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Reliability of Dynamic Systems Under Limited Information

Richard V. Field, Jr. and Mircea D. Grigoriu

Prepared by
Sandia National Laboratories
Albuquerque, New Mexico 87185 and Livermore, California 94550

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Richard V. Field, Jr.
Applied Mechanics Development
Sandia National Laboratories
P.O. Box 5800
Albuquerque, NM 87185-0847
rvfield@sandia.gov

Mircea D. Grigoriu
Civil and Environmental Engineering
Cornell University
Ithaca, NY 14853-3501
mdg12@cornell.edu

Abstract

A method is developed for reliability analysis of dynamic systems under limited information. The available information includes one or more samples of the system output; any known information on features of the output can be used if available. The method is based on the theory of non-Gaussian translation processes and is shown to be particularly suitable for problems of practical interest. For illustration, we apply the proposed method to a series of simple example problems and compare with results given by traditional statistical estimators in order to establish the accuracy of the method. It is demonstrated that the method delivers accurate results for the case of linear and nonlinear dynamic systems, and can be applied to analyze experimental data and/or mathematical model outputs. Two complex applications of direct interest to Sandia are also considered. First, we apply the proposed method to assess design reliability of a MEMS inertial switch. Second, we consider re-entry body (RB) component vibration response during normal re-entry, where the objective is to estimate the time-dependent probability of component failure. This last application is directly relevant to re-entry random vibration analysis at Sandia, and may provide insights on test-based and/or model-based qualification of weapon components for random vibration environments.

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Reliability of Dynamic Systems Under Limited Information

1 Introduction

Consider the input/output relationship illustrated by Fig. 1 and assume that: (1) input $\mathbf{Y}$ is a stochastic process with specified probability law; and (2) $\mathcal{S}$ is a deterministic dynamic system. Accordingly, system output $\mathbf{X}$ is a stochastic process. Our objective is to estimate properties of the output in order to assess system performance; examples include the mean, variance, and/or covariance of $\mathbf{X}$. A more difficult task, and more relevant from the perspective of engineering applications, is the estimation of system reliability, *i.e.*, the probability that $\mathbf{X}$ remains within a "safe set" during some specified time interval, denoted by $[0, \tau]$. The complement of this event is the probability of system failure in $[0, \tau]$. For example, consider the case of a re-entry body (RB) subject to an applied stochastic pressure field, $\mathbf{Y}$, that models the dynamic excitation provided by the re-entry environment, where output $\mathbf{X}$ denotes the resulting stress response at the mounting point of critical internal component. An appropriate safe set is defined by the value for the yield stress of the mounting point material, and system failure occurs if the value for $\mathbf{X}$ exceeds the yield stress at any time prior to time τ, the duration of the re-entry event. Alternative failure modes, *e.g.*, structural fatigue, can also be studied in this framework.

Numerous methods have been developed to approximate the statistics and/or probability law of the output assuming various properties of the system are known. For example, if $\mathcal{S}$ is known to have a linear functional form, the second-moment properties of $\mathbf{X}$ can be obtained by classical methods of linear random vibration; see, for example, [18], [20] or [25]. If, in addition, input $\mathbf{Y}$ is known to follow a Gaussian distribution, so does the output and the probability law of $\mathbf{X}$ is completely defined by its second-moment properties. For non-Gaussian input, the complete probability law of $\mathbf{X}$ is typically unavailable, but it is possible to calculate higher-order statistics of the output for certain classes of non-Gaussian input. For example, moments of any order of $\mathbf{X}$ can be obtained when $\mathbf{Y}$ can be expressed as a polynomial function of a filtered Gaussian process (see [13], Section 5.3.2).

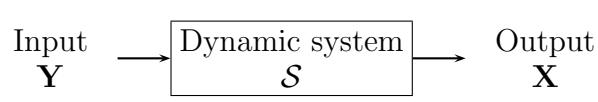

Figure 1. Input/output relationship for dynamic systems.

If $\mathcal{S}$ is a nonlinear dynamic system, analytical and numerical solutions for the output distribution are limited to small dimensional systems. The path integral method [19] can be applied to propagate the density of the state vector in time, provided that the state can be expressed as a diffusion process with known functional form. The Fokker-Planck equation can also be used, but numerical solutions are required for most problems and the method is typically limited to state vectors with dimension four or less [24, 26]. The methods of perturbation [6] and stochastic averaging (see [17], Section 4.7) can be used provided that the degree of nonlinearity of $\mathcal{S}$ is small and input $\mathbf{Y}$ has small intensity with a broad band spectrum, respectively. Equivalent linearization [23] and moment closure techniques [12] can be used to estimate statistics of $\mathbf{X}$, but can deliver inaccurate results; their usefulness must therefore be assessed on a case-by-case basis. Monte Carlo simulation is the only general method that can deliver the accurate approximations for the statistics of $\mathbf{X}$ irrespective of the dimension of $\mathcal{S}$ and properties of $\mathbf{Y}$, but the method proves intractable when applied to realistic, large dimensional systems.

As demonstrated, current methods are limited to small dimensional systems and/or require knowledge of the system that, for problems of practical interest, is unavailable. To illustrate, we consider again the example of RB normal re-entry. An analyst may have a mathematical model for input, $\mathbf{Y}$, and RB structure, $\mathcal{S}$, to make predictions of output $\mathbf{X}$. Often, the models for $\mathbf{Y}$ and $\mathcal{S}$ involve a large number of equations that can only be solved numerically with a computer, requiring many hours to obtain an accurate solution. The information on $\mathbf{X}$ is therefore limited to a few samples of the output allowed under budget and/or time constraints. Likewise, an experimentalist may collect a series of measurements of the aircraft's response at a few important locations during a series of flights. In both cases, the complete representation for $\mathcal{S}$ is unavailable so that properties of $\mathcal{S}$, *e.g.*, the degree of nonlinearity, are unknown.

We propose a new method to approximate the statistics of output $\mathbf{X}$, including the probability that $\mathbf{X}$ remains within a set of acceptable performance during a specified time interval. The method is based on the crossing theory of translation processes [11] and is useful for problems of practical interest because all that is required for calculation is one or more independent samples of the system output. At present, the method is developed assuming stationary output and, when only one sample of the system output is available, $\mathbf{X}$ must also be assumed ergodic. However, it is possible to extend the method to a special class of non-stationary, non-ergodic output. It is shown that the proposed method is applicable for the case of linear or nonlinear systems, and is particularly useful when information on certain features of the output, *e.g.*, knowledge that $\mathbf{X}$ takes values in a bounded rectangle or is nonnegative almost surely, is available. Further, the method can be applied when $\mathbf{X}$ represents outputs from a mathematical model for a physical system, or actual experimental measurements of physical system response. To assess the accuracy of the proposed method, we compare with results given by classical statistical estimators.

The organization of this report is as follows. We review the essentials of crossing theory of stochastic processes for reliability analysis in Section 2, and develop the necessary statistical estimators used in this study in Section 3. The proposed method based on crossings of non-Gaussian translation processes is defined in Section 4 and applied to a series of example problems in Section 5 to demonstrate the accuracy and applicability of the proposed method. Section 5.4 is a particular interest, as we apply the method to a complex, real-world application of relevance to re-entry random vibration analysis at Sandia. A brief extension of the proposed method to a special class of non-stationary, non-ergodic output is discussed in Appendix A.

2 System reliability by crossing theory

Let $\mathbf{X}(t)$, $t \geq 0$, be an $\mathbb{R}^d$–valued mean-square differentiable stochastic processes that denotes system output (refer to Fig. 1). Denote by

$$D = \left\{ \mathbf{x} \in \mathbb{R}^d \colon g(\mathbf{x}) \leq 0 \right\} \subseteq \mathbb{R}^d, \tag{1}$$

the safe set for output $\mathbf{X}(t)$, where $g \colon \mathbb{R}^d \to \mathbb{R}$ is a deterministic, measurable function that defines system performance. For example, if the system fails whenever the output exits a sphere of radius $a > 0$ centered at the origin in $\mathbb{R}^d$, then $g(\mathbf{x}) = \|\mathbf{x}\| - a$ defines an appropriate safe set. Our objective is to estimate

$$p_{\mathrm{F}}(\tau) = 1 - P\left(\mathbf{X}(t) \in D,\ 0 \leq t \leq \tau\right), \tag{2}$$

where $\tau \geq 0$ denotes the system lifetime. We refer to $p_{\mathrm{F}}(\tau)$ as the probability of system failure within lifetime τ; $1 - p_{\mathrm{F}}(\tau)$ is a measure for system reliability.

The use of Eq. (2) for applications is oftentimes impractical because the complete probability law for $\mathbf{X}$ is required. One approximate solution for $p_{\mathrm{F}}(\tau)$ is based on crossing theory of stochastic processes; the approach is attractive because only the marginal probability law for $\mathbf{X}$ and its time derivative are required for calculations [5]. Let random variable $N_D(\tau) \geq 0$ denote the number of departures, or outcrossings, of stochastic process $\mathbf{X}(t)$ from safe set D during time interval $[0, \tau]$; one D–outcrossing of $\mathbf{X}(t) \in \mathbb{R}^d$ is illustrated by Fig. 2. Assuming infrequent D–outcrossings and $P\left(\mathbf{X}(0) \in D\right) = 1$, *i.e.*, the system is safe at $t = 0$ almost surely, a common approximation for Eq. (2) is given by (see [22], Section 7.2)

$$p_{\mathrm{F}}(\tau) \approx 1 - \exp\left(-\mathrm{E}\left[N_D(\tau)\right]\right), \tag{3}$$

where $\mathrm{E}[A]$ denotes the expected value of random variable A. For the special case when $\mathbf{X}$ is stationary,

$$p_{\mathrm{F}}(\tau) \approx 1 - e^{-\nu_D \cdot \tau}, \tag{4}$$

where $\nu_D = \mathrm{E}\left[N_D(1)\right]$ is the mean D–outcrossing rate of $\mathbf{X}$.

