna parameters and constraints will be consistent with a sinc() function pattern, with

$\psi_{d,\min}$ = 5 deg.,
$\psi_{d,\max}$ = 60 deg.,
$\phi_{a,limit}$ = –1.1312,
$\phi_{b,limit}$ = 1.1312, and
$\theta_{el,nom}$ = 7 degrees. (110)

Our outputs are then calculated to be

r_{c0} = 70 km,
$r_{f0,temp}$ = 91.8 km,
$r_{n0,temp}$ = 31.4 km, and
$\psi_{d,ant}$ = 5.25 deg. (111)

Specific swath calculations can be made from these parameters as previously indicated.

Conclusions

We repeat the following observations.

- SNR is affected by spherical wavefront spreading with range, atmospheric attenuation, and antenna beam illumination. Antenna beam illumination is a function of antenna boresight pointing.

- Geometry and trigonometric calculations are complicated by the curved earth, and atmospheric refraction of the radar signal.

- The aforementioned characteristics affect a combined signal gain model.

- The combined gain model can be used to calculate a maximum swath width that meets some minimum gain criteria with respect to a reference range, allowing antenna pointing to float.

- The combined gain model can be used to calculate an optimum antenna pointing angle for a fixed range swath.

- The combined gain model can be used to calculate a maximum swath width that meets some minimum gain criteria with respect to a reference range, for a fixed antenna pointing angle.

- Several of the various calculations involved in optimizing swaths and pointing require function inversions that require numerical techniques to solve.

- Several hardware and software limits will also complicate the various optimizations.

"In business, words are words; explanations are explanations, promises are promises, but only performance is reality."
-- Harold S. Geneen

Appendix A – Calculating Depression Angles for Specified Gain

We desire to calculate a depression angle ψ_d that corresponds to a particular combined gain. That is, we desire to find the depression angle ψ_d that satisfies

$$G_{combined}(\psi_d, \psi_{d,ant}) = g \,, \tag{A1}$$

where g is a constant.

The nature of this function for values for g of interest is that there are at least two values for ψ_d that satisfy this. We are interested in the first value above the antenna boresight, and the first value below the antenna boresight.

We define the inverse function to be

$$\begin{aligned} \psi_{d,a} &= {}^{a}G^{-1}_{combined}(g, \psi_{d,ant}) = \text{the depression angle above boresight, and} \\ \psi_{d,b} &= {}^{b}G^{-1}_{combined}(g, \psi_{d,ant}) = \text{the depression angle below boresight.} \end{aligned} \tag{A2}$$

These are rather difficult to calculate directly. The inverse function depends on antenna pointing and both depression angle and range, which are of course related in a fairly complicated way. Essentially, we wish to solve for either $\psi_{d,a}$ or $\psi_{d,b}$ in the respective equations

$$\begin{aligned} G_{combined}(\psi_{d,a}, \psi_{d,ant}) &= \left[G_{el,ant}\left(\frac{\psi_{d,a} - \psi_{d,ant}}{\theta_{el,nom}} \right) \right]^2 G_{range}(R)\, G_{atm}(R) = g \text{ , or} \\ G_{combined}(\psi_{d,b}, \psi_{d,ant}) &= \left[G_{el,ant}\left(\frac{\psi_{d,b} - \psi_{d,ant}}{\theta_{el,nom}} \right) \right]^2 G_{range}(R)\, G_{atm}(R) = g \,. \end{aligned} \tag{A3}$$

The nature of $G_{combined}(\psi_d, \psi_{d,ant})$ is that at the antenna boresight direction, $G_{combined}(\psi_d, \psi_{d,ant})$ is decreasing with decreasing ψ_d, and increasing with increasing ψ_d. Moreover, $G_{combined}(\psi_d, \psi_{d,ant})$ is a fairly smooth function, and even typically fairly linear in the interesting neighborhoods where $G_{combined}(\psi_d, \psi_{d,ant}) = g$.

