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A Non-local, Ordinary-State-Based Viscoelasticity Model for Peridynamics

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Prepared by
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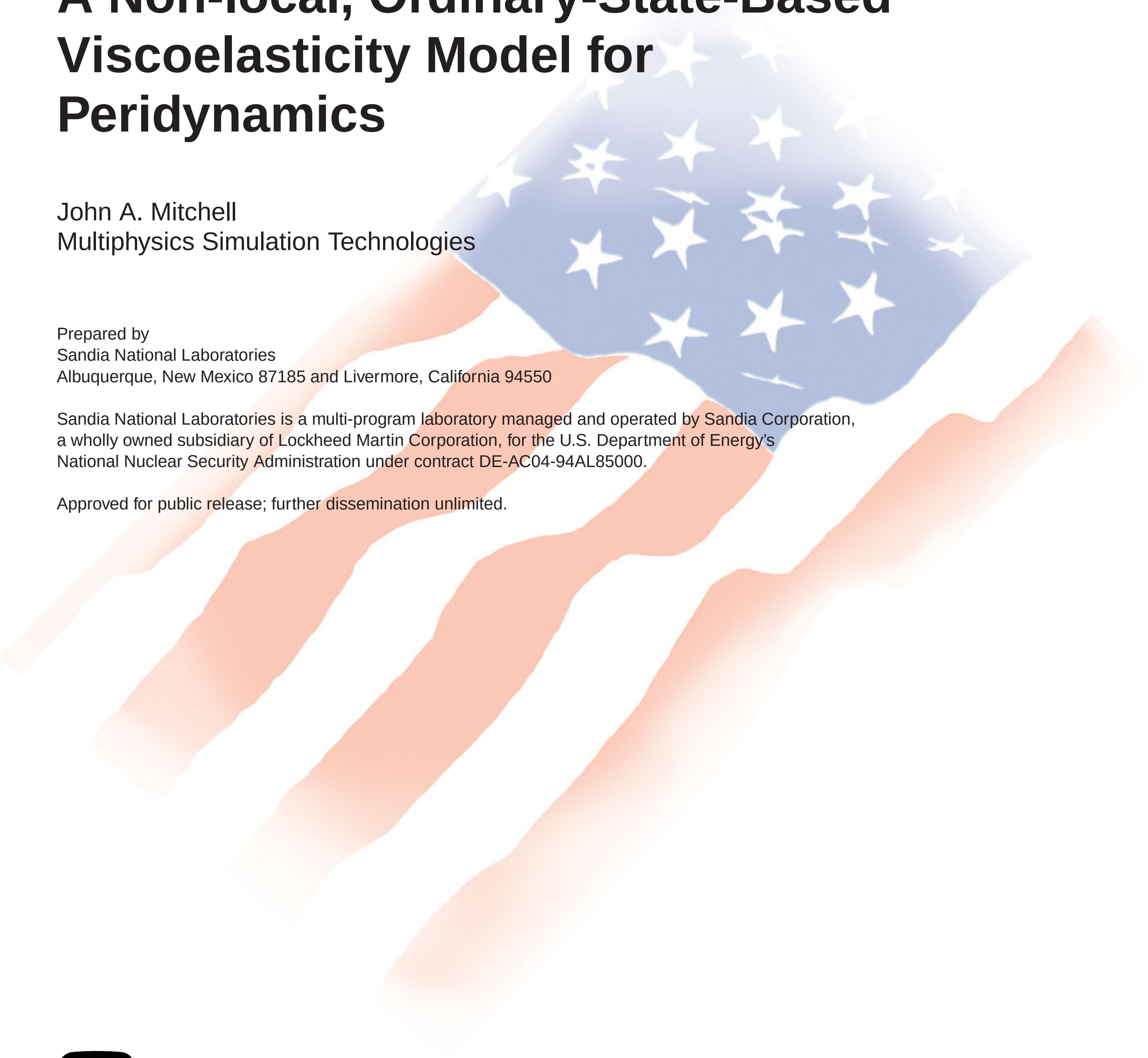

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Abstract

A non-local, ordinary-state-based, peridynamics viscoelasticity model is developed. In this model, viscous effects are added to deviatoric deformations and the bulk response remains elastic. The model uses internal state variables and is conceptually similar to linearized isotropic viscolelasticity in the local theory [7]. The modulus state, which is used to form the Jacobian matrix in Newton-Raphson algorithms, is presented. The model is shown to satisfy the 2nd law of thermodynamics and is applicable to problems in solid continuum mechanics where fracture and rate effects are important; it inherits all the advantages for modeling fracture associated with peridynamics [4, 5]. By combining this work with the previously published ordinary-state-based plasticity model [2], the model may be amenable to viscoplasticity problems where plasticity and rate effects are simultaneously important. Also, the model may be extended to include viscous effects for spherical deformations as well. The later two extensions are not presented and may be the subject of further work.

Acknowledgment

The author gratefully acknowledges David Littlewood, Michael Parks, Rich Lehoucq and Stewart Silling for the many helpful discussions, observations and suggestions concerning my work on peridynamics. Thanks also to John Aidun and Randy Summers for providing the opportunity and support to work on peridynamics.

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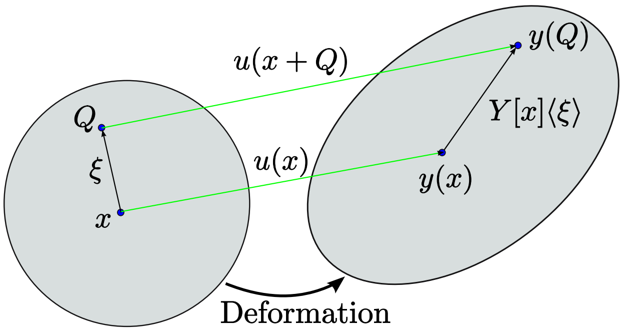

Figure 1. Peridynamics: bonds and states

1 Introduction and Approach

This report focuses on the recent development of a viscoelasticity model for peridynamics. It builds upon the state-based peridynamics theory [5] and is conceptually similar to the plasticity model developed by Mitchell [2]. The model uses internal state variables which are analogous to back strain(s) in the local theory. In its most elementary form, the model is conceptually similar to the *standard linear solid* [8] used in the local theory.