For the remainder of the discussion, we assume $\mathbf{X}(t)$ is a scalar process, denoted by $X(t)$, and consider safe set $D = (-\infty, a)$. The D–outcrossings of $X(t)$ in this case correspond to crossings of level a with a positive slope, referred to as a–upcrossings of $X(t)$. An a–upcrossing event for one sample of $X(t)$ is illustrated by Fig. 3. Outcrossings for a more general safe set, *e.g.*, a–downcrossings and/or b–upcrossings of $D = (a, b)$, can also be calculated. The restriction to scalar processes is not a limitation for reliability studies since, if $\mathbf{X}$ denotes the vector output of a dynamic system and acceptable performance implies $\mathbf{X}$ belongs to a safe set D as defined by Eq. (1), then an 0–upcrossing of scalar process $X = g(\mathbf{X})$ defines system failure.

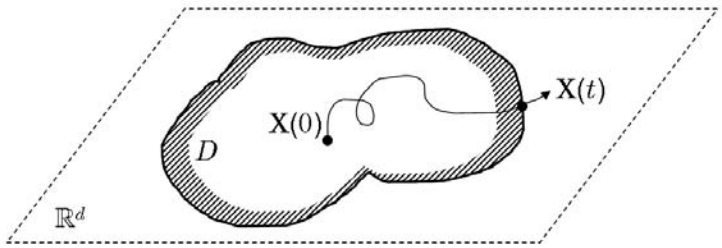

Figure 2. D–outcrossing of one sample of $\mathbf{X}(t) \in \mathbb{R}^d$.

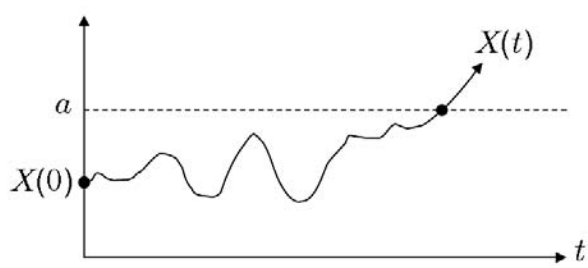

Figure 3. An a–upcrossing event of one sample of $X(t)$.

By Eqs. (3) and (4), our estimate for the probability of system failure is completely characterized by the mean D–outcrossing rate of X. We therefore focus on methods to estimate ν_D under limited information; two such methods are considered. Method 1 is based on the statistics of output X. We demonstrate that this approach works well when data on X is abundant and/or the value of the threshold a is small. However, for the more realistic case of limited data on a highly reliable system, *i.e.*, a relatively large threshold a, Method 1 may provide no information on system failure because observation of a failure event is extremely unlikely in this case. Method 2 uses a translation process model to approximate system output X; the crossing theory of translation processes is then used to estimate ν_D. The translation model is calibrated to the available data and any prior knowledge on X, and the method provides an estimate of system failure regardless of the amount of available data or value for threshold a. In contrast to Method 1 which depends on the joint distribution of $(X(t), X(s))$ for $s > t$, we show that Method 2 depends only on the second-moment properties and marginal distribution of X. The obvious disadvantage of Method 2 is that the functional form of the translation assumed may be incorrect, or that X may not be a translation process. The generalities of Methods 1 and 2 are discussed in Sections 3 and 4, respectively.

3 Output statistics

Suppose $X(t)$, $t \geq 0$, is a real-valued stochastic process with finite moments and unknown probability law. To simplify the discussion, we assume X is stationary and ergodic, where the latter implies that X satisfies

$$\mathrm{E}\left[h(X(t))\right] = \lim_{\tau \to \infty} \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} h(X(u))\,\mathrm{d}u \tag{5}$$

almost surely for any real-valued measurable function h such that $\mathrm{E}\left[h(X(t))\right] < \infty$ (see [14], p. 120). Output statistics for a special class of non-stationary, non-ergodic processes are considered in Appendix A. The available information on X is assumed limited to one sample, denoted by $(X_1 = X(t_1), X_2 = X(t_2), \ldots, X_n = X(t_n))$, where $t_1 < t_2 < \cdots < t_n$ and $n > 1$ is the sample length. We assume the time step, $\Delta t = t_k - t_{k-1}$ for $k = 2, 3, \ldots, n$, is constant and sufficiently small to accurately characterize the frequency content of $X(t)$, the output of a dynamic system.

Let $\mu = \mathrm{E}\left[X(t)\right]$, $\gamma_p = \mathrm{E}\left[(X(t) - \mu)^p\right]$ for $p = 2, 3, \ldots$, $r(u) = \mathrm{E}\left[X(t)\,X(t+u)\right]$, and F denote the mean, central moments, correlation function, and marginal CDF of X, respectively. We develop unbiased estimators for these quantities based on the available data, and calculate the relationship between the variance and/or coefficient of variation of these estimators and sample size, n. This relationship can be used, for example, to assess the sensitivity of the estimators to the particular sample used for calculations, or to calculate the value for n needed to achieve estimators of specified accuracy.

3.1 Moments

Let

$$M_n = \frac{1}{n} \sum_{i=1}^{n} X_i \quad \text{and} \quad \Gamma_{p,n} = \frac{1}{n} \sum_{i=1}^{n} (X_i - M_n)^p \tag{6}$$

be statistical estimators for μ and $\gamma_p = \mathrm{E}\left[(X(t) - \mu)^p\right]$, the mean and central moments of $X(t)$, respectively, where $p \geq 2$ is an integer. For $p = 2$, $\Gamma_{2,n}$ provides an estimate for the variance of $X(t)$; $\Gamma_{3,n}/(\Gamma_{2,n})^{3/2}$ and $\Gamma_{4,n}/(\Gamma_{2,n})^2$ provide estimates for the coefficients of skewness and kurtosis of $X(t)$, respectively.

The estimators defined by Eq. (6) have some desirable properties. For example, M_n is unbiased, *i.e.*, $\mathrm{E}\left[M_n\right] = \mu$, and $\Gamma_{2,n}$ is asymptotically unbiased as $n \to \infty$. The variance of

M_n and $\Gamma_{p,n}$ are given by

$$\mathrm{Var}\left[M_n\right] = \frac{\sigma^2}{n}\left[1 + 2\sum_{k=1}^{n-1}\left(1 - \frac{k}{n}\right)\rho_k\right] \quad \text{and}$$

$$\mathrm{Var}\left[\Gamma_{p,n}\right] = \frac{1}{n^2}\sum_{i,j=1}^{n}\mathrm{E}\left[(X_i - M_n)^p\,(X_j - M_n)^p\right] - \gamma_p^2 \tag{7}$$

respectively, where ρ_k denotes the correlation coefficient of $(X_{i+k},\, X_i)$. For the special case where $\{X_i\}$ is an independent identically distributed (iid) time series, $\mathrm{Var}\left[M_n\right] = \gamma_2/n$. If, in addition, the series has zero mean and the sample size n is sufficiently large that $M_n \approx 0$, then

$$\mathrm{Var}\left[\Gamma_{p,n}\right] \approx \frac{1}{n}\left(\gamma_{2p} - \gamma_p^2\right). \tag{8}$$

3.2 Correlation and marginal distribution

Let

$$R_n(k\,\Delta t) = \frac{1}{n-k}\sum_{i=1}^{n-k} X_{i+k}\,X_i,\ \ k = 0, \ldots, n' < n, \ \text{and}$$

$$F_n(x) = \frac{1}{n}\sum_{i=1}^{n} 1(X_i \le x),\ \ \forall x \in \mathbb{R}, \tag{9}$$

be statistical estimators for $r(u) = \mathrm{E}\left[X(t)\,X(t+u)\right]$ and F, the correlation function and marginal probability law of $X(t)$, respectively, where $1(A) = 1$ if event A is true and zero otherwise (see [2], Section 5.2.5 and [3], Section 11.4). The estimators defined by Eq. (9) are unbiased, *i.e.*, $\mathrm{E}\left[R_n(k\,\Delta t)\right] = r(k\,\Delta t)$ for $k = 0, \ldots, n' < n$, and $\mathrm{E}\left[F_n(x)\right] = F(x)$, $\forall x \in \mathbb{R}$. The variance of R_n and F_n, given by

$$\mathrm{Var}\left[R_n(k\,\Delta t)\right] = \frac{1}{(n-k)^2}\sum_{i,j=1}^{n-k}\mathrm{E}\left[X_{i+k}\,X_i\,X_{j+k}\,X_j\right] - r(k\,\Delta t)^2 \quad \text{and}$$

$$\mathrm{Var}\left[F_n(x)\right] = \frac{1}{n^2}\sum_{i,j=1}^{n}\mathrm{E}\left[1\left(X_i \le x\right)\,1\left(X_j \le x\right)\right] - F(x)^2, \tag{10}$$

provide a measure of the accuracy of the estimators for r, and F, respectively.

For the special case where $\{X_i\}$ is an iid time series, we have

$$\begin{aligned} \mathrm{Var}\left[R_n(0)\right] &= \frac{1}{n}\,\left[\mathrm{E}\left[X_1^4\right] - (\mathrm{E}\left[X_1^2\right])^2\right] \quad \text{and} \\ \mathrm{Var}\left[F_n\right] &= \frac{1}{n}\,F(x)\,\left[1 - F(x)\right], \end{aligned} \tag{11}$$

demonstrating that both estimators improve with increasing sample size n.

3.3 Mean crossing rate

The mean a–upcrossing rate of mean-square differentiable process $X(t)$ is given by (see [22], p. 290)

$$\nu(a) = \int_0^\infty u\, f(a,u)\,\mathrm{d}u, \tag{12}$$

where $f(x,u)$ denotes the joint PDF of vector $(X(t),\, \dot{X}(t))$, and $\dot{X}(t) = \mathrm{d}X(t)/\mathrm{d}t$. For the special case where X is Gaussian with zero mean and variance σ^2, we have [21]

$$\nu(a) = \frac{b}{2\pi\sigma}\, e^{-a^2/2\sigma^2} \tag{13}$$

where b denotes the standard deviation of process $\dot{X}$. Assuming Δt is sufficiently small so that at most one a–upcrossing occurs during the interval $(t,\, t+\Delta t)$ for any $t \geq 0$,

$$\tilde{\nu}(a) = \frac{1}{\Delta t}\, P(X_i \leq a,\, X_{i+1} > a) \tag{14}$$

provides an approximation for $\nu(a)$ defined by Eq. (12).