We will use this to advantage in an iterative technique to solve for either $\psi_{d,a}$ or $\psi_{d,b}$. The basics of the iterative technique that we will use are given in Appendix C.

<u>Above Boresight</u>

The key to convergence to $\psi_{d,a}$ is to start with a good initial guess. Recall that we generally expect a monotonic slope for $G_{combined}(\psi, \psi_{d,ant})$ over the interval $\lfloor \psi_{d,a}, \psi_{d,ant} \rfloor$, and indeed somewhat beyond in both directions. Consequently, we can start with something near the edge of the useful beam, namely an initial estimate of

$$\hat{\psi}_{d,a} = \psi_{d,ant} + (0.95)\,\theta_{el,nom}\,\phi_{a,limit}\,. \tag{A4}$$

We must ensure that all new depression angle estimates remain within the valid interval of the antenna model and perhaps even on the correct side of boresight. In addition, it would be prudent to place a limit on the maximum number of iterations, and define convergence to be perhaps within 10% of the desired gain value.

These are combined in the iterative procedure given as follows.

<u>*Step 1.*</u> First we check if the answer is even within the limits of the antenna beam model by calculating and checking the gain at the edge of the antenna model, namely

If $\psi_{d,ant} + \theta_{el,nom}\,\phi_{a,limit} > \psi_{d,horizon}$, and

$G_{combined}\left(\psi_{d,ant} + \theta_{el,nom}\,\phi_{a,limit}\,, \psi_{d,ant}\right) \geq g$,

then we assume $\psi_{d,a} = \psi_{d,ant} + \theta_{el,nom}\,\phi_{a,limit}$, and we are finished,

else we go on to Step 2. (A5)

<u>*Step 2.*</u> Initialize the seed angle to an angle near the edge of the antenna beam model, limited by the horizon, as

$$\hat{\psi}_{d,a} = \max\left(\psi_{d,ant} + (0.95)\,\theta_{el,nom}\,\phi_{a,limit} \;,\; \psi_{d,horizon} \right), \tag{A6}$$

and exit criteria

$N_{\max}$ = maximum number of iterations, and

$S_{convergence}$ = convergence threshold. (A7)

<u>*Step 3.*</u> Calculate the sample points and update parameters as

$\psi_1 = \hat{\psi}_{d,a}$,

$\psi_2 = \psi_1 + \theta_{el,nom}/20$, a reasonable offset,

$\Delta\psi_{1,\max} = \theta_{el,nom}/N_{\max}$, a reasonable maximum step size, and

$\mu = 1$. (A8)

Step 4. Calculate the step

$$\Delta\psi_1 = \mu \frac{\varepsilon(\psi_1)}{m(\psi_1)} = \operatorname{sgn}\left(\mu \frac{\varepsilon(\psi_1)}{m(\psi_1)}\right) \min\left(\left|\mu \frac{\varepsilon(\psi_1)}{m(\psi_1)}\right|, \Delta\psi_{1,\max}\right), \tag{A9}$$

where the constituent values are

$$m(\psi_1) = \frac{G_{combined}(\psi_2, \psi_{d,ant}) - G_{combined}(\psi_1, \psi_{d,ant})}{\psi_2 - \psi_1},$$
$$\varepsilon(\psi_1) = G_{combined}(\psi_1, \psi_{d,ant}) - g\,. \tag{A10}$$

Step 5. Calculate the updated estimate,

$$\hat{\psi}_{d,a} = \psi_1 - \Delta\psi_1, \tag{A11}$$

and constrain it to

$$\hat{\psi}_{d,a} = \max\left(\hat{\psi}_{d,a} \;,\; \psi_{d,ant} + \theta_{el,nom}\,\phi_{a,limit} \;,\; \psi_{d,horizon} \right), \text{ and}$$
$$\hat{\psi}_{d,a} = \min\left(\hat{\psi}_{d,a} \;,\; \psi_{d,ant} \right). \tag{A12}$$