Section 2 provides an introduction to the kinematics associated with peridynamics. Nearly all variables relevant to ordinary state-based materials are presented. Peridynamic treatment of the momentum equation is briefly presented with an emphasis on its relationship with explicit time integration of the momentum equation. Section 3 defines an ordinary material and introduces the modulus state for ordinary materials.

The main content of this report is presented in Sections 4 and 5. A *standard linear solid* is presented for peridynamics in Section 4. Viscoelasticity is introduced via an evolution equation for a deviatoric extension state-like field. The auxiliary field is out of phase with the deviatoric extension state and creates the viscoelastic effect. The section develops an integration algorithm for a load step and demonstrates that the model satisfies the 2nd law of thermodynamics. The section goes on to linearize the *standard linear solid* for implicit dynamics and quasistatic calculations. Section 5 follows the steps taken in Section 4 and generalizes the *standard linear solid* to include an arbitrary number of auxiliary fields. Each auxiliary field has two material parameters: relaxation time and relaxation modulus. Time integration, linearization, and analysis of the 2nd law of thermodynamics for the generalized model is given by a straightforward extension of results obtained for the peridynamics *standard linear solid.*

2 Notation and Preliminary Development

In this Section, a review of the kinematics associated with peridynamics is given. The notation used closely follows [3, 5]. Figure 1 may be helpful for following the discussion.

2.1 Definitions

A bond vector ξ is defined between an arbitrary point x in an undeformed body, and a second point $Q \in H_x$, where $H_x = \{x' \mid 0 < \|x - x'\| \leq \delta\}$ denotes the neighborhood of x. The horizon at x is denoted by δ.

$$\text{Bond:} \qquad \xi = Q - x. \tag{1}$$

States are functions that act on bonds. There are scalar, vector and modulus states used and developed in this report. These correspond with functions that act on bonds to produce scalars, vectors and 2nd order tensors, respectively.

The deformed configuration of a body is described using the following states. Note that these are vector states unless specifically identified as a scalar state.

$$\text{Scalar reference state:} \qquad \begin{aligned} \underline{x} &= |X|\langle\xi\rangle \\ &= |Q - x| \\ &= |\xi| \end{aligned} \tag{2}$$

$$\text{Deformation state:} \qquad \begin{aligned} Y[x]\langle\xi\rangle &= y(x+\xi) - y(x) \\ &= y(Q) - y(x), \end{aligned} \tag{3}$$

where x is a point and should not to be confused with the scalar reference state $\underline{x}$ and $y(x)$ is the location of the point x in the deformed configuration. The angle brackets $\langle\cdot\rangle$ are used to indicate the action of a state on a particular bond ξ while the $[\cdot]$ notation is used to indicate that the associated state depends upon the quantity within the square brackets.

$$\text{Scalar deformation state:} \qquad |Y|\langle\xi\rangle = [Y[x]\langle\xi\rangle \cdot Y[x]\langle\xi\rangle]^{\frac{1}{2}} \tag{4}$$

$$\text{Deformed direction state:} \qquad M(Y) = \frac{Y}{|Y|}, \quad |Y| \neq 0 \tag{5}$$

Note that $M(Y)$ is a unit vector directed along Y.

$$\text{Displacement state:} \qquad \begin{aligned} U[x]\langle\xi\rangle &= u(x+\xi) - u(x) \\ &= u(Q) - u(x) \end{aligned} \tag{6}$$

$$\text{Scalar extension state:} \qquad e(Y) = |Y| - |X| \tag{7}$$

The following definitions require the use of a dot product between two states. The formal definition for the dot product between two states is given in [5] and is denoted here as $a \bullet b$ for states a and b. The value of the dot product depends upon the point $x \in \mathbb{R}^3$, its neighborhood H_x, and the order and values of the states a and b. To evaluate the dot product, the states a and b are evaluated at x. The dot product consists of an integral over all bonds ξ in the neighborhood H_x. The integrand for this integral is given by contracting $a[x]\langle\xi\rangle$ and $b[x]\langle\xi\rangle$. For scalar states the contraction is multiplication. For vector states, this contraction is the usual Euclidean dot product of vectors. These two cases cover all dot products between states in this report.

The following two scalars are used in conjunction with a constitutive model and the scalar extension state e to evaluate internal forces.

$$\text{Weighted volume:} \qquad \begin{aligned} m &= (\omega \underline{x}) \bullet \underline{x} \\ &= \int_{H_x} \omega(|\xi|)|\xi|^2 \, dV_Q \end{aligned} \tag{8}$$

$$\text{Dilatation:} \qquad \begin{aligned} \theta(Y) &= \frac{3}{m}(\omega \underline{x}) \bullet e \\ &= \frac{3}{m}\int_{H_x} \omega |X|\langle\xi\rangle |Y|\langle\xi\rangle \, dV_Q - 3 \end{aligned} \tag{9}$$

A new state, ω, called the *influence function* is introduced in the proceeding two equations. A formal definition for ω is given in [5, Definition 2.1]. Note that the definitions for weighted volume and dilatation introduce the dot product between two states. Each state in the dot product is defined at a point x within the undeformed body, and the integral associated with the dot product is over the neighborhood, H_x. Therefore the dot product produces a quantity (in these two cases a scalar) that depends upon the point x. One application of ω is as a weight function in the above dot products.