Let

$$V_n(a) = \frac{1}{n\,\Delta t}\,\sum_{i=1}^{n} 1\left(X_i \leq a,\, X_{i+1} > a\right) \tag{15}$$

denote an estimator for $\tilde{\nu}(a)$. It follows that $\mathrm{E}\left[V_n(a)\right] = \tilde{\nu}(a)$ so that $V_n(a)$ is an unbiased estimator for $\tilde{\nu}(a)$. The variance of $V_n(a)$, given by

$$\mathrm{Var}\left[V_n(a)\right] = \frac{1}{(n\,\Delta t)^2}\,\sum_{i,j=1}^{n} \mathrm{E}\left[1\left(X_i \leq a,\, X_{i+1} > a\right)\, 1\left(X_j \leq a,\, X_{j+1} > a\right)\right] - \tilde{\nu}(a)^2, \tag{16}$$

demonstrates that the accuracy of $V_n(a)$ defined by Eq. (15) depends on the sample size, n, the threshold, a, and the correlation structure of $X(t)$. We note that, unlike the estimators

defined in Sections 3.1 and 3.2 which depend only on the statistics of X, the estimator for $\tilde{\nu}(a)$ defined by Eq. (15) depends on the joint statistics of vector $(X_i,\ X_{i+1})$. A large sample size may therefore be required before $V_n(a)$ is sufficiently accurate for applications.

To illustrate the properties of estimator $V_n(a)$, we apply it to the AR(1) time series, *i.e.*, the first order, stationary, auto-regressive process defined by [4]

$$X_{i+1} = \bar{\rho}\, X_i + (1-\bar{\rho})^{1/2}\, W_i,\ \ i = 0, 1, \ldots, \tag{17}$$

where $|\,\bar{\rho}\,| < 1$ is a deterministic constant that defines the correlation length of the sequence, $\{W_i,\ i \geq 1\}$ is a sequence of independent $N(0,1)$ random variables that are independent of $X_0 \sim N(0,1)$, and time step $\Delta t = 1$. For the special case where $\bar{\rho} = 0$, $\{X_i,\ i \geq 1\}$ is an independent standard Gaussian time series. The mean a–upcrossing rate of the AR(1) process is given by (see [13], Section 3.6.1)

$$\nu(a) = \tilde{\nu}(a) = \int_{-\infty}^{a} \left[1 - \Phi\left(\frac{a - \bar{\rho}\, u}{(1-\bar{\rho})^{1/2}} \right) \right] \phi(u)\, \mathrm{d}u, \tag{18}$$

where Φ and ϕ denote the CDF and PDF, respectively, of a $N(0,1)$ random variable, and the first equality follows since $\{X_i\}$ is a discrete-time stochastic process with $\Delta t = 1$.

For $\tilde{\nu}(a) \neq 0$, let

$$\mathrm{COV}\left[V_n(a)\right] = \frac{\sqrt{\mathrm{Var}\left[V_n(a)\right]}}{\mathrm{E}\left[V_n(a)\right]} = \frac{1}{\tilde{\nu}(a)} \sqrt{\mathrm{Var}\left[V_n(a)\right]} \tag{19}$$

denote the coefficient of variation of estimator $V_n(a)$ defined by Eq. (15), where quantity $\mathrm{Var}\left[V_n(a)\right]$ is defined by Eq. (16). Table 1 lists values for $\mathrm{COV}\left[V_n(a)\right]$ with $\bar{\rho} = 0$, 0.7, and 0.9, thresholds $a = 1$, 2, and 3, and samples of size $n = 10$, 50, and 100. Results illustrate that $\mathrm{COV}\left[V_n(a)\right]$ decreases with increasing values for n, and increases with increasing values for a. For example, with $\bar{\rho} = 0.9$ the coefficient of variation of $V_{10}(1)$ and $V_{100}(2)$ are approximately the same.

The accuracy of $V_n(a)$ also depends on the correlation length, $\bar{\rho}$. In general, to achieve a specified value for $\mathrm{COV}\left[V_n(a)\right]$ we need a longer sample for a correlated series than for an independent series. For example, the coefficient of variation of $V_n(3)$ is approximately 3 for $(n = 70, \bar{\rho} = 0)$ and $(n = 100, \bar{\rho} = 0.7)$ and approximately 5 for $(n = 30, \bar{\rho} = 0)$ and $(n = 100, \bar{\rho} = 0.9)$. Hence, one sample of a correlated AR(1) process of size $n = 100$ with coefficients $\bar{\rho} = 0.7$ and $\bar{\rho} = 0.9$ carries the same information as one sample of an independent AR(1) process of size $n = 70$ and $n = 30$, respectively. This example motivates the need for alternative methods to estimate the a–upcrossing rate for the case of high threshold, a, and/or long correlation length, $\bar{\rho}$.

Table 1. Coefficient of variation of $V_n(a)$ for AR(1) process.

Sample size, n	Threshold, a	$\bar{\rho} = 0$	$\bar{\rho} = 0.7$	$\bar{\rho} = 0.9$
	1	0.6849	0.9630	1.4732
10	2	2.0538	2.6623	3.5038
	3	8.5965	9.8242	15.198
	1	0.3010	0.4332	0.7330
50	2	0.9167	1.2083	1.8952
	3	3.8441	4.4822	7.2857
	1	0.2124	0.2967	0.5134
100	2	0.6481	0.8704	1.4199
	3	2.7181	3.1055	5.1499

4 Translation model

Let $X(t)$, $t \geq 0$, be a real-valued stationary and ergodic stochastic process with unknown probability law; the procedures considered in Appendix A can be used for a special class of non-stationary, non-ergodic output. As in Section 3, the available information on X includes one sample, denoted by $(X_1, X_2, \ldots, X_n)$; prior knowledge on X may or may not be available. Our objective is to construct an approximation for X with known second-moment properties and marginal probability law in order to estimate system performance. The approximation is consistent with the available information on X and is based on a translation process model. Estimates based on translation processes are particularly useful if prior knowledge is available. For example, we may know that X has bounded range and/or its probability law belongs to a family of distribution functions.

4.1 Model definition

Let $X_{\mathrm{T}}(t)$, $t \geq 0$, be a translation process defined by

$$X_{\mathrm{T}}(t) = F^{-1} \circ \Phi[G(t)] = h[G(t)], \tag{20}$$

where F denotes an arbitrary distribution function, Φ denotes the CDF of a $N(0,1)$ random variable, and $G(t)$ denotes a mean-square differentiable stationary Gaussian process with zero mean, unit variance, and known correlation function. It can be shown that: (i) X_{T} is stationary in the strict sense and has marginal distribution F; and (ii) X_{T} has an a–upcrossing at time t if, and only if, G has an $h^{-1}(a)$–upcrossing at time t [11]. By (ii),

$$\nu(a) = \frac{\beta}{\sqrt{2\pi}}\, \phi\left[\Phi^{-1} \circ F(a)\right] \tag{21}$$

is the mean a–upcrossing rate of X_{T}, where ϕ denotes the PDF of a $N(0,1)$ random variable and β^2 denotes the variance of process $\dot{G}(t) = \mathrm{d}G(t)/\mathrm{d}t$.

4.2 Model calibration to available information

Let $F(x;\boldsymbol{\theta})$ denote the marginal CDF of X_T defined by Eq. (20), explicitly written as a function of $x \in \mathbb{R}$ with parameter vector $\boldsymbol{\theta}$. The selection of $F(x;\boldsymbol{\theta})$ is a two-step process:

1. Choose the functional form for F based on any prior knowledge about X; and
2. Given a functional form for F, use the available sample, $(X_1, \ldots, X_n)$, to estimate parameter vector, $\boldsymbol{\theta}$; let $\hat{\boldsymbol{\theta}}$ denote the estimate for $\boldsymbol{\theta}$.

For illustration, three specific cases are considered: (i) no prior knowledge of X is available; (ii) the range of X is known to be bounded; and (iii) the marginal distribution of X is known to be symmetric. We note that many problems of practical interest are consistent with cases (i)–(iii); methods of model selection, developed in [8] and [15], may be applied to more general types of prior knowledge.

4.2.1 No prior knowledge on X

With no prior knowledge about the probability law for X, we apply the standard statistical estimator developed in Section 3 so that $F = F_n$ as defined by Eq. (9). This approach can be useful if the sample size, n, is large, or the threshold a is small. However, for sufficiently large a and/or sufficiently small n such that $X_i < a$, $i = 1, \ldots, n$, we have $F_n(a) = 1$; in this case, the estimator defined by Eq. (9) may be of little practical use.

4.2.2 Range of X is known to be bounded

Suppose it is known that output X takes values in bounded interval $[d_1, d_2]$. This type of prior knowledge is common for systems subject to limiting constraints, *e.g.*, the displacement response of a SDOF oscillator constrained by two rigid barriers at locations d_1 and d_2. One appropriate model for this case is the distribution of a beta random variable, given by (see [2], pp. 129–133)

$$F(x; q, \lambda) = \frac{1}{B(q, \lambda)\,(d_2 - d_1)^{q+\lambda-1}} \int_{d_1}^{\bar{x}} (y - d_1)^{q-1}\,(d_2 - y)^{\lambda-1}\,\mathrm{d}y, \tag{22}$$

where $\boldsymbol{\theta} = (q, \lambda)^{\mathrm{T}}$ denote deterministic shaping parameters, $\bar{x} = (x - d_1)/(d_2 - d_1)$, and $B(q, \lambda) = \Gamma(q)\,\Gamma(\lambda)/\Gamma(q + \lambda)$ and $\Gamma(\,\cdot\,)$ denote the beta and gamma functions, respectively (see [1], Sections 6.1 and 6.2). The method of maximum likelihood can be used to calculate $\hat{\boldsymbol{\theta}} = (\hat{q}, \hat{\lambda})^{\mathrm{T}}$, an estimate for $\boldsymbol{\theta}$ based on the available data (see [16], Chapter 25).

The coefficients of skewness and kurtosis, denoted by $\gamma_3/\gamma_2^{3/2}$ and γ_4/γ_2^2 respectively and defined in Section 3.1, are illustrated by Fig. 4 for $q \in (0, 6]$ and $\lambda \in (0, 2]$. The broad range of values shown make the model defined by Eq. (22) very attractive for applications.