Step 6. Check on exit criteria, which we choose to be when the combined gain is perhaps close enough to the desired value, that is

If $\left|G_{combined}(\hat{\psi}_{d,a}, \psi_{d,ant})/g - 1\right| \le S_{convergence}$,
then we are finished, and we assume $\psi_{d,a} = \hat{\psi}_{d,a}$,
else if we have exceeded the maximum number of iterations,
then proceed to Step 7.
else go back to Step 3. (A13)

Step 7. In the event of failed convergence, decide if we still have a useable answer

If $G_{combined}(\hat{\psi}_{d,a}, \psi_{d,ant}) \ge g$,
then we are finished, and we assume $\psi_{d,a} = \hat{\psi}_{d,a}$,
else we have failed to achieve a useable answer, and
we finish with a default assignment $\psi_{d,a} = \psi_{d,ant} - \theta_{el,nom}/2$.
(A14)

<u>Below Boresight</u>

As with the 'Above Boresight' case, the key to convergence to $\psi_{d,b}$ is to start with a good initial guess. However, we do expect to not have a monotonic slope for $G_{combined}(\psi, \psi_{d,ant})$ over the interval $\lfloor \psi_{d,ant}, \psi_{d,b} \rfloor$. Therefore, our initial guess needs to be far enough beyond the 'hump' so that the combined gain is monotonic between the initial guess and the actual angle $\psi_{d,b}$. Consequently, we can start with something near the edge of the useful beam, namely an initial estimate of

$$\hat{\psi}_{d,n0} = \psi_{d,ant} + (0.95)\,\theta_{el,nom}\,\phi_{b,limit}\,. \tag{A15}$$

We must ensure that all new depression angle estimates remain within the valid interval of the antenna model and perhaps even on the correct side of boresight. In addition, it would be prudent to place a limit on the maximum number of iterations, and define convergence to be perhaps within 10% of the desired gain value.

These are combined in the iterative procedure given as follows.

<u>*Step 1.*</u> First we check if the answer is even within the limits of the antenna beam model by calculating and checking the gain at the edge of the antenna model, namely

If $G_{combined}\left(\psi_{d,ant} + \theta_{el,nom}\,\phi_{b,limit}\,, \psi_{d,ant}\right) \geq G_{\min}$,
then we assume $\psi_{d,n0} = \psi_{d,ant} + \theta_{el,nom}\,\phi_{b,limit}$, and we are finished,
else we go on to Step 2. (A16)

<u>*Step 2.*</u> Initialize the seed angle to an angle near the edge of the antenna beam model as

$$\hat{\psi}_{d,n0} = \psi_{d,ant} + (0.95)\,\theta_{el,nom}\,\phi_{b,limit}\,, \tag{A17}$$

and exit criteria

$N_{\max}$ = maximum number of iterations, and
$S_{convergence}$ = convergence threshold. (A18)

<u>*Step 3.*</u> Calculate the sample points and update parameters as

$\psi_1 = \hat{\psi}_{d,n0}$,
$\psi_2 = \psi_1 - \theta_{el,nom}/20$, a reasonable offset,

$\Delta\psi_{1,\max} = \theta_{el,nom} / N_{\max}$, a reasonable maximum step size, and
$\mu = 1$. (A19)

Step 4. Calculate the step

$$\Delta\psi_1 = \mu \frac{\varepsilon(\psi_1)}{m(\psi_1)} = \operatorname{sgn}\left(\mu \frac{\varepsilon(\psi_1)}{m(\psi_1)} \right) \min\left(\left| \mu \frac{\varepsilon(\psi_1)}{m(\psi_1)} \right|, \Delta\psi_{1,\max} \right), \tag{A20}$$

where the constituent values are

$$m(\psi_1) = \frac{G_{combined}(\psi_2, \psi_{d,ant}) - G_{combined}(\psi_1, \psi_{d,ant})}{\psi_2 - \psi_1},$$
$$\varepsilon(\psi_1) = G_{combined}(\psi_1, \psi_{d,ant}) - g \,. \tag{A21}$$