A key element for constitutive model development is the additive decomposition of the scalar extension state (7) into spherical and deviatoric parts.

$$e(Y) = e^i(Y) + e^d(Y) \tag{10}$$

$$\text{Spherical extension state:} \qquad e^i(Y) = \frac{\theta(Y)|X|}{3} \tag{11}$$

$$\text{Deviatoric extension state:} \qquad \begin{aligned} e^d(Y) &= e(Y) - e^i(Y) \\ &= e(Y) - \frac{\theta(Y)|X|}{3} \\ &= |Y| - |X| - \frac{\theta(Y)|X|}{3} \end{aligned} \tag{12}$$

2.2 Peridynamic treatment of the momentum equation for solids

Solution algorithms used for the momentum equation under finite element and peridynamic discretizations are very similar. The internal force for peridynamics is identical to that used in nonlinear finite element codes in the sense that forces are summed at a point. In quasi-statics, the balance of forces at a point defines equilibrium, while in dynamics the sum of forces equals the product of density and acceleration at that point. In each case, it is necessary to evaluate the sum of forces at a point arising from deformations of the solid. The method for evaluation of this force at a point is what differentiates peridynamics [4, 5] from local continuum mechanics. This section is a brief introduction to the the internal force vector associated with peridynamics with an emphasis on its relation to constitutive models.

Using the state-based theory of peridynamics, the momentum equation is given as:

$$\begin{aligned} \rho(x)\ddot{u}(x,t) &= \int_{H_x} \{T(Y[x])\langle p-x\rangle - T(Y[p])\langle x-p\rangle\}\, dV_p + b(x,t). \\ &= f(x,Y,t) + b(x,t) \end{aligned} \tag{13}$$

The above equation is the peridynamic representation of Newton's 2nd law for the Lagrangian point x and defines the internal force vector $f(x,Y,t)$. The function $b(x,t)$ denotes a body force per unit volume. $T(Y[x])$ is a vector force state function (to be defined later) and is somewhat analogous to the stress tensor in local continuum mechanics in the sense that it represents the material response due to deformations. For a specific material, $T(Y[x])$ would be represented by a constitutive model. Note the notation $T(Y[x])$ is used to denote the force state dependency upon the deformation state Y at the point x. This dependency is used later to linearize the internal force vector about a deformation state $Y^0[x]$.

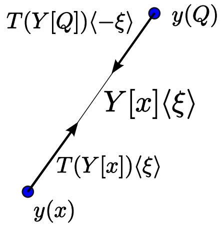

Figure 2. Ordinary material.

3 Ordinary State Based Constitutive Models

The vector force state for an ordinary material is defined as:

$$T(Y) = t(Y)\, M(Y), \tag{14}$$

where M is the deformed direction state (5). An ordinary material is such that $T(Y)\langle\xi\rangle$ acts along the direction defined by $Y[x]\langle\xi\rangle$, with a magnitude given by the scalar force state $t(Y)\langle\xi\rangle$, that is, the bond force between two points x and Q acts along the line connecting the two points. A graphical representation of an ordinary material is shown if Figure 2.

The modulus state, K, is used to calculate the Jacobian associated with Newton iterations. It represents a change in force at a point in the body for a given configuration y^0 when a small change to the configuration is superposed. The modulus state is given by linearizing the vector force state, $T(Y)$, using the Fréchet derivative. $T(Y)$ is linearized about the deformation state Y^0 associated with configuration y^0. Note that U is the displacement state (6) associated with a small displacement field $u(x)$ superposed upon a deformed configuration y^0.

$$T(Y^0 + U) = T(Y^0) + \nabla T \bullet U \tag{15}$$

$$\begin{aligned}
\nabla T \bullet U &= \frac{\partial}{\partial \epsilon} T(Y^0 + \epsilon U)|_{\epsilon=0} \\
&= M(Y^0)\frac{\partial}{\partial \epsilon} t(Y^0 + \epsilon U)|_{\epsilon=0} + t(Y^0)\frac{\partial}{\partial \epsilon} M(Y^0 + \epsilon U)|_{\epsilon=0} \\
&= K \bullet U
\end{aligned} \tag{16}$$

The above relations define the modulus state for an ordinary material. Note that it consists of two parts: 1) a constitutive model component defined by the Fréchet derivative of the scalar force state; 2) a geometric component that arises due to rotation of a force carrying bond.

4 Two Parameter Peridynamic Standard Linear Solid

The viscoelasticity model developed here utilizes an additive decomposition of the scalar extension state into its scalar volumetric and deviatoric parts. The model applies viscoelasticity concepts from the local theory to the deviatoric part of the scalar extension state in peridynamics. Just as in the case of plasticity [2], there are state variables and associated evolution equations for those state variables. Using the additive decomposition, the constitutive model remains reliant upon the elastic form of the constitutive model for the deviatoric force state. There is no restriction or reason why similar concepts cannot also be applied to the volumetric piece of the scalar extension state. Conceptually, applying viscoelasticity to the deviatoric part creates a logical intermediate step between a fluid, which has little or no elastic resistance to shear but may resist compressive volumetric deformations, and a solid which has both volumetric and deviatoric resistance to deformation. Because isotropic ordinary state based constitutive models use a scalar extension state decomposed into volumetric and deviatoric parts, the viscoelasticity model is conceptually identical to one dimensional spring/dashpot systems. The model extends the additive decomposition of the deviatoric extension state to include a back extension state. For the case of a *standard linear solid*, there is one back extension state and an associated evolution equation, relaxation time, and relaxation modulus. In the generalized model, an arbitrary number of *Maxwell* models are added in parallel with the *standard linear solid*, each with its own evolution equation, relaxation time, and relaxation modulus. This is conceptually identical to the way many local viscoelasticity models are handled.