4.2.3 Marginal distribution of X is known to be symmetric

Suppose the distribution of X is known to be symmetric about $x = \mu$, but no information on the range of X is available. In this case, we require our model for F defined by Eq. (20)

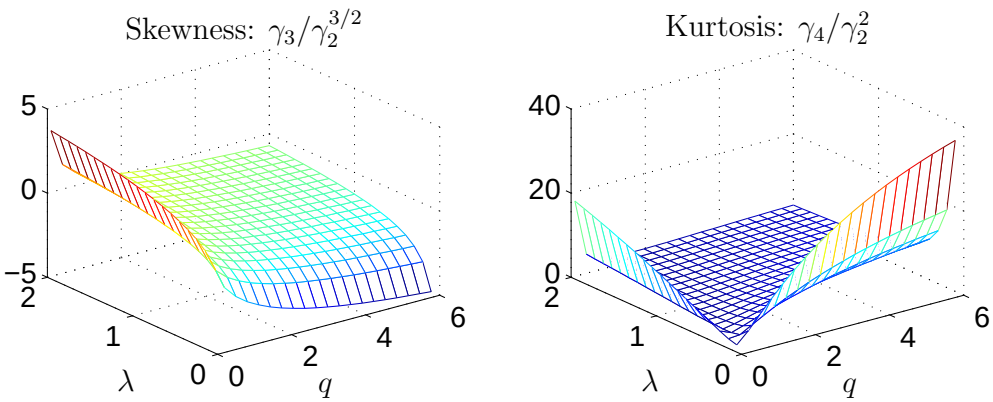

Figure 4. Relationship between coefficients of skewness and kurtosis and shape parameters q and λ for the beta distribution.

to have zero skewness and be defined over the entire real line. One appropriate model in this case is the distribution of a student-t random variable, given by (see [27], Section A.2)

$$F(x;\theta) = \frac{\Gamma(\frac{\theta+1}{2})}{\sqrt{\pi\,\theta}\,\Gamma(\frac{\theta}{2})} \int_{-\infty}^{x} \left[1 + \frac{1}{\theta}\,(y-\mu)^2\right]^{-(\theta+1)/2} \mathrm{d}y, \tag{23}$$

where μ denotes the mean of the distribution, and $\boldsymbol{\theta} = \theta > 0$ is a deterministic shape parameter. As before, the method of maximum likelihood can be applied to provide $\hat{\boldsymbol{\theta}} = \hat{\theta}$, an estimate for parameter θ (see [16], Chapter 28).

The coefficient of kurtosis of the student-t random variable as a function of parameter θ is illustrated by Fig. 5, demonstrating the flexibility of the model defined by Eq. (23). We note that as $\theta \to \infty$, F defined by Eq. (23) approaches the distribution of a $N(0,1)$ random variable.

4.3 Mean crossing rate

Let

$$G_i = \Phi^{-1} \circ F(X_i;\hat{\boldsymbol{\theta}}),\ i = 1,\ldots,n, \tag{24}$$

define a collection of random variables, where $F(x;\hat{\boldsymbol{\theta}})$ is the marginal CDF of X_{T} defined by Eq. (20) calibrated to the available information on X as described in Section 4.2, and

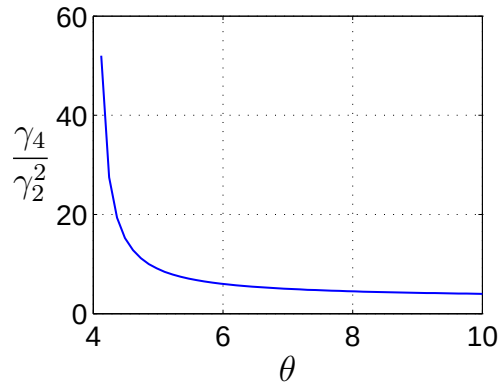

Figure 5. Relationship between coefficient of kurtosis and shape parameter θ for the student-t distribution.

Φ denotes the CDF of a $N(0,1)$ random variable. The collection $(G_1, \ldots, G_n)$ denotes the approximate Gaussian image of $(X_1, \ldots, X_n)$, the available sample of output $X(t)$. An approximation for the mean a–upcrossing rate of X_{T} defined by Eq. (21) is given by the following random variable

$$V_{\mathrm{T},n}(a) = \sqrt{\frac{\Gamma_{2,n}}{2\pi}}\, \phi\left[\Phi^{-1} \circ F(a; \hat{\boldsymbol{\theta}})\right], \tag{25}$$

where $\Gamma_{2,n}$ is an estimator for β^2, the variance of $\dot{G}(t)$ defined by Eq. (21); for calculations, we apply Eq. (6) to sequence $(\dot{G}_1, \ldots, \dot{G}_{n-2})$, where

$$\dot{G}_k = \frac{1}{2\,\Delta t}\left(G_{k+2} - G_k\right), \;\; k = 1, \ldots, n-2. \tag{26}$$

5 Applications

We next consider several applications to demonstrate the two methods for estimating mean outcrossing rates defined in Sections 3 and 4. The stationary, ergodic response of a linear and nonlinear oscillator are discussed in Sections 5.1 and 5.2, respectively. These first two applications provide benchmark tests for the proposed method since the distribution functions and mean crossing rates of the oscillator response are available analytically. We next consider the stationary, ergodic response of a nonlinear vibro-impact system in Section 5.3, where the system output is known to take values in a bounded interval; this system has been successfully used to model the dynamics of certain MEMS devices at Sandia. In Section 5.4, $X(t)$ denotes the non-stationary, non-ergodic vibration response of RB components during normal re-entry, where the available sample of X comes from either actual flight test data or outputs from a complex finite element model for the RB. The applications studied in Section 5.4 are constructed to be directly relevant to random vibration analysis at Sandia.

For each application, a single sample of the system output is assumed available, and prior knowledge on the output may or may not be known. We apply estimators $V_n(a)$ defined by Eq. (15) and $V_{\mathrm{T},n}(a)$ defined by Eq. (25); for the latter, we choose marginal CDF F defined by Eq. (20) to be consistent with any available prior knowledge on the system output. The coefficient of variation of each estimator is used to assess the accuracy of the methods.

5.1 Linear oscillator

Let X denote the stationary response of a linear single degree-of-freedom (SDOF) oscillator to W, a stationary, zero-mean Gaussian white noise with one-sided spectral density $1/\pi$. The oscillator has damping ratio $0 < \zeta < 1$, natural frequency $\omega_0 > 0$, and its displacement satisfies the following differential equation

$$\ddot{X}(t) + 2\,\zeta\,\omega_0\,\dot{X}(t) + \omega_0^2\,X(t) = W(t),\ t \geq 0, \tag{27}$$

with initial conditions $X(0) \sim N(0,\sigma)$ and $\dot{X}(0) \sim N(0,\omega_0\,\sigma)$, where $\sigma^2 = 1/(4\,\zeta\,\omega_0^3)$. Parameters $\zeta = 0.1$ and $\omega_0 = 1$ are used for calculations.

Because the input is Gaussian and the system is linear, output X is a stationary Gaussian process with zero mean, variance σ^2, and mean a–upcrossing rate (see [22], Sections 5.2.1 and 7.3.1)

$$\nu(a) = \frac{\omega_0}{2\pi}\,\exp\left(-\,\frac{a^2}{\sigma^2}\right). \tag{28}$$

We next apply methods developed in Sections 3 and 4 to estimate Eq. (28) under various types of available information on X.

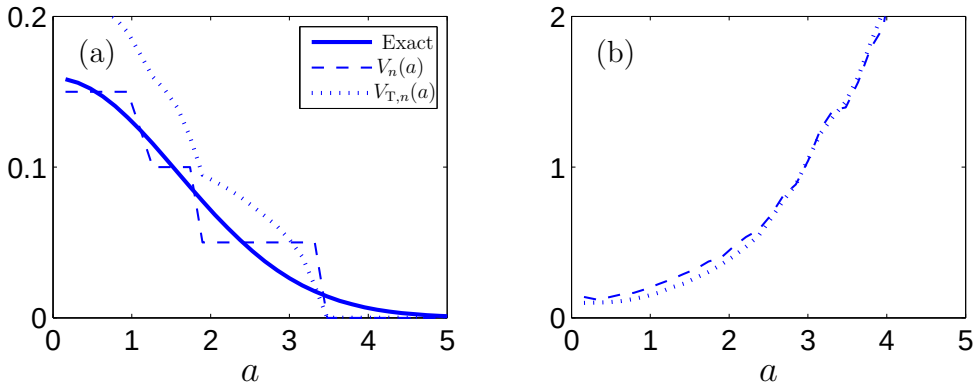

Figure 6. Estimates of $\nu(a)$ for the response of the linear oscillator with no prior knowledge and $n = 4{,}000$: (a) based on one sample of X, and (b) the coefficient of variation of each estimate.

5.1.1 No prior knowledge on X

We first assume information on X is limited to a single sample of length n with time step $\Delta t = 0.01$. The exact and approximate mean a–upcrossing rates of output $X(t)$ are illustrated by Fig. 6(a) for sample size $n = 4{,}000$ and threshold range $a \in (0, 5]$. Results indicate that $V_n(a)$ and $V_{\mathrm{T},n}(a)$ both provide poor estimates of the mean a–upcrossing rates of $X(t)$. Further, no information is provided by either method for $a > 3.5$.

Estimates of the coefficient of variation of both estimators are illustrated by Fig. 6(b); results from 100 independent Monte Carlo samples of length n were used for calculations. Estimates of $\mathrm{COV}\,[V_n(a)]$ and $\mathrm{COV}\,[V_{\mathrm{T},n}(a)]$ are nearly identical and increase with increasing threshold, a.

5.1.2 X is known to be Gaussian with unknown variance

We next assume that, in addition to one sample of X, it is known that output X is a stationary and ergodic Gaussian process with zero mean and unknown variance. In this case, we apply estimator $V_{\mathrm{T},n}(a)$ defined by Eq. (25) with marginal CDF F defined by Eq. (20) equal to the CDF of a $N(0, (\Gamma_{2,n})^{1/2})$ random variable, where $\Gamma_{2,n}$ denotes the sample variance of output X as defined by Eq. (6). The estimates of $\nu(a)$ by estimator $V_n(a)$ are identical to the results presented in Section 5.1.1.