Step 5. Calculate the updated estimate,

$$\hat{\psi}_{d,b} = \psi_1 - \Delta\psi_1, \tag{A22}$$

and constrain it to

$$\hat{\psi}_{d,b} = \min\left(\hat{\psi}_{d,b} \;,\; \psi_{d,ant} + \theta_{el,nom}\, \phi_{b,limit} \right), \text{ and}$$
$$\hat{\psi}_{d,b} = \max\left(\hat{\psi}_{d,b} \;,\; \psi_{d,ant} \right). \tag{A23}$$

Step 6. Check on exit criteria, which we choose to be when the combined gain is perhaps close enough to the desired value, that is

If $\left| G_{combined}(\hat{\psi}_{d,b}, \psi_{d,ant}) / g - 1 \right| \le S_{convergence}$,
then we are finished, and we assume $\psi_{d,b} = \hat{\psi}_{d,b}$,
else if we have exceeded the maximum number of iterations,
then proceed to Step 7.
else go back to Step 3. (A24)

Step 7. In the event of failed convergence, decide if we still have a useable answer

If $G_{combined}(\hat{\psi}_{d,b}, \psi_{d,ant}) \ge g$,
then we are finished, and we assume $\psi_{d,b} = \hat{\psi}_{d,b}$,
else we have failed to achieve a useable answer, and
we finish with a default assignment $\psi_{d,b} = \psi_{d,ant} + \theta_{el,nom} / 2$.
(A25)

"Everything depends upon execution; having just a vision is no solution."
-- Stephen Sondheim

Appendix B – Calculating Depression Angle for Equal Gain at Swath Edges

We desire to find the single antenna boresight angle $\psi_{d,ant}$ that allows the following equation to be true, namely

$$G_{combined}\left(\psi_{d,f0},\psi_{d,ant}\right)=G_{combined}\left(\psi_{d,n0},\psi_{d,ant}\right). \tag{B1}$$

To solve this numerically, we create a new error function that is overtly dependent on $\psi_{d,ant}$ and write this as

$$\varepsilon\left(\psi_{d,ant}\right)=G_{combined}\left(\psi_{d,f0},\psi_{d,ant}\right)-G_{combined}\left(\psi_{d,n0},\psi_{d,ant}\right). \tag{B2}$$

The task at hand is to find the $\psi_{d,ant}$ that minimizes $\left|\varepsilon\left(\psi_{d,ant}\right)\right|$, ideally allowing

$$\varepsilon\left(\psi_{d,ant}\right)=0\,, \tag{B3}$$

acknowledging that real antenna pointing limits might not allow us to achieve perfect equality.

We must ensure that all new depression angle estimates remain within the valid interval of acceptable angles. In addition, it would be prudent to place a limit on the maximum number of iterations, and define convergence to be gains perhaps within 10% of each other.

An iterative solution might be calculated as follows

<u>*Step 1.*</u> Select an initial antenna depression angle as perhaps the midpoint of the two swath edges

$$\hat{\psi}_{d,ant}=\frac{\psi_{d,f0}+\psi_{d,n0}}{2}\,, \tag{B4}$$

and exit criteria

$N_{\max}$ = maximum number of iterations,
$S_{convergence}$ = convergence threshold, and
$G_{\min}$ = minimum tolerable gain. (B5)

Step 2. Calculate the sample points and update parameters as

$$\psi_1 = \hat{\psi}_{d,ant},$$
$$\psi_2 = \psi_1 - \theta_{el,nom}/20, \text{ a reasonable offset,}$$
$$\Delta\psi_{1,\max} = \theta_{el,nom}/N_{\max}, \text{ a reasonable maximum step size, and}$$
$$\mu = 1. \tag{B6}$$