The remainder of this section is devoted to the detailed development of the two parameter peridynamic *standard linear solid*.

4.1 Elastic stored energy and decomposition of scalar deformation state

A key ingredient in the viscoelastic constitutive model is the decomposition of the scalar extension state into dilatation and deviatoric parts, as well as the additive decomposition of the deviatoric extension state into elastic e^{de} and back extension $e^{db(i)}$ parts. This decomposition is represented by the schematic of the standard linear solid shown in Figure 3. Notation using the superscript $db(i)$ is in anticipation of the more generalized version of this model in Section 5.

The total deviatoric extension state (12) is additively decomposed into elastic and back parts as:

$$e^{d} = e^{de} + e^{db(i)}. \tag{17}$$

Using the above decomposition for the deviatoric extension state, an elastic stored energy

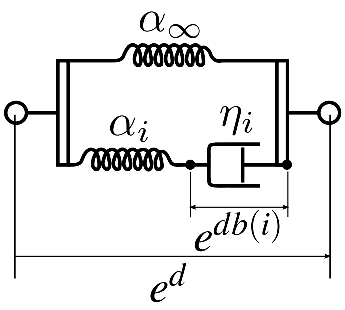

Figure 3. Standard Linear Solid

functional W is assumed (see [5], Eq. 104)

$$W(\theta, e^d, e^{db(i)}) = \frac{k\theta^2}{2} + \frac{\alpha_\infty}{2} e^d \bullet \omega e^d + \frac{\alpha_i}{2}(e^d - e^{db(i)}) \bullet \omega (e^d - e^{db(i)}). \tag{18}$$

The scalar volumetric and deviatoric force states, t^i and t^d respectively, are derived using W as follows:

$$t^i = \frac{\partial W}{\partial \theta}\frac{\partial \theta}{\partial e^i}, \quad t^d = \frac{\partial W}{\partial e^d}. \tag{19}$$

With these expressions, the scalar force state t is given by

$$\begin{aligned} t &= t^i + t^d \\ &= -\frac{3p}{m}\omega \underline{x} + (\alpha_\infty + \alpha_i)e^d - \alpha_i \omega e^{db(i)}, \end{aligned} \tag{20}$$

where $p = -k\theta$ is the the peridynamic pressure and k is the bulk modulus. Note that

$$\frac{\partial \theta}{\partial e^i} = \frac{3\omega \underline{x}}{m} \tag{21}$$

is given using the definition for the dilatation (9) and the decomposition of the scalar extension state (10). The elastic parameters α_∞ and α_i are defined by the following relationships:

$$\begin{aligned} \alpha &= \frac{15\mu}{m} \\ &= \alpha_\infty + \alpha_i, \end{aligned} \tag{22}$$

where μ is the elastic shear modulus and m is the weighted volume (8). To prevent unbounded creep of this model, $\alpha_\infty > 0$ must be used. It is assumed that $\alpha_i > 0$, otherwise the model degenerates to a *Kelvin* model [7] which is not practical for implementation in explicit or implicit codes with kinematically driven constitutive models. Therefore, the following additional constraint is given:

$$0 < \alpha_i < \frac{15\mu}{m}. \tag{23}$$

4.2 Evolution equation

Introduction of an evolution equation for the scalar deviatoric back extension is the key ingredient of the viscoelasticity model. The evolution equation is given based upon the following assumptions:

$$\begin{aligned} t^{d(i)} &= \eta_i \dot{e}^{db(i)} \\ &= \alpha_i(e^d - e^{db(i)}) \end{aligned} \tag{24}$$

where $t^{d(i)}$ is the scalar deviatoric force state in the dashpot and by Newton's 3rd law also the force in the adjoining spring. Therefore, the evolution equation for the *standard linear solid*, is given by:

$$\dot{e}^{db(i)} = \frac{1}{\tau_i^b}(e^d - e^{db(i)}), \tag{25}$$

where $\tau_i^b = \frac{\eta_i}{\alpha_i}$ is a time constant associated with material response. The state of viscoelastic materials is history dependent. It is assumed that $e^{db(i)}(t) \to 0$ as $t \to -\infty$. This assumption will play a role in the convolution integral discussed in the following section. Selection of τ_i^b is constrained by thermodynamics, and expected or relevant material response. Note that for physical and practical reasons $\eta_i > 0$. If $\eta_i < 0$, then the viscous effect would not oppose motion and therefore this case is disallowed. If $\eta_i = 0$, then the model degenerates to two springs in parallel which is physically sound but under practical conditions causes $\frac{1}{\tau_i^b}$ to be undefined and therefore this case is disallowed. Since $\alpha_i > 0$, it follows that $\tau_i^b > 0$.

Note that the evolution equation (25) is driven by e^d which is particularly suitable for peridynamics codes which use either explicit and implicit time integration.

4.3 Time integration over a load step

Existence of a solution to the evolution equation depends upon τ_i^b and the form of $e^d(t)$. A solution is sought over the range $t \in (0, \infty)$. It is assumed that a solution exists; when it does, it takes the following form [7]:

$$e^{db(i)}(t) = \frac{1}{\tau_i^b}\int_{-\infty}^{t} e^{\frac{-(t-s)}{\tau_i^b}} e^d(s)\, ds \tag{26}$$

Conceptually, the benefit of the above convolution integral is that it allows for an accounting of the complete time history of material loading. However, this is not convenient for implementation as a constitutive model. In particular, it is inconvenient to integrate from $s = -\infty$ to $s = t$ for every time step $t = t_{n+1}$. This would require knowledge of the complete time history of $e^d(t)$ which is not known in practice. In explicit or implicit codes, $e^d(t)$ is only given or implied at two discrete points in time (previous step and current step) and therefore the solution must be adapted to this use case. The approach taken here converts the requirement that the complete time history of $e^d(t)$ be given into an equivalent requirement for an initial condition on $e^{db(i)}(t)$. This is perhaps equivalently difficult to deliver in practice but from a numerical implementation point of view this value is easily taken as zero. The resulting model retains all the generality associated with (26) but in a more convenient form.