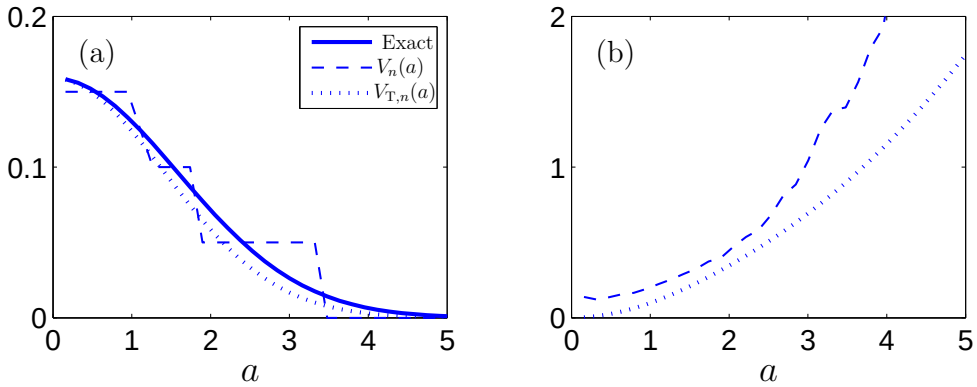

Figure 7. Estimates of $\nu(a)$ for the response of the linear oscillator with prior knowledge and $n = 4{,}000$: (a) based on one sample of X, and (b) the coefficient of variation of each estimate.

The exact and approximate mean a–upcrossing rates of $X(t)$ are illustrated by Fig. 7(a); the results using the calibrated translation model developed in Section 4 are clearly superior for all values for a considered. In addition, as demonstrated by Fig. 7(b), $\mathrm{COV}\left[V_{\mathrm{T},n}(a)\right] < \mathrm{COV}\left[V_n(a)\right]$ meaning that the estimate of the mean a–upcrossing rate of $X(t)$ defined by Eq. (25) is less sensitive to the particular sample used for calculations than is the estimate defined by Eq. (15).

Similar results are illustrated by Fig. 8 for the case of $n = 40{,}000$; the scenario demonstrates the benefit of a longer sample, *i.e.*, larger value for n. Results indicate significant improvement in the accuracy of estimator $V_n(a)$. The estimates of the mean a–upcrossing rate of $X(t)$ by both methods are nearly identical for all values for a considered, and the coefficients of variation of both $V_n(a)$ and $V_{\mathrm{T},n}(a)$ have decreased when compared to the results for $n = 4{,}000$ illustrated by Fig. 7.

In summary, either method provides adequate estimates for $\nu(a)$ when the available sample is long, *i.e.*, $n = 40{,}000$. However, estimator $V_n(a)$ is inadequate when the available sample is short, demonstrating that the additional information on the properties of $X(t)$ available in Section 5.1.2 is essential when dealing with short samples. Further it is observed that, in general, $\mathrm{COV}\left[V_{\mathrm{T},n}(a)\right] \leq \mathrm{COV}\left[V_n(a)\right]$, meaning that the estimate of the mean a–upcrossing rate of $X(t)$ defined by Eq. (25) is less sensitive to the particular sample used for calculations than is the estimate defined by Eq. (15).

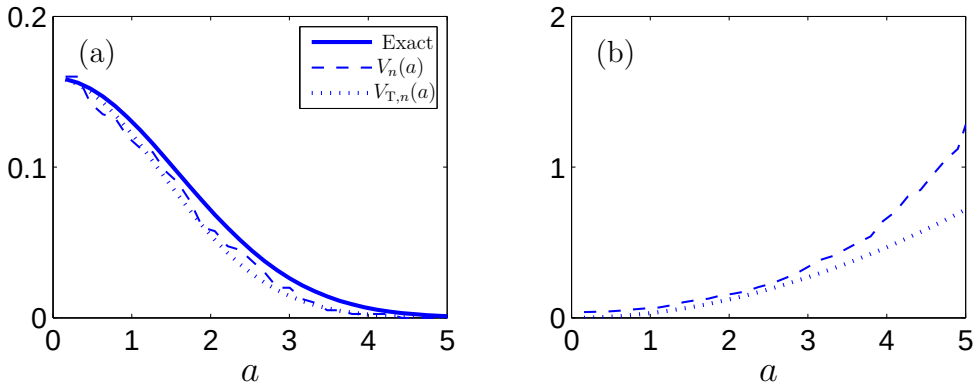

Figure 8. Estimates of $\nu(a)$ for the response of the linear oscillator with prior knowledge and $n = 40{,}000$: (a) based on one sample of X, and (b) the coefficient of variation of each estimate.

5.2 Duffing oscillator

Next let X denote the stationary response of a nonlinear SDOF oscillator to a stationary, Gaussian white noise with one-sided spectral density $1/\pi$. The oscillator displacement satisfies the following differential equation

$$\ddot{X}(t) + c\,\dot{X}(t) + \omega_0^2\,X(t)\left[1 + \epsilon\,X(t)^2\right] = W(t),\ t \geq 0, \tag{29}$$

where $c > 0$ denotes a damping coefficient, and constants $\omega_0 > 0$ and ϵ define the initial frequency and degree of nonlinearity of the oscillator, respectively. Parameters $c = 0.3$, $\omega_0 = 1$, and $\epsilon = 0.4$ are used for calculations.

The stationary joint density function of vector $(X(t),\, \dot{X}(t))$ has an analytical solution (see [22], p. 220) given by $f(x_1, x_2) = f_1(x_1)\, f_2(x_2)$, where

$$\begin{aligned} f_1(x_1) &= \sqrt{2\pi}\,\beta_0\,q\,\exp\left[-\frac{1}{2\,\sigma_0^2}\left(x_1^2 + \frac{\epsilon}{2}\,x_1^4\right)\right] \\ f_2(x_2) &= \frac{1}{\sqrt{2\pi}\beta_0}\,\exp\left[-\frac{1}{2\,\beta_0^2}\,x_2^2\right] \end{aligned} \tag{30}$$

constants

$$\sigma_0^2 = \frac{1}{2\,c\,\omega_0^2}, \quad \beta_0^2 = \omega_0^2\,\sigma_0^2 = \frac{1}{c}, \quad q^{-1} = \frac{\pi}{\sqrt{2\pi\,c\,\epsilon}}\,\exp\left[\frac{1}{8\,\epsilon\,\sigma_0^2}\,K_{1/4}\left(\frac{1}{8\,\epsilon\,\sigma_0^2}\right)\right], \tag{31}$$

and $K_{1/4}(\cdot)$ denotes the modified Bessel function of order 1/4. The functional form of $f(x_1, x_2)$ demonstrates that, at stationarity: (i) stochastic processes $X(t)$ and $\dot{X}(t)$ are independent and symmetric about $x = 0$; and (ii) $X(t)$ and $\dot{X}(t)$ are non-Gaussian and Gaussian, respectively.

Let $a > 0$ denote a critical threshold for $X(t)$. The mean a–upcrossing rate of $X(t)$, at stationarity, is

$$\nu(a) = \frac{1}{2\sqrt{c\,\pi}}\, f_1(a), \tag{32}$$

where f_1 is defined by Eq. (30). Because $f_1(x)$ is symmetric about $x = 0$, the mean $(-a, a)$–outcrossing rate of $X(t)$ is equal to $2\,\nu(a)$, where $\nu(a)$ is defined by Eq. (32).

5.2.1 No prior knowledge on X

We first assume information on X is limited to a single sample of length $n = 4{,}000$ with time step $\Delta t = 0.01$. The exact and approximate mean $(-a, a)$–outcrossing rates of $X(t)$ are illustrated by Fig. 9(a) for threshold range $a \in (0, 3]$. Results indicate that $V_n(a)$ and $V_{\mathrm{T},n}(a)$ both give poor estimates of the mean $(-a, a)$–outcrossing rates of $X(t)$. Further, no information is provided by either method for $a > 2.5$. Estimates of the coefficient of variation of both estimators are illustrated by Fig. 9(b); results from 100 independent Monte Carlo samples of length n were used for calculations. Estimates of $\mathrm{COV}\,[V_n(a)]$ and $\mathrm{COV}\,[V_{\mathrm{T},n}(a)]$ are nearly identical and increase with increasing threshold, a.

5.2.2 Marginal distribution of X is known to be symmetric

We next assume that, in addition to the one sample of X, the distribution of X is known to be symmetric about $x = 0$. In this case, we apply estimator $V_{\mathrm{T},n}(a)$ defined by Eq. (25) with marginal CDF F defined by Eq. (20) equal to the student-t distribution defined by Eq. (23). The estimates of $\nu(a)$ by estimator $V_n(a)$ are identical to the results presented in Section 5.2.1.

The exact and approximate mean $(-a, a)$–outcrossing rates of $X(t)$ are illustrated by Fig. 10(a); the results using the calibrated translation model developed in Section 4 are clearly superior for all values for a considered. In addition, as demonstrated by Fig. 10(b), $\mathrm{COV}\,[V_{\mathrm{T},n}(a)] \leq \mathrm{COV}\,[V_n(a)]$ meaning that the estimate of the mean $(-a, a)$–outcrossing rate of $X(t)$ defined by Eq. (25) is less sensitive to the particular sample used for calculations than is the estimate defined by Eq. (15).