Step 3. Calculate the step

$$\Delta\psi_1 = \mu\frac{\varepsilon(\psi_1)}{m(\psi_1)} = \operatorname{sgn}\left(\mu\frac{\varepsilon(\psi_1)}{m(\psi_1)}\right)\min\left(\left|\mu\frac{\varepsilon(\psi_1)}{m(\psi_1)}\right|, \Delta\psi_{1,\max}\right), \tag{B7}$$

where the constituent values are

$$m(\psi_1) = \frac{\varepsilon(\psi_2) - \varepsilon(\psi_1)}{\psi_2 - \psi_1},$$
$$\varepsilon(\psi_1) = G_{combined}(\psi_{d,f0}, \psi_1) - G_{combined}(\psi_{d,n0}, \psi_1), \text{ and}$$
$$\varepsilon(\psi_2) = G_{combined}(\psi_{d,f0}, \psi_2) - G_{combined}(\psi_{d,n0}, \psi_2). \tag{B8}$$

Step 4. Calculate the updated estimate $\hat{\psi}_{d,ant} = \psi_1 - \Delta\psi_1$.

$$\hat{\psi}_{d,ant} = \psi_1 - \Delta\psi_1, \tag{B9}$$

and constrain it to with pointing limits, and swath edge limits, namely

$$\psi_{d,\min} \le \hat{\psi}_{d,ant} \le \psi_{d,\max}, \text{ and if additionally possible}$$
$$\psi_{d,f0} \le \hat{\psi}_{d,ant} \le \psi_{d,n0}. \tag{B10}$$

Step 5. Check on exit criteria, which we choose to be when the respective combined gains are close enough to each other, that is

If $\left|G_{combined}(\psi_{d,f0}, \hat{\psi}_{d,ant}) / G_{combined}(\psi_{d,n0}, \hat{\psi}_{d,ant}) - 1\right| \le S_{convergence}$,
then we are finished, and we assume $\psi_{d,ant} = \hat{\psi}_{d,ant}$,
else if we have exceeded the maximum number of iterations,
then proceed to Step 6.
else go back to Step 2. (B11)

Step 6. In the event of failed convergence, decide if we still have a useable answer

If $G_{combined}\left(\psi_{d,f0}, \hat{\psi}_{d,ant}\right) \geq G_{\min}$ and $G_{combined}\left(\psi_{d,n0}, \hat{\psi}_{d,ant}\right) \geq G_{\min}$,

then we are finished, and we assume $\psi_{d,ant} = \hat{\psi}_{d,ant}$,

else we have failed to achieve a useable answer, and
we finish with a default assignment $\psi_{d,ant} = \left(\psi_{d,f0} + \psi_{d,n0}\right)/2$.

(B12)

"To aim is not enough, you must hit!"
-- German Proverb

Appendix C – Solving Equations with Basic Iterative Calculations

Consider an equation where a not quite arbitrary function is set equal to some constant, as in the equation

$$F(\psi) = f\,, \tag{C1}$$

where

$F(\psi)$ = not quite arbitrary function of ψ,
ψ = the independent argument within the domain of the function, and
f = a constant within the range of $F(\psi)$. (C2)

The inverse function is described as

$$\psi = F^{-1}(f). \tag{C3}$$

When we are unable to find a closed-form solution for $F^{-1}(f)$, then iterative techniques might get us arbitrarily close.

We stipulate that over the domain of interest, $F(\psi)$ is smooth and monotonic. That is not to say that $F(\psi)$ is entirely smooth and monotonic, but rather that $F(\psi)$ is smooth and monotonic over some limited interval of ψ in which we will search for an answer.

