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Approximations to Wire Grid Inductance

Larry K. Warne, Kimball O. Merewether, and William A. Johnson
Electromagnetics and Plasma Physics Analysis Dept.

Sandia National Laboratories
P. O. Box 5800
Albuquerque, NM 87185-1152

Abstract

By using a multipole-conformal mapping expansion for the wire currents we examine the accuracy of approximations for the transfer inductance of a one dimensional array of wires (wire grid). A simple uniform fit is constructed by introduction of the decay factor from bipolar coordinates into existing formulas for this inductance.

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1 INTRODUCTION

The problem of field leakage through an array of cylinders is the basic canonical periodic shield [1]. Simple solutions can be found for this problem when the cylinder radius is small compared with the spacing [2, 3, 4]. However because of the basic nature of the problem, it is of interest, not only to assess the accuracy of these conformal mapping approximate solutions, but also to construct an easily used solution which remains accurate over the entire range of radius to spacing ratios.

This paper considers the effects of line multipole additions to the simple filament approximation in representing the elements of a one dimensional wire grid array. Here we look at the limit of small wire radius a (and general ratios of wire radius to wire half spacing w) to determine which of the existing approximations to the wire array inductance is most accurate. We also construct a simple and accurate uniform fit using the simple filament approximation along with the decay factor from bipolar coordinates [5].

2 MULTIPOLE CONFORMAL MAPPING REPRESENTATION

The array of wires is along the x axis (but a distance $y = s \to 0$ above) each carrying z directed current I with wire spacing $2w$ and wire radius a. One wire is positioned at $x = 0$ and all wires are parallel with the z axis. A uniform field $H_x \sim \mp I/(4w)$ is generated by the wire currents as $y \to \pm\infty$ and the asymptotic form of the potential is $A_z \sim \mp \frac{\mu_0 I}{4w} y$. A conformal mapping solution for the vector potential is [4]

$$A_z = -\frac{\mu_0 I}{2\pi} \ln \left| e^{-i\pi z/w} - e^{\pi s/w} \right| + \frac{\mu_0 I}{4w} (y + s) \quad , \tag{1}$$

where $z = x + iy$. The multipole moments in this case can be written as

$$A_{zm} = -\frac{\mu_0}{2\pi} p_m \mathrm{Re} \left[\frac{\partial^m}{\partial s^m} \left\{ \ln \left(e^{-i\pi z/w} - e^{\pi s/w} \right) - \frac{\pi}{2w} (-iz + s) \right\} \right]_{s=0} \quad , \tag{2}$$

where the filament or monopole term is $m = 0$, the dipole term is $m = 1$, and the quadrupole term is $m = 2$, etc. Therefore we can write

$$A_z \, 2\pi/(\mu_0 I) = -\ln \left| e^{-i\pi z/w} - 1 \right| + \frac{\pi}{2w} y + \frac{p_1 \pi}{Iw} \mathrm{Re} \left[\frac{1}{e^{-i\pi z/w} - 1} + \frac{1}{2} \right]$$

$$+ \frac{p_2 \pi^2}{Iw^2} \mathrm{Re} \left[\frac{e^{-i\pi z/w}}{\left(e^{-i\pi z/w} - 1 \right)^2} \right] + \cdots \quad , \tag{3}$$

We now let $p'_m = p_m \pi^m / (I w^m)$ and use the match points

$$z = -iae^{im\pi/M} \, , \, m = 0, 1, ..., M \quad , \tag{4}$$

All odd moments vanish in this case because of the symmetry about the x axis. Thus we only use $m = 0, 1, ..., M/2$. The case $M = 2$ thus gives

$$A_z \left(z = -ia \right) 2\pi/(\mu_0 I) = -\ln \left(1 - e^{-\pi a/w} \right) - \frac{\pi a}{2w} + p'_2 \frac{e^{-\pi a/w}}{\left(e^{-\pi a/w} - 1 \right)^2} \quad , \text{and}$$

$$A_z\left(z=a\right)\,2\pi/\left(\mu_0 I\right)=-\ln\left|e^{-i\pi a/w}-1\right|+p_2'\mathrm{Re}\left[\frac{e^{-i\pi a/w}}{\left(e^{-i\pi a/w}-1\right)^2}\right]\quad . \tag{5}$$