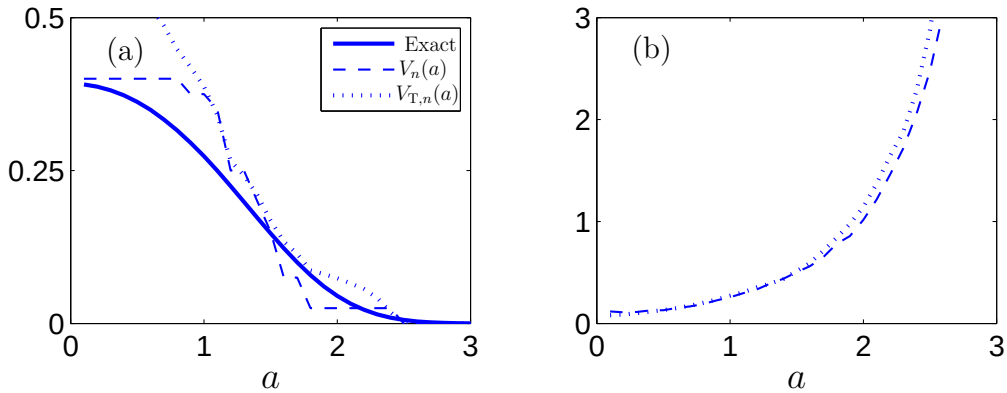

Figure 9. Estimates of $2\nu(a)$ for the response of the Duffing oscillator with no prior knowledge: (a) based on one sample of X, and (b) the coefficient of variation of each estimate.

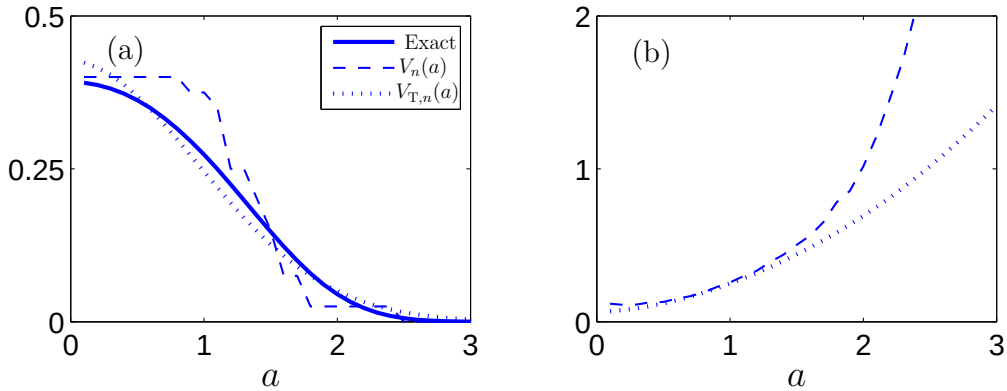

Figure 10. Estimates of $2\nu(a)$ for the response of the Duffing oscillator with prior knowledge: (a) based on one sample of X, and (b) the coefficient of variation of each estimate.

5.3 Vibro-impact system

We next consider the response of a SDOF oscillator constrained by two rigid barriers; this type of system is often referred to in the literature as a vibro-impact system [7]. Let X denote the stationary response of a vibro-impact system to a stationary Gaussian white noise with zero mean and one-sided spectral density $1/\pi$; the driving noise is denoted by W. We consider the case of two perfectly rigid barriers located at $X = \pm d$, where $d > 0$ is a known constant, and assume $|X(0)| < d$.

In between impact events, *i.e.*, when $|X(t)| < d$, the oscillator behaves in a linear fashion; in this case, X satisfies the following differential equation

$$\ddot{X}(t) + 2\,\zeta\,\omega_0\,\dot{X}(t) + \omega_0^2\,X(t) = W(t) - \mu,\ \ t \geq 0, \tag{33}$$

where $0 < \zeta < 1$ and $\omega_0 > 0$ denote the damping ratio and natural frequency of the oscillator, respectively, and $\mu \geq 0$ is a prescribed parameter. For the initial conditions to Eq. (33), we ignore inertial and damping effects and solve Eq. (33) at $t = 0$, *i.e.*,

$$\Big(X(0),\, \dot{X}(0)\Big) = \left(\max\left\{-\frac{\mu}{\omega_0^2},\, -d\right\},\, 0\right). \tag{34}$$

By Eq. (34), cases $X(0) = -\mu/\omega_0^2$ and $X(0) = -d$ correspond to when the oscillator is initially free of the barrier at $-d$ and in contact with the barrier at $-d$, respectively. Hence, we use parameter μ to control the initial position of the oscillator.

To model the effects of the rigid barriers, suppose the first impact with either barrier occurs at time $t' > 0$; the following three-step procedure can be used to generate one sample of $X(t)$, for $0 \leq t \leq \tau$:

1. Solve Eq. (33) with initial conditions defined by Eq. (34) over interval $[0, t')$;
2. Solve Eq. (33) with initial conditions $(d,\, -\eta\,\dot{X}(t^-))$ over interval $(t', \min\{t'',\, \tau\})$, where $t^- < t'$ denotes the time just prior to the first impact event, t'' denotes the time of the second impact event, and $0 < \eta < 1$ is the deterministic coefficient of restitution; and
3. Repeat step 2 until time τ is achieved.

The vibro-impact system described has been used to model, for example, the dynamic response of a micro-electro-mechanical system (MEMS) switch to random excitation [10]. For this application, the switch consists of a sensor mass suspended between two voltage sources by a series of flexible supports. The time-varying location of the sensor mass is modeled by $X(t)$, parameters ζ and ω_0 represent the effects of the flexible supports, and the

voltage sources are assumed rigid and located at $X = \pm d$. If the sensor mass travels near enough to either voltage source, *i.e.*, if $|X(t)| > a$ where $a = d - \epsilon$ and $\epsilon > 0$ is a small prescribed constant, the switch is assumed closed at time t; the switch is open otherwise. A controlled static load is also applied to the switch, modeled by parameter μ. We select a value for μ based on whether or not we want the switch to close; for $\mu \approx 0$ and $\mu \gg 0$, it is desirable for the switch to remain open and closed, respectively.

With this application in mind, we consider two cases. For Case 1, $\mu = 0$ and an appropriate measure on the probability of failure is given by

$$p_{\mathrm{F},1}(\tau) = 1 - P(|X(t)| \leq a,\ 0 \leq t \leq \tau, \mid \mu = 0). \tag{35}$$

By Eq. (35), failure occurs if the sensor mass travels to within a distance less than ϵ from either voltage source at any time during $t \in [0, \tau]$. For Case 1, we are therefore interested in estimates for the mean $(-a, a)$–outcrossing rate of X. For Case 2, $\mu = 25$ meaning that it is desirable for the switch to remain closed such that the sensor mass stays in contact with the voltage source located at $-d$. An appropriate measure on the probability of failure for the switch in this case is given by

$$p_{\mathrm{F},2}(\tau) = 1 - P(X(t) \leq -a,\ 0 \leq t \leq \tau \mid \mu = 25), \tag{36}$$

meaning that failure occurs if the sensor mass travels a distance more than ϵ away from the voltage source located at $-d$ at any time during $t \in [0, \tau]$. For Case 2, we are therefore interested in estimates for the mean $-a$–upcrossing rate of X.

For Cases 1 and 2, the available information on X is given by: (i) one sample of X, denoted by $(X_1, \ldots, X_n)$; and (ii) prior knowledge that X takes values in $[-d, d]$ almost surely. The available sample of X for Cases 1 and 2 are illustrated by Figs. 11 and 12, respectively, for sample size $n = 1{,}000$; parameters $d = 0.2$, $\zeta = 0.1$, $\eta = 0.9$, $\tau = 10$, $\Delta t = 0.01$, and $\omega_0 = 10$ are used for calculations. Thresholds assuming $a = 0.15$ are also shown and denoted by dashed lines. We note that numerous impacts between the oscillator and barrier at $-d = -0.2$ for Case 2 are clearly evident.

We next apply estimators $V_n(a)$ defined by Eq. (15) and $V_{\mathrm{T},n}(a)$ defined by Eq. (25) to provide estimates for the mean outcrossing rate of X for Cases 1 and 2. For $V_{\mathrm{T},n}(a)$, we set marginal CDF F defined by Eq. (20) equal to the beta distribution defined by Eq. (22) to be consistent with the prior knowledge on X.

5.3.1 Case 1

The approximate mean $(-a, a)$–outcrossing rates of $X(t)$ by estimators $V_n(a)$ and $V_{\mathrm{T},n}(a)$ are illustrated by Fig. 13(a) for threshold range $a \in [0.05, 2)$. Also shown is the mean $(-a, a)$–outcrossing rate of $X(t)$ based on one sample of X computed for 1,000,000 time steps; we

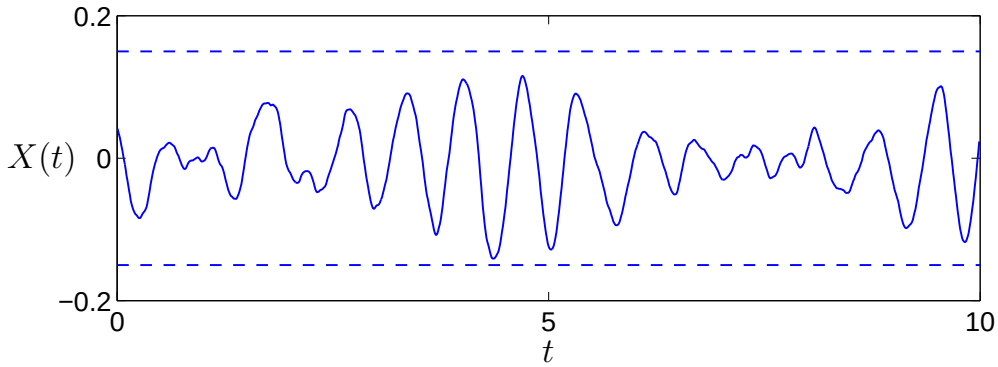

Figure 11. Available sample of $X(t)$ for Case 1.

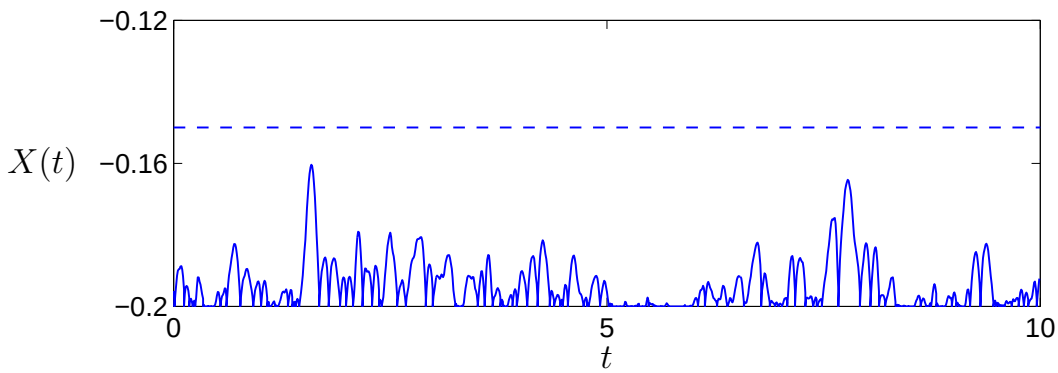

Figure 12. Available sample of $X(t)$ for Case 2.