To facilitate the development, we create a new error function. That is

$$\varepsilon(\psi) = F(\psi) - f\,. \tag{C4}$$

We desire to find the value for ψ which causes the error function to equal zero, that is, some specific ψ_0 that causes

$$\varepsilon(\psi_0) = F(\psi_0) - f = 0\,. \tag{C5}$$

Consider two seed values for ψ in the vicinity of the solution, and the respective error function values, that is

$\varepsilon(\psi_1)$ = first function point, and
$\varepsilon(\psi_2)$ = second function point. (C6)

If these values are sufficiently close together, then we may estimate the slope of the function in this region as

$$m(\psi_1) = \frac{d}{d\psi}\varepsilon(\psi_1) \approx \frac{\varepsilon(\psi_2) - \varepsilon(\psi_1)}{\psi_2 - \psi_1} = \frac{F(\psi_2) - F(\psi_1)}{\psi_2 - \psi_1}. \tag{C7}$$

In general, the slope $m(\psi_1)$ may be of either sign.

If the function is sufficiently linear in the vicinity of the solution, then we can estimate a value for ψ_0 that is closer to the solution. We do this by calculating

$$\hat{\psi}_0 = \psi_1 - \frac{\varepsilon(\psi_1)}{m(\psi_1)}. \tag{C8}$$

More generally, we calculate

$$\hat{\psi}_0 = \psi_1 - \Delta\psi_1, \tag{C9}$$

where

$$\Delta\psi_1 = \frac{\varepsilon(\psi_1)}{m(\psi_1)} = \text{the step from } \psi_1 \text{ to the estimate } \hat{\psi}_0. \tag{C10}$$

The iteration is accomplished by subsequent to this calculation, updating $\psi_1 \leftarrow \hat{\psi}_0$, selecting a new ψ_2, and repeating to find a new $\hat{\psi}_0$. This is repeated until some exit criterion is met. This is detailed in the following procedure

<u>*Step 1.*</u> Select an initial $\hat{\psi}_0$.

<u>*Step 2.*</u> Select $\psi_1 = \hat{\psi}_0$ and accordingly a ψ_2.

<u>*Step 3.*</u> Calculate $\Delta\psi_1$.

<u>*Step 4.*</u> Calculate the new estimate $\hat{\psi}_0 = \psi_1 - \Delta\psi_1$.

<u>*Step 5.*</u> If convergence criteria is met,
then exit iteration loop,
else return to step 2.

The update equation given above attempts to jump to the answer in a single iteration. In attempting to do so, it is fairly sensitive to nonlinearities in the vicinity of the solution as well as seed values. This sensitivity can be reduced at the expense of slower convergence by introducing a convergence factor, and letting

$$\Delta\psi_1 = \mu \frac{\varepsilon(\psi_1)}{m(\psi_1)}, \tag{C11}$$

where

$$\mu = \text{convergence factor, with typically } 0 < \mu \le 1. \tag{C12}$$

We might also employ heuristics that adjust μ as a function of something, like perhaps even iteration number.

Another technique might be to limit the step size $\Delta\psi_1$. For example, we might calculate

$$\Delta\psi_1 = \mu \frac{\varepsilon(\psi_1)}{m(\psi_1)} = \operatorname{sgn}\left(\mu \frac{\varepsilon(\psi_1)}{m(\psi_1)}\right) \min\left(\left|\mu \frac{\varepsilon(\psi_1)}{m(\psi_1)}\right|, \Delta\psi_{1,\max}\right), \tag{C13}$$

where

$$\Delta\psi_{1,\max} = \text{the maximum allowable step size.} \tag{C14}$$

A reasonable step size limit might be one guaranteed to keep $\hat{\psi}_0$ in the same interval of ψ such that there is no slope change in between $F(\hat{\psi}_0)$ and $F(\psi_0)$.

Additionally, we stipulate that heuristics might be developed and implemented that limit the magnitude of $m(\psi_1)$, $\hat{\psi}_0$, or both.

"Performance, and performance alone, dictates the predator in any food chain."
-- SEAL Team saying

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