Taking the difference and setting the result to zero gives

$$p_2'=-\frac{\frac{1}{2}\ln\left\{4\sin^2\left(\frac{\pi a}{2w}\right)\right\}-\ln\left(1-e^{-\pi a/w}\right)-\frac{\pi a}{2w}}{\frac{1}{4\sin^2\left(\frac{\pi a}{2w}\right)}+\frac{e^{-\pi a/w}}{\left(e^{-\pi a/w}-1\right)^2}}=\frac{\frac{1}{2}\ln\left\{4\sinh^2\left(\frac{\pi a}{2w}\right)\right\}-\frac{1}{2}\ln\left\{4\sin^2\left(\frac{\pi a}{2w}\right)\right\}}{\frac{1}{4\sin^2\left(\frac{\pi a}{2w}\right)}+\frac{1}{4\sinh^2\left(\frac{\pi a}{2w}\right)}}\quad . \tag{6}$$

Alternatively let us take $I=0$ but

$$A_{z0}=\mu_0 H_0 y \tag{7}$$

representing a uniform field H_0 in the x direction. Then we have

$$A_z\,2\pi/\left(\mu_0 2wH_0\right)=\pi y/w+\frac{p_1\pi}{2wH_0w}\mathrm{Re}\left[\frac{1}{e^{-i\pi z/w}-1}+\frac{1}{2}\right]$$

$$+\frac{p_2\pi^2}{2wH_0w^2}\mathrm{Re}\left[\frac{e^{-i\pi z/w}}{\left(e^{-i\pi z/w}-1\right)^2}\right]+\frac{p_3\pi^3}{Iw^3}\mathrm{Re}\left[\frac{\left(e^{-i\pi z/w}+1\right)e^{-i\pi z/w}}{\left(e^{-i\pi z/w}-1\right)^3}\right]+\cdots\quad . \tag{8}$$

Again set $p_m'=p_m\pi^m/\left(2wH_0w^m\right)$ and let $z=-iae^{im\pi/M}$, $m=0,1,...,M$. All even moments vanish in this case because of the odd symmetry about the x axis. Thus, we only use $m=0,1,...,\left(M-1\right)/2$. The case $M=1$ thus gives

$$A_z\left(z=-ia\right)\,2\pi/\left(\mu_0 2wH_0\right)=-\pi a/w+p_1'\left[\frac{1}{e^{-\pi a/w}-1}+\frac{1}{2}\right]\quad . \tag{9}$$

Setting this result to zero gives

$$p_1'=\frac{\pi a/w}{\frac{1}{e^{-\pi a/w}-1}+\frac{1}{2}}=-4\left(\frac{\pi a}{2w}\right)\tanh\left(\frac{\pi a}{2w}\right)\quad . \tag{10}$$

Expanding the dipole and quadrupole coefficients for $\left(\frac{\pi a}{2w}\right)\to 0$ gives

$$p_1'\sim -4\left(\frac{\pi a}{2w}\right)^2\left[1-\frac{1}{3}\left(\frac{\pi a}{2w}\right)^2+\cdots\right]\quad \text{, and} \tag{11}$$

$$p_2'\sim\frac{2}{3}\left(\frac{\pi a}{2w}\right)^4+\cdots\quad . \tag{12}$$

We take the sum of these two problems to represent the situation when a uniform x directed field H_0, where $2wH_0=I$, exists at a large distance below the x axis and zero field exists at a large distance above the x axis. The single linear array of wires has been chosen to have the uniform x directed fields $2wH_0=\mp I/2$ for $y\to\pm\infty$. Thus the linear combination of the single array with current I (including the filament and quadrupole terms) and one half of the uniform field solution (including uniform field and dipole terms) gives the desired representation. The total potential is thus