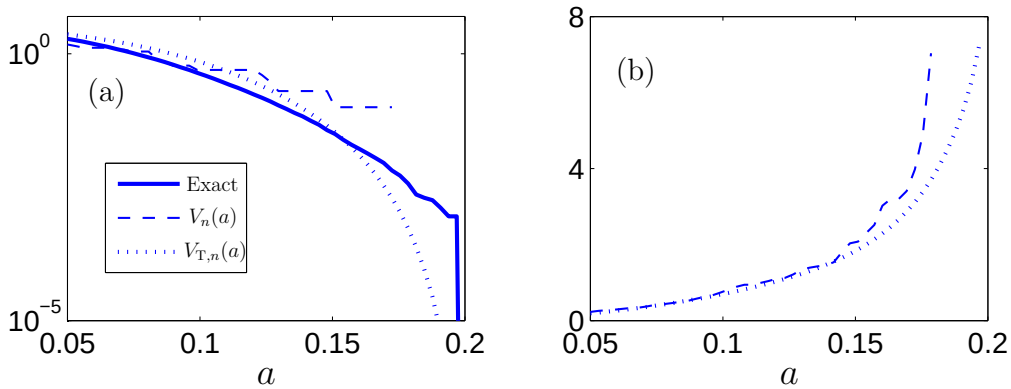

Figure 13. Estimates of the mean $(-a, a)$–outcrossing rate of the response of the vibro-impact system for Case 1: (a) based on one sample of X, and (b) the coefficient of variation of each estimate.

refer to this result as "Exact." The estimates nearly coincide for $a \leq 0.12$. For $a > 0.12$, no information is provided by estimator $V_n(a)$, while an approximation for the mean $(-a, a)$–outcrossing rate of X is provided by estimator $V_{\mathrm{T},n}(a)$ for all thresholds considered. We note $V_{\mathrm{T},n}(a) \to 0$ as $a \to d$ since $F(d) = 1$ (see Eq. (25)); the performance of $V_{\mathrm{T},n}(a)$ can therefore be poor for thresholds very near the barrier. Estimates of the coefficient of variation of both estimators are illustrated by Fig. 13(b); results from 100 Monte Carlo samples of length n were used for calculations.

5.3.2 Case 2

The approximate mean $-a$–outcrossing rates of $X(t)$ by estimators $V_n(a)$ and $V_{\mathrm{T},n}(a)$ are illustrated by Fig. 14(a) for threshold range $-a \in (-0.2, -0.1]$. Also shown is the mean $-a$–outcrossing rate of $X(t)$ based on one sample of X computed for 1,000,000 time steps; we refer to this results as "Exact." By Fig. 14(a), the estimate of $\nu(a)$ by $V_n(a)$ becomes unstable for $a > -0.15$. Estimates of $\mathrm{COV}\,[V_n(a)]$ and $\mathrm{COV}\,[V_{\mathrm{T},n}(a)]$ are illustrated by Fig. 13(b) quantifying the sensitivity of both methods to the particular sample used for calculations. Results from 100 Monte Carlo samples of length n were used to calculate values for $\mathrm{COV}\,[V_n(a)]$ and $\mathrm{COV}\,[V_{\mathrm{T},n}(a)]$.

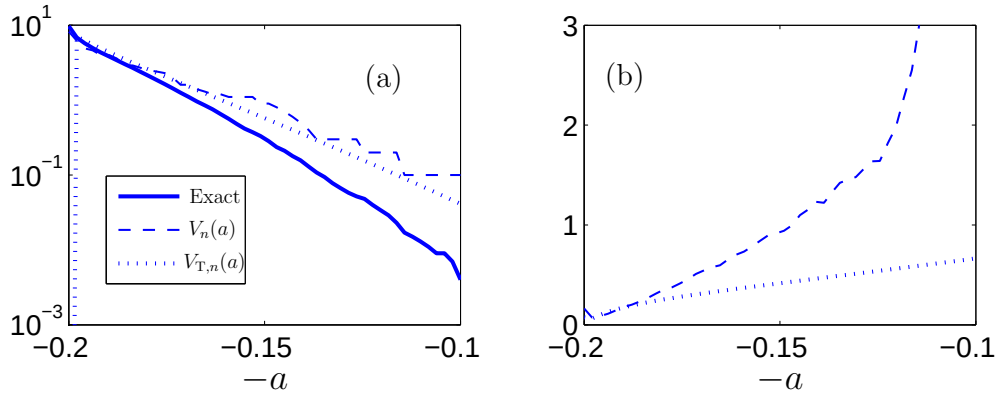

Figure 14. Estimates of the mean $-a$–upcrossing rate of the response of the vibro-impact system for Case 2: (a) based on one sample of X, and (b) the coefficient of variation of each estimate.

5.4 RB component vibration response

Finally, let X denote the vibration response of an RB weapon component during normal re-entry. We will consider two scenarios, discussed in Sections 5.4.1 and 5.4.2, respectively. Both scenarios are constructed to be directly relevant to structural dynamics applications here at Sandia.

For scenario 1, X represents measured accelerometer data from a single re-entry flight test of the W76, EFIA-1 (FCET-32). To avoid sensitivity issues, a uniform normalizing factor has been applied to the data. For scenario 2, X is the output from a high-fidelity finite element (FE) model simulation of a generic RB during normal re-entry, where a random pressure field applied to the outside surface of the aeroshell provides the dynamic excitation to the system. All calculations for scenario 2 were performed using Salinas, and because of budget and/or time constraints, we assume that only a single run of the FE code is possible, meaning that one sample of stochastic output X is made available. Further details of both the FE model and the random pressure field model for the re-entry environment are provided in [9].

In both scenarios considered, a single sample of output X is available, and prior knowledge about the underlying physical phenomena suggests that X should come from a distribution that is symmetric about its mean, and is not necessarily bounded. Further, we assume that the weapon component fails when output X exceeds threshold a, a critical level of acceleration. Our objective is to use the sample and prior knowledge on the system output

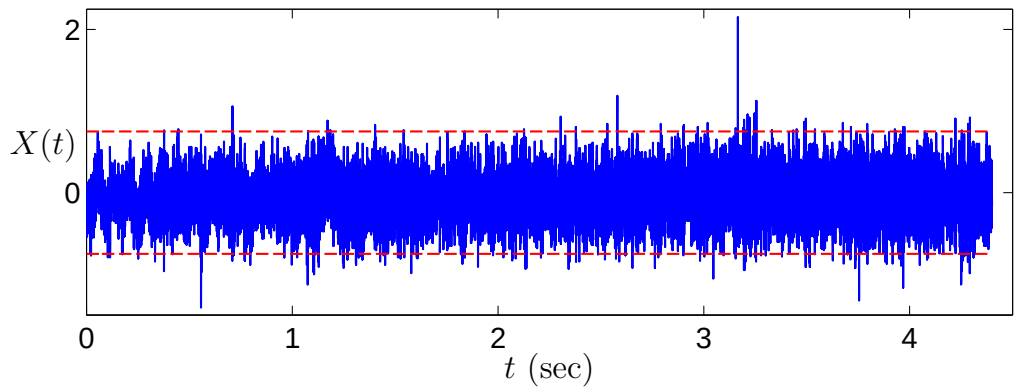

Figure 15. Measured (normalized) response from W76 EFIA-1 (FCET-32), $X(t)$, with constant thresholds $\pm a$, for $a = 0.75$.

to estimate the probability that X will exceed threshold a during the re-entry event; these estimates will provide information on the time-dependent reliability of the RB component. Our approach is to apply estimators $V_n(a)$ defined by Eq. (15) and $V_{\mathrm{T},n}(a)$ defined by Eq. (25) to approximate the mean $(-a, a)$–outcrossing rate of X. For the latter, we set marginal CDF F defined by Eq. (20) equal to the student-t distribution defined by Eq. (23) to be consistent with the prior knowledge on X.

5.4.1 Scenario 1: flight data

The available sample of X, *i.e.*, the measured flight data, is illustrated by Fig. 15 where, as mentioned, a uniform normalizing factor has been applied. The sample has length $n = 22{,}000$ and time step $\Delta t = 0.0002$ sec. Several extreme values can be observed in the data, *e.g.*, near $t = 0.5$ sec and $t = 3.1$ sec. Expert opinion suggests that these extreme values are valid data points and should not be considered "drop outs." Further, we note that the coefficient of kurtosis of X is approximately 3.3 (see Section 3.1), suggesting that the measured vibration data is slightly non-Gaussian. The dashed lines shown in Fig. 15 represent thresholds $\pm a$ for the case of $a = 0.75$.

Upon close inspection of Fig. 15, the sample variance of X slowly varies in time, meaning that X should not be assumed stationary; we therefore apply techniques from Appendix A to properly scale the available data. We apply Eqs. (37) and (38) with window size $w_1 = 250$ assuming X can be represented by $k = 1$ segment. The scaled flight data, denoted by $(Z_1^{(1)}, \ldots, Z_{n-2\,w_1}^{(1)})$, is illustrated by Fig. 16. The thresholds at $X = \pm 0.75$, scaled by Eq. (40), are also shown.