Subscripts n denote function values evaluated at time t_n, and subscripts $n+1$ denote function values at time $t_{n+1} = t_n + \Delta t$. Given $\{e^d_{n+1}, e^d_n, e^{db}_n\}$, the problem is to find e^{db}_{n+1}. To proceed, the convolution integral (26) is first integrated by parts over $s \in (-\infty, t_{n+1})$ using the assumption that $e^{db(i)}(t) \to 0$ as $t \to -\infty$. In the next step, the integral is broken into two separate integrals over domains $s \in (-\infty, t_n)$ and $s \in (t_n, t_{n+1})$. In the last step, $e^d(t_{n+1})$ is written as $e^d(t_{n+1}) = e^d_n + \Delta e^d$. These steps (shown below) produce a more practical form of the convolution integral.

$$\begin{aligned} e^{db(i)}_{n+1} &= e^d_{n+1} - \int_{-\infty}^{t_{n+1}} e^{\frac{-(t_{n+1}-s)}{\tau^b_i}} \dot{e}^d(s)\, ds \\ &= e^d_n + \Delta e^d - e^{\frac{-\Delta t}{\tau^b_i}} \int_{-\infty}^{t_n} e^{\frac{-(t_n-s)}{\tau^b_i}} \dot{e}^d(s)\, ds - \int_{t_n}^{t_{n+1}} e^{\frac{-(t_{n+1}-s)}{\tau^b_i}} \dot{e}^d(s)\, ds. \end{aligned} \qquad (27)$$

In the above expression, recognize that the third term can be written in the following form:

$$e^{\frac{-\Delta t}{\tau^b_i}} \int_{-\infty}^{t_n} e^{\frac{-(t_n-s)}{\tau^b_i}} \dot{e}^d(s)\, ds = e^{\frac{-\Delta t}{\tau^b_i}} (e^d_n - e^{db(i)}_n) \qquad (28)$$

Substituting this expression gives:

$$e^{db(i)}_{n+1} = e^d_n(1 - e^{\frac{-\Delta t}{\tau^b_i}}) + e^{db(i)}_n e^{\frac{-\Delta t}{\tau^b_i}} + \Delta e^d - \int_{t_n}^{t_{n+1}} e^{\frac{-(t_{n+1}-s)}{\tau^b_i}} \dot{e}^d(s)\, ds \qquad (29)$$

The final step is to estimate the scalar deviatoric extension state $e^d(t)$ by linearly interpolating over the time step as follows:

$$e^d(t) \approx e^d_n + \frac{\Delta e^d}{\Delta t}(t - t_n), \qquad t_n \le t \le t_{n+1}. \qquad (30)$$

With this assumption on $e^d(t)$, the integral (29) can be evaluated which gives the update for $e^{db(i)}_{n+1}$:

$$\begin{aligned} e^{db(i)}_{n+1} &= e^d_n(1 - e^{\frac{-\Delta t}{\tau^b_i}}) + e^{db(i)}_n e^{\frac{-\Delta t}{\tau^b_i}} + \Delta e^d - \frac{\Delta e^d}{\Delta t}(1 - e^{\frac{-\Delta t}{\tau^b_i}})\tau^b_i \\ &= e^d_n(1 - e^{\frac{-\Delta t}{\tau^b_i}}) + e^{db(i)}_n e^{\frac{-\Delta t}{\tau^b_i}} + \beta^i \Delta e^d \end{aligned} \qquad (31)$$

where β^i is given by:

$$\beta^i = 1 - \frac{\tau^b_i}{\Delta t}(1 - e^{\frac{-\Delta t}{\tau^b_i}}). \qquad (32)$$

After computing e^{db}_{n+1} with this update, it is straightforward to compute the deviatoric force state from (20).

4.4 2nd Law of thermodynamics

The purpose of this section is to demonstrate that the proposed nonlocal viscoelasticity model does not violate the 2nd law of thermodynamics. This is shown by demonstrating that the rate of entropy production is positive. The approach used is equivalent to demonstrating that power due to viscoelastic deformations is always positive thus satisfying the *dissipation inequality* [1]. Following the procedure developed in [2, 6], the 1st and 2nd laws of thermodynamics are combined to produce the following condition

$$t^i \bullet \dot{e}^i + t^d \bullet \dot{e}^d - \dot{\theta}_T \eta - \dot{\phi} \geq 0, \tag{33}$$

where t^i and t^d are the spherical and deviatoric force states (20) respectively, e^i and e^d are the spherical and deviatoric extension states (11) and (12) respectively, ϕ is the free energy, θ_T is the absolute temperature, and η is entropy. To proceed, a specific form for the free energy will be taken as follows:

$$\phi = W(\theta, e^d, e^{db(i)}) + h(\theta_T) \tag{34}$$

where W is the elastic stored energy functional (18), θ is the dilatation (9), $e^{db(i)}$ is the back extension portion of the deviatoric extension state e^d, and h is an arbitrary function of the temperature θ_T. With this functional, (33) is written as:

$$\left(t^i - \frac{\partial W}{\partial \theta}\frac{\partial \theta}{\partial e^i}\right) \bullet \dot{e}^i + \left(t^d - \frac{\partial W}{\partial e^d}\right) \bullet \dot{e}^d - \frac{\partial W}{\partial e^{db(i)}} \bullet \dot{e}^{db(i)} - \left(\eta + \frac{\partial h}{\partial \theta_T}\right)\dot{\theta}_T \geq 0 \tag{35}$$