$$A_z\, 2\pi/\left(\mu_0 2wH_0\right) = -\ln\left|e^{-i\pi z/w}-1\right| + \frac{\pi}{2w}y + p_2'\mathrm{Re}\left[\frac{e^{-i\pi z/w}}{\left(e^{-i\pi z/w}-1\right)^2}\right] + \cdots$$

$$+\frac{1}{2}\frac{\pi}{w}y + \frac{1}{2}p_1'\mathrm{Re}\left[\frac{1}{e^{-i\pi z/w}-1}+\frac{1}{2}\right] + \cdots \quad . \tag{13}$$

3 TRANSFER INDUCTANCE OF ARRAY

The transfer inductance of the grid can be defined by

$$L = \Phi/H_0 \tag{14}$$

where Φ is the $-x$ directed magnetic flux per unit length passing between the wires and the point $y \to +\infty$. Thus, we can write

$$\Phi = A_z\left(z = ia\right) - A_z\left(z \to i\infty\right) \tag{15}$$

Therefore the inductance is

$$L\left(\frac{\pi}{\mu_0 w}\right) = -\ln\left(e^{\pi a/w}-1\right) + \pi a/w + \frac{1}{2}p_1'\left[\frac{1}{e^{\pi a/w}-1}+\frac{1}{2}\right] + \frac{p_2' e^{\pi a/w}}{\left(e^{\pi a/w}-1\right)^2} + \cdots - \frac{1}{4}p_1' - \cdots \tag{16}$$

which can be rewritten as

$$L\left(\frac{\pi}{\mu_0 w}\right) = -\ln\left\{2\sinh\left(\frac{\pi a}{2w}\right)\right\} + \left(\frac{\pi a}{2w}\right) + \frac{1}{4}p_1'\coth\left(\frac{\pi a}{2w}\right) + \frac{1}{4}p_2'\mathrm{csch}^2\left(\frac{\pi a}{2w}\right) + \cdots - \frac{1}{4}p_1' - \cdots \quad . \tag{17}$$

Inserting the coefficients gives

$$L\left(\frac{\pi}{\mu_0 w}\right) = -\ln\left\{2\sinh\left(\frac{\pi a}{2w}\right)\right\} + \frac{\frac{1}{2}\ln\left\{\sinh^2\left(\frac{\pi a}{2w}\right)/\sin^2\left(\frac{\pi a}{2w}\right)\right\}}{1+\sinh^2\left(\frac{\pi a}{2w}\right)/\sin^2\left(\frac{\pi a}{2w}\right)} + \cdots + \left(\frac{\pi a}{2w}\right)\tanh\left(\frac{\pi a}{2w}\right) - \cdots \quad . \tag{18}$$

Expanding for $\left(\frac{\pi a}{2w}\right) \to 0$ gives

$$L\left(\frac{\pi}{\mu_0 w}\right) \sim \ln\left(\frac{w}{\pi a}\right) + \left(\frac{\pi a}{2w}\right)^2 + \cdots \quad . \tag{19}$$

Therefore we see that the simple small radius a approximate formula

$$L\left(\frac{\pi}{\mu_0 w}\right) \approx \ln\left(\frac{w}{\pi a}\right) \tag{20}$$

is more accurate than the commonly used filament result [2, 3]

$$L\left(\frac{\pi}{\mu_0 w}\right) \approx -\ln\left(1-e^{-\pi a/w}\right) \sim \ln\left(\frac{w}{\pi a}\right) + \left(\frac{\pi a}{2w}\right) - \frac{1}{6}\left(\frac{\pi a}{2w}\right)^2 \tag{21}$$

in the $\left(\frac{\pi a}{2w}\right) \to 0$ limit.