Estimates for the mean $(-a, a)$–outcrossing rate of X using $V_n(a)$ defined by Eq. (15) and $V_{\mathrm{T},n}(a)$ defined by Eq. (25) are illustrated by Fig. 17 for thresholds $0 < a \leq 2$. For comparison, results are also shown assuming the measured data to be Gaussian, *i.e.*, Eq. (13) applied to $Z^{(1)}$; the latter approach is common in industry when processing experimental data. The three methods considered give similar results for $a \leq 1$. For $a > 1$, estimates based on $V_n(a)$ become unreliable; this is consistent with Fig. 15 where few outcrossings are observed for $a > 1$. Estimates based on $V_{\mathrm{T},n}(a)$ and the Gaussian model differ for $a > 1$, and this difference increases with increasing threshold, a. This difference occurs because the flight data is not Gaussian as evidenced by the extreme values and a coefficient of kurtosis greater than 3. Estimator $V_{\mathrm{T},n}(a)$ is preferable in this case since, by Eq. (4), it provides conservative estimates of system reliability when compared to the Gaussian model.

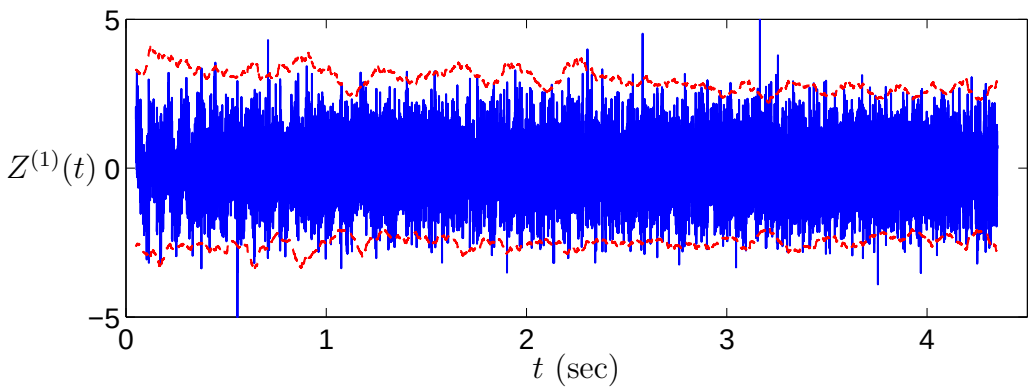

Figure 16. Scaled flight data, $Z^{(1)}(t)$, with scaled time-varying threshold, $\alpha(t)$.

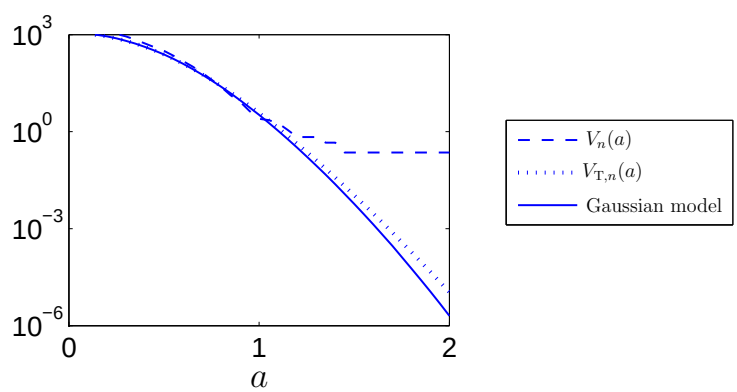

Figure 17. Estimates of the mean $(-a, a)$–outcrossing rate of measured flight data, $X(t)$.

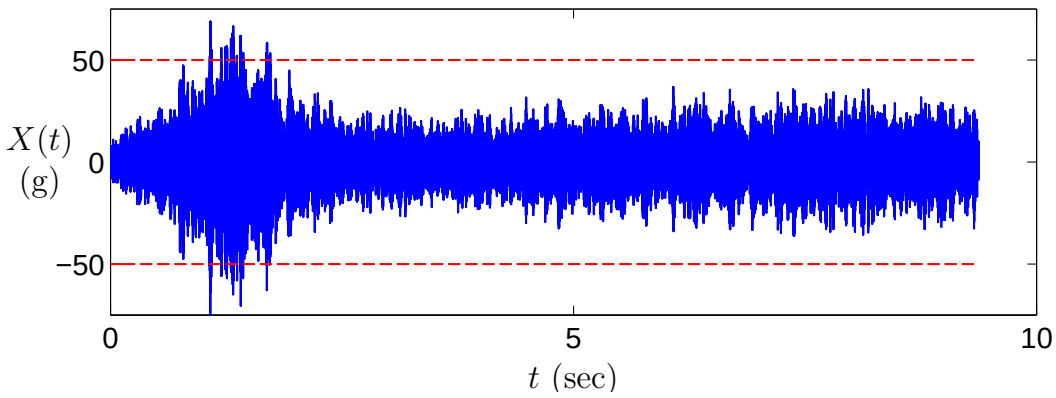

Figure 18. Output from Salinas FE model, $X(t)$, with constant threshold, $a = 50$ g (taken from [9]).

5.4.2 Scenario 2: FE model output

Next let X be a prediction of component vibration response, *i.e.*, the output from a single run of a Salinas FE model simulation of an RB during normal re-entry. For calculations, the FE model is linear but, as demonstrated by the results in Sections 5.2 and 5.3, the proposed method can be applied to nonlinear models for the RB as well. The available sample of X is illustrated by Fig. 18; the sample has length $n = 75{,}000$ and time step $\Delta t = 1.25 \times 10^{-4}$ sec. The dashed lines shown in Fig. 18 represent thresholds $X = \pm a$ for the case of $a = 50$ g. For calculations, X is taken from [9].

By inspection of Fig. 18, the sample variance of X is time-varying, meaning that X cannot be assumed stationary. As in Section 5.4.1, we apply techniques from Appendix A to properly scale the available data. The scaled output, denoted by $(Z_1^{(1)}, \ldots, Z_{n-2\,w_1}^{(1)})$, is illustrated by Fig. 19; the thresholds at $X = \pm 50$ g, scaled by Eq. (40), are also shown.

The proposed method, *i.e.*, estimator $V_{\mathrm{T},n}(a)$ defined by Eq. (25), is applied to approximate the mean $(-a, a)$–outcrossing rate of X. These results can then used to provide estimates for the time-dependent probability of failure of the RB component, *i.e.*, estimates for Eq. (2) with safe set $D = (-a, a)$. Calculation of these probabilities follow from Eq. (39).

Estimates for the time-dependent probability of failure of the RB component are illustrated by Fig. 20 for thresholds 25 g $\leq a \leq$ 100 g and times $0 \leq \tau \leq 10$ sec. In general, the results demonstrate that the probability of failure, denoted by p_{F}, increases with decreasing threshold a and increasing time, τ. For example, with threshold $a = 90$ g, the probability of failure during the first $\tau = 4$ sec of the re-entry event is $p_{\mathrm{F}} = 0.2$. If $a = 90$ g and $\tau = 7$ sec,

the probability of failure increases to $p_F = 0.8$. In summary, assuming we have a high level of confidence in the models for the RB and re-entry environment, the results summarized by Fig. 20 demonstrate how one might accomplish model-based qualification of RB components for re-entry random vibration given one or more outputs from a FE model.

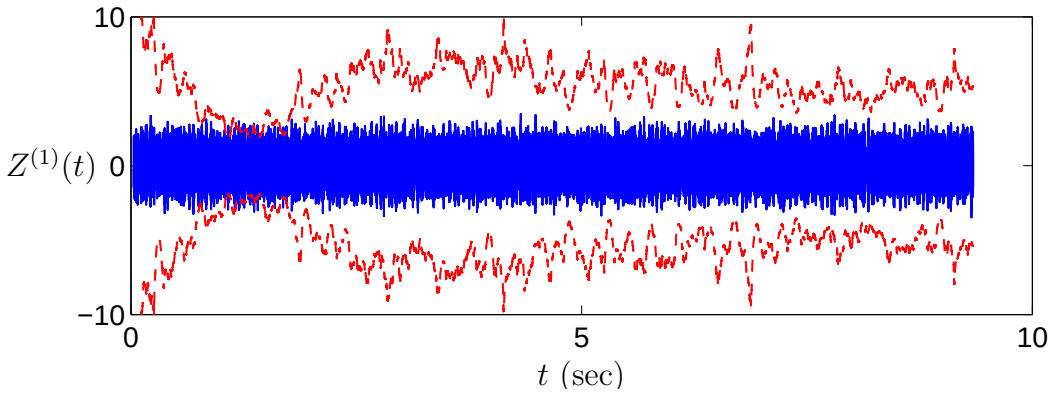

Figure 19. Scaled model output, $Z^{(1)}(t)$, with scaled time-varying threshold, $\alpha(t)$.

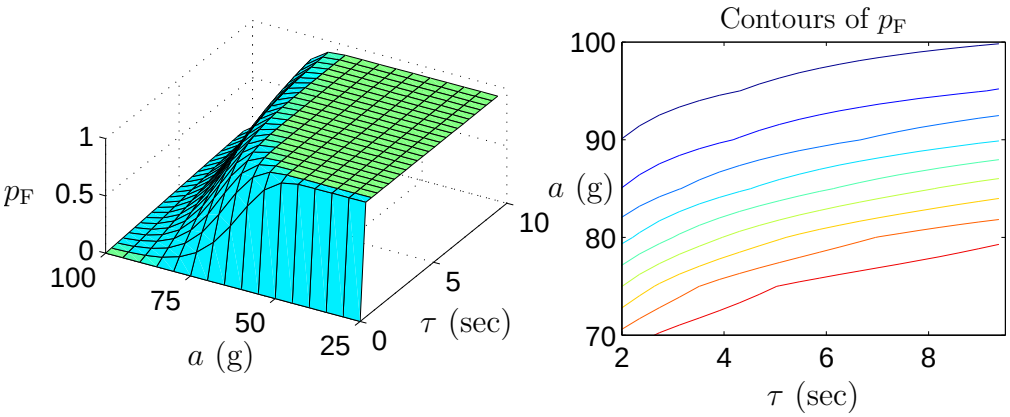

Figure 20. Estimates of time-dependent probability of component failure, p_F, as a function of threshold, a, and time, τ, based on FE model output.

6 Conclusions

A method has been developed for reliability analysis of dynamic systems under limited information. The available information included one or more samples of the system output; any known information on properties of the output was used when available. The proposed method was based on the theory of non-Gaussian translation processes, where the type of translation considered was consistent with any prior knowledge on the system output. This feature of the method made it particularly useful for problems of practical interest. For illustration, we applied the proposed method