From this equation, the following relations are established:

$$t^i = \frac{\partial W}{\partial \theta}\frac{\partial \theta}{\partial e^i}, \quad t^d = \frac{\partial W}{\partial e^d}, \quad \eta = -\frac{\partial h}{\partial \theta_T}, \quad -\frac{\partial W}{\partial e^{db(i)}} \bullet \dot{e}^{db(i)} \geq 0 \tag{36}$$

The first two of the above relations were used in Section 4.1 to derive the elastic material response (see (20)). The last condition is of particular interest here because it is the power dissipation P_d associated with viscoelasticity. Using this expression and the evolution equations (24) and (25), the viscoelastic power dissipated is

$$\begin{aligned} P_d &= -\frac{\partial W}{\partial e^{db(i)}} \bullet \dot{e}^{db(i)} \\ &= t^{d(i)} \bullet \dot{e}^{db(i)} \\ &= \eta_i \dot{e}^{db(i)} \bullet \dot{e}^{db(i)}. \end{aligned} \tag{37}$$

Therefore $P_d \geq 0$ for all values $\eta_i \geq 0$ and the model satisfies the 2nd law constraint (33).

4.5 Relaxation response: time integration of single bond

When developing and studying viscoelasticity models, it is common to consider creep and relaxation response. The creep response is measured/simulated by applying a fixed load

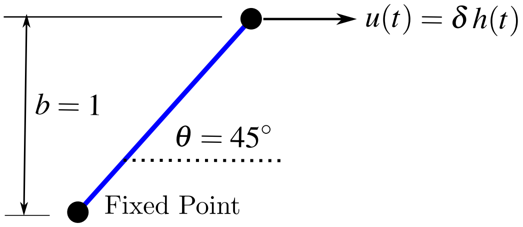

Figure 4. Single Bond: Relaxation Schematic

and measuring/simulating the strain/displacement evolution over time. The creep response can only be computed using a quasi-static or dynamic simulation because displacements are unknown. On the other hand, the relaxation response is determined by applying and holding a step displacement input and measuring or simulating the force relaxation over time. Therefore a relaxation response can be calculated on a single bond using the time integration algorithm developed above. The basic idea is to use (25) to evolve the back extension state for a given step extension state input. Once the back extension state is computed, the elastic extension state can be computed using the additive decomposition and then the force state is computed using the elastic constitutive model.

It is also possible to write down an exact solution for the evolution of the scalar force state t^d directly and thus compare the computed numerical solution with the exact solution. For a constant step input e^d_0, of the deviatoric extension state, the governing equation for evolution of t^d is

$$\dot{t}^d + \frac{1}{\tau^b_i} t^d = \frac{\alpha_\infty \omega}{\tau^b_i} e^d_0, \tag{38}$$

where the right hand side term e^d_0 is given by the step input. This equation is found by combining the elastic constitutive model (20) with the evolution equation (24). An initial condition t^d_0 is needed for the scalar force state. At $t = 0$, the initial deviatoric scalar force state induced by the step input is carried by the elastic portion of the *standard linear solid*, that is, the back extension in the dashpot is zero. Therefore,

$$\begin{aligned} t^d_0 &= \alpha \omega e^d_0 \\ &= (\alpha_\infty + \alpha_i) \omega e^d_0. \end{aligned} \tag{39}$$

As the back extension state evolves, the scalar force state relaxes towards its long term value which is dictated by α_∞. In the discussion to follow, it is useful to define $\alpha_i = \lambda_i \alpha$ where $0 < \lambda_i < 1$. When λ_i is close to zero, the model response is very nearly elastic whereas when λ_i is close to one, the model approximates the *Maxwell* model which has unbounded creep. For both cases, the analytic solution over time is given by

$$t^d(t) = e^d_0(\alpha_\infty + \alpha_i e^{\frac{-t}{\tau^b_i}}). \tag{40}$$

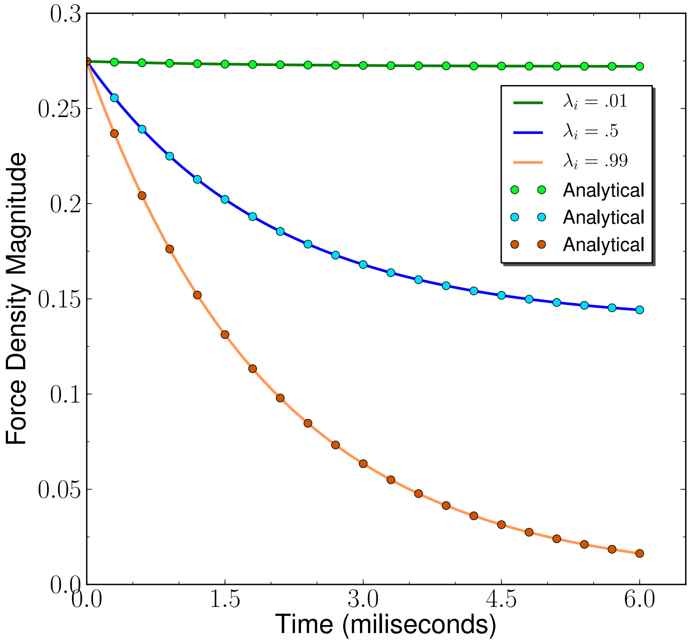

Figure 5. Single Bond: Relaxation Response

The above cases can be used to verify calculations in a peridynamics implementation of the *standard linear solid*. Figure 4 depicts a simulation of the scalar force state in a single bond. The initial condition is applied to the upper point, that is, it is given an initial infinitesimal displacement of δ in the x-direction, where $h(t)$ is the heaviside step function. The relaxation response for $\tau_b^i = 2$ mili-seconds is shown in Figure 5 for three cases: $\lambda_i = .01$, $\lambda_i = .5$, and $\lambda_i = .99$. In practical terms, these cases span the full range of possible responses for the model. Solid lines were computed with the time integration algorithm developed in Section 4.3; dots were computed with the analytical solution given in this section. When $\lambda_i = .99$, $\alpha_\infty \approx \alpha$ and the relaxation force state tends to 0 after long times $t >> \tau_i^b$. When $\lambda_i = .01$, $\alpha_\infty \approx 0$, viscous effects are nearly eliminated, and the response tends to be elastic.