Figure 1 shows the normalized transfer inductance as a function of a/w. The filament curve, labeled with zeros, is (21) (without the small $\frac{\pi a}{2w}$ expansion) which is also the result obtained if the multipole expansions of the previous section are terminated without the dipole, quadrupole, etc., terms. The dipole curve, labeled with ones, is the result obtained from the previous section if the expansions are terminated without the quadrupole, etc., terms. Similarly, the quadrupole curve, labeled with twos, is (18), etc. Finally, the small a approximation curve, labeled as thin wire, is (20). Thus, although (20) is more accurate for small values of a/w, the filament approximation (21) provides a more uniform fit to the actual value of the transfer inductance.

4 SMOOTHED CONFORMAL MAPPING SOLUTION

The attenuation resulting from the region between cylinders, when they are closely spaced, is difficult to represent by means of the multipole expansion. Previously [4] a "smoothed" conformal transformation has been used to treat this problem. This approximate solution is found by first finding the solution of the transcendental equation

$$\csc\left[\frac{\pi a}{2w}(1+\lambda)\right] = \coth\left[\frac{\pi a}{2w\lambda}(1+\lambda)\right] \tag{22}$$

and then finding

$$\left(\frac{\pi}{\mu_0 w}\right) L = \ln\left[\csc\left\{\frac{\pi a}{2w}(1+\lambda)\right\}\right] \quad . \tag{23}$$

This result is shown as the dotted curve in Figure 1. However because the "smoothed" conductor solution is a good approximation to the cylinder only when the conductors are not in close proximity, and also because these results involve a transcendental equation, we will examine a simple fit in the next section.

5 BIPOLAR COORDINATES (TWO CYLINDERS)

Here we use the exponential decay from the bipolar system of coordinates (representing two cylinders) times the array result to construct an accurate fit to the transfer inductance which holds for all ratios of radius to spacing. The bipolar system [5] uses the description (note here that $x+w$ is the same coordinate as x in previous sections)

$$x = a_0 \frac{\sinh u}{\cosh u - \cos v}$$

$$y = a_0 \frac{\sin v}{\cosh u - \cos v} \tag{24}$$

or $(x - a_0 \coth u)^2 + y^2 = a_0^2 \mathrm{csch}^2 u$ and $x^2 + (y - a_0 \cot v)^2 = a_0^2 \csc^2 v$, where $-\infty < u < \infty$ and $0 < v < 2\pi$. We choose cylindrical coordinate surfaces $u = \pm u_0$, with radii a and center-to-center spacing $2w$, by means of $a = a_0 \mathrm{csch}(u_0)$ and $w = a_0 \coth(u_0)$ or

$$u_0 = \ln\left[w/a + \sqrt{w^2/a^2 - 1}\right] \quad \text{, and}$$

$$a_0 = \sqrt{w^2 - a^2} \quad . \tag{25}$$

In this coordinate system Laplace's equation becomes [5]

$$\nabla^2 A_z = \frac{1}{a_0^2}\left(\cosh u - \cos v\right)^2 \left(\frac{\partial^2 A_z}{\partial u^2} + \frac{\partial^2 A_z}{\partial v^2}\right) + \frac{\partial^2 A_z}{\partial z^2} = 0 \quad . \tag{26}$$

In the two dimensional case the last term is zero and we have have the same solutions as in Cartesian coordinates. We take the lowest order mode that is constant on the conducting surfaces

$$A_z = A\cos\left(\frac{\pi u}{2u_0}\right) e^{\pi(v-\pi)/(2u_0)} + C \quad , \tag{27}$$

where C is a constant. The difference potential between $u = 0$ and the surface $u = \pm u_0$ is thus

$$\Delta A_z = Ae^{-\pi(\pi - v)/(2u_0)} = A\exp\left[-\pi\frac{\text{Arctan}\left(y/\sqrt{w^2 - a^2}\right)}{\ln\left(w/a + \sqrt{w^2/a^2 - 1}\right)}\right] \quad . \tag{28}$$

There is thus a decay from the illuminated side $y < 0$ to the shadow side $y > 0$ of the array. If we assume this decay takes place over the interval $-h < y < h$, we have an overall decay factor of the square of the exponential in (28) with $y = h$.