4.6 Linearization: analytical formulae for the modulus state

To facilitate implicit solution methods, it is necessary to evaluate the modulus state. The peridynamics *standard linear solid* presented here is geometrically nonlinear with respect to displacements. The context is one in which a configuration Y^0 (deformation state) corresponds with an un-converged iterate to a solution for step $n+1$ in a nonlinear solution algorithm that requires linearization of the internal force. This is accomplished using the linearization framework [3] by Silling and uses many of the elementary Fréchet derivative results [2] for computing the modulus state. Note that the modulus state derived here is based upon the discrete time integration algorithm given in Section 4.3.

The linearization begins with the definition of the time discretized deviatoric force state (20):

$$\begin{aligned} t^d_{n+1} &= (\alpha_\infty + \alpha_i)\omega e^d_{n+1} - \alpha_i \omega e^{db(i)}_{n+1} \\ &= \alpha\omega e^d_{n+1} - \alpha_i \omega e^{db(i)}_{n+1}. \end{aligned} \tag{41}$$

The Fréchet derivative of t^d_{n+1} requires evaluation of $\nabla e^d_{n+1} \bullet U$ and $\nabla e^{db(i)}_{n+1} \bullet U$ of which the former was given in [2, Eq. 14].

$$\nabla t^d_{n+1} \bullet U = \alpha\omega \nabla e^d_{n+1} \bullet U - \alpha_i \omega \nabla e^{db(i)}_{n+1} \bullet U \tag{42}$$

Recall $e^{db(i)}_{n+1}$ given in (31), and observe that with respect to the Fréchet derivative, e^d_n, and $e^{db(i)}_n$ are constant. Therefore,

$$\nabla e^{db(i)}_{n+1} \bullet U = \beta^i \nabla e^d_{n+1} \bullet U \tag{43}$$

and the Fréchet derivative of $t^{d(i)}_{n+1}$ is defined by:

$$\begin{aligned} \nabla t^{d(i)}_{n+1} \bullet U &= (\alpha - \alpha_i \beta^i)\omega \nabla e^d_{n+1} \bullet U \\ &= \alpha(1 - \lambda_i \beta^i)\omega \nabla e^d_{n+1} \bullet U \\ &= \alpha\omega\Gamma \nabla e^d_{n+1} \bullet U \end{aligned} \tag{44}$$

where

$$\Gamma = (1 - \lambda_i \beta^i), \quad \alpha_i = \lambda_i \alpha. \tag{45}$$

Repeated here for convenience, the value for β^i is

$$\beta^i = 1 - \frac{\tau^b_i}{\Delta t}(1 - e^{\frac{-\Delta t}{\tau^b_i}}). \tag{46}$$

Using (44), the modulus state for the peridynamics *standard linear solid* is

$$\begin{aligned} K\langle \zeta, \xi \rangle &= \left(\frac{9k - \alpha\, m\, \Gamma}{m^2} \right) \omega(|\zeta|)\omega(|\xi|)|\zeta|\,|\xi| M(Y^0)\langle \zeta \rangle \otimes M(Y^0)\langle \xi \rangle \\ &+ \alpha\Gamma\omega(|\zeta|) M(Y^0)\langle \zeta \rangle \otimes M(Y^0)\langle \xi \rangle\, \Delta(\zeta - \xi) \\ &+ t(Y^0)\langle \zeta \rangle \left(\frac{I - M(Y^0)\langle \xi \rangle \otimes M(Y^0)\langle \xi \rangle}{|Y^0|\langle \xi \rangle} \right) \Delta(\zeta - \xi). \end{aligned} \tag{47}$$

5 Generalized Peridynamic Standard Linear Solid

The peridynamics *standard linear solid* is generalized to include an arbitrary number of time constants τ_i^b, and relaxation moduli α_i. This extension is given by simply adding an arbitrary number of Maxwell models (spring/dashpot) in parallel with the peridynamics *standard linear solid.* Most results presented in this section follow directly from the analogous results described in Section 4.

5.1 Elastic stored energy and decomposition of scalar deformation state

The elastic stored energy functional W, given in (18), is generalized to include N Maxwell model mechanisms as follows:

$$W(\theta, e^d, e^{db(1)}, ..., e^{db(N)}) = \frac{k\theta^2}{2} + \frac{\alpha_\infty}{2} e^d \bullet \omega e^d + \sum_{i=1}^{N} \frac{\alpha_i}{2}(e^d - e^{db(i)}) \bullet \omega(e^d - e^{db(i)}). \tag{48}$$