6 FIT FOR TRANSFER INDUCTANCE

The fit to the transfer inductance will be taken as the standard filament result (21) times the decay from bipolar coordinates (28). The distance $h = O(a) = ca$ is taken to account for the missing decay in the filament result

$$\left(\frac{\pi}{\mu_0 w}\right) L \approx -\ln\left(1 - e^{-\pi a/w}\right)\exp\left[-2\pi\frac{\text{Arctan}\left(c/\sqrt{w^2/a^2 - 1}\right)}{\ln\left(w/a + \sqrt{w^2/a^2 - 1}\right)}\right] \quad . \tag{29}$$

If we take c to be a constant, the best choice seems to be $c \approx 0.71$, resulting in relative errors of less than 12% when compared to the multipole results (over their range of validity). This can be improved by adding variation with a/w (for example $c \approx 1/2 + a/(3w)$ works well). A similar but slightly better choice is

$$c \approx \frac{1}{2}\sqrt{1 + \frac{\pi a}{2w}} \quad . \tag{30}$$

Figure 1 shows the line multipole moment results (labeled with the number included, from zero to eight) for the sheet transfer inductance compared with this fit (dashed curve). Figure 2 shows the relative errors between the multipole results and the fit.

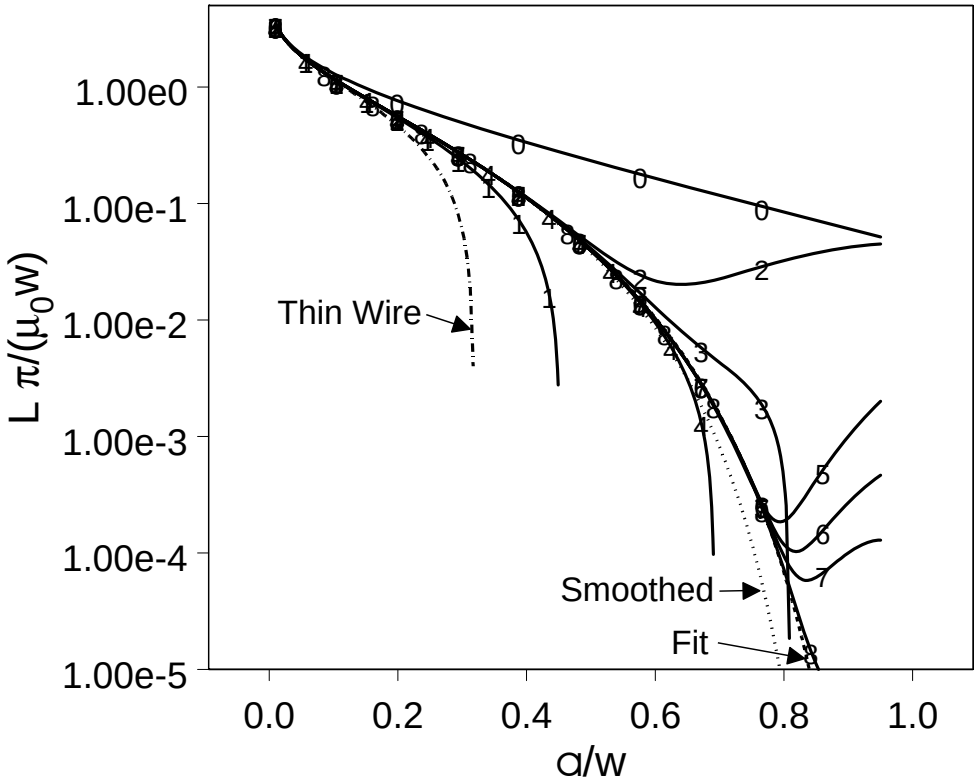

Figure 1. Multipole moment calculations for sheet transfer inductance. Numbers labeling the curves indicate how many line multipole moments were included. The dashed black curve is a fit consisting of the product of the filament solution (zero multipole moment) times the decay found from bipolar coordinates. The dotted curve is the smoothed conformal mapping solution. The thin wire approximation is shown as the dot-dash curve.

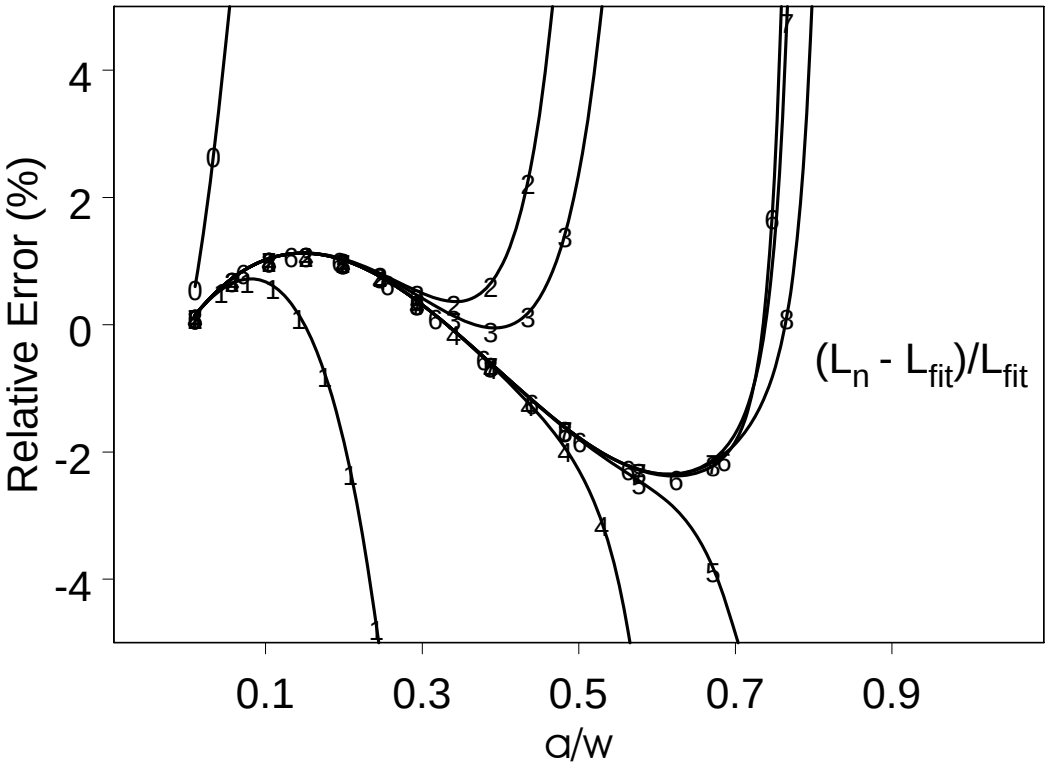

Figure 2. Relative error between sheet inductance computed with multipole moments and fit.

7 CONCLUSION

Line multipole moments have been used in conjunction with conformal mapping to describe the magnetic field penetration of a periodic array of perfectly conducting cylinders (two-dimensional wire mesh). This rigorous approach describes the transfer inductance of the array in closed form for small radius to spacing ratios and can be used to obtain power series representations for this quantity. Numerical summation of the multipole representation extends the range of validity to larger ratios. A simple fit to the transfer inductance has been constructed by introducing exponential decay from the bipolar coordinate system (representing the region between a pair of cylinders) into the typical filament result for the transfer inductance of the array. This fit yields accurate results for all ratios of radius to spacing.

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