The scalar volumetric and deviatoric force states, t^i and t^d respectively, are derived similarly as before using W as follows:

$$t^i = \frac{\partial W}{\partial \theta}\frac{\partial \theta}{\partial e^i}, \quad t^d = \frac{\partial W}{\partial e^d}, \tag{49}$$

where

$$\begin{aligned} t &= t^i + t^d \\ &= -\frac{3p}{m}\omega \underline{x} + \alpha_\infty e^d + \sum_{i=1}^{N} \alpha_i \omega(e^d - e^{db(i)}) \\ &= t^i + \alpha_\infty e^d + \sum_{i=1}^{N} t^{d(i)}, \end{aligned} \tag{50}$$

where $p = -k\theta$ is the the peridynamic pressure and k is the bulk modulus. The elastic parameter α_∞ and relaxation moduli α_i are defined by the following relationships:

$$\begin{aligned} \alpha &= \frac{15\mu}{m} \\ &= \alpha_\infty + \sum_{i=1}^{N} \alpha_i, \end{aligned} \tag{51}$$

where μ is the elastic shear modulus, and m is the weighted volume. It is convenient to normalize the relaxation moduli as $\alpha_i = \alpha\lambda_i$ so that

$$\alpha_\infty = \alpha(1 - \sum_{i=1}^{N} \lambda_i). \tag{52}$$

As before, α_∞ is bounded below by zero to prevent unbounded creep and thus the constraint on the normalized relaxation moduli is given as

$$\sum_{i=1}^{N} \lambda_i < 1. \tag{53}$$

Also, λ_i, each of which corresponds to a spring, must be positive. Otherwise, the i^{th} Maxwell mechanism would consist of a dashpot only which would produce a degenerate evolution equation that is incompatible with the extension-state-driven relaxation approach here. Combining this constraint with (53) fully defines the range of allowable values for each λ_i.

$$0 < \lambda_i < 1. \tag{54}$$

5.2 Evolution equations

For each relaxation time constant τ_i^b, an evolution equation (25) is given. Motivated by Newton's 3rd law, the force state $t^{d(i)}$ in each mechanism is assumed to be the same in both the spring and dashpot. Mathematically, this is expressed in the following equations:

$$\begin{aligned} t^d &= \alpha_\infty e^d + \sum_{i=1}^{N} t^{d(i)} \\ &= \alpha_\infty e^d + \alpha \sum_{i=1}^{N} \lambda_i \omega (e^d - e^{db(i)}) \\ &= \alpha_\infty e^d + \alpha \sum_{i=1}^{N} \eta_i \dot{e}^{db(i)}, \end{aligned} \tag{55}$$

where for each i

$$\eta_i \dot{e}^{db(i)} = \alpha_i (e^d - e^{db(i)}). \tag{56}$$

To maintain consistency with the two parameter peridynamic *standard linear solid*, the evolution equation for each i is written as

$$\dot{e}^{db(i)} = \frac{1}{\tau_i^b}(e^d - e^{db(i)}), \tag{57}$$

where $\tau_i^b = \frac{\eta_i}{\alpha_i} = \frac{\eta_i}{\alpha\lambda_i}$.

5.3 Time integration over a load step

Results from section 4.3 are directly applicable. The time discretized form of (55) is written as

$$t^d_{n+1} \;=\; \alpha_\infty e^d_{n+1} + \alpha \sum_{i=1}^{N} \lambda_i \omega (e^d_{n+1} - e^{db(i)}_{n+1}), \tag{58}$$

where the update of $e^{db(i)}_{n+1}$ for each mechanism i is given by (31).

5.4 2nd Law of thermodynamics

The approach here follows directly from the treatment in Section 4.4. The viscoelastic power dissipated P_d is

$$\begin{aligned} P_d &= -\sum_{i=1}^{N} \frac{\partial W}{\partial e^{db(i)}} \bullet \dot{e}^{db(i)} \\ &= \sum_{i=1}^{N} t^{d(i)} \bullet \dot{e}^{db(i)} \\ &= \sum_{i=1}^{N} \eta_i \dot{e}^{db(i)} \bullet \dot{e}^{db(i)}. \end{aligned} \tag{59}$$

Therefore $P_d \geq 0$ for all values $\eta_i \geq 0$ and the model satisfies the 2nd law constraint (33).

5.5 Linearization: analytical formulae for the modulus state

The modulus state for the generalized model follows directly from results given in Section 4.6. The modulus state (47) can be re-used with the parameter Γ given in (45) replaced with the following value:

$$\Gamma = 1 - \sum_{i=1}^{N} \lambda_i \beta^i. \tag{60}$$

The resulting modulus state and value for β^i is repeated here for convenience:

$$\beta^i = 1 - \frac{\tau^b_i}{\Delta t}(1 - e^{\frac{-\Delta t}{\tau^b_i}}),$$

$$
\begin{aligned}
K\langle\zeta,\xi\rangle &= \left(\frac{9k-\alpha\, m\,\Gamma}{m^2}\right)\omega(|\zeta|)\omega(|\xi|)|\zeta|\,|\xi| M(Y^0)\langle\zeta\rangle\otimes M(Y^0)\langle\xi\rangle \\
&+ \alpha\Gamma\omega(|\zeta|)M(Y^0)\langle\zeta\rangle\otimes M(Y^0)\langle\xi\rangle\,\Delta(\zeta-\xi) \\
&+ t(Y^0)\langle\zeta\rangle\left(\frac{I-M(Y^0)\langle\xi\rangle\otimes M(Y^0)\langle\xi\rangle}{|Y^0|\langle\xi\rangle}\right)\Delta(\zeta-\xi). \qquad (61)
\end{aligned}
$$

6 Summary

The peridynamics viscoelasticity model developed here is applicable to problems in solid continuum mechanics where fracture and rate effects are important. The model is applicable to both explicit and implicit time integration of the momentum equation and inherits all the advantages for modeling fracture that are associated with peridynamics. It was shown that the model satisfies the dissipation inequality associated with 2nd law of thermodynamics. All of the necessary ingredients for implementation in explicit or implicit time integrated peridynamics codes are presented. This includes time discretization of evolution equations and their updates, and the modulus state required for computation of the Jacobian used in Newton iterations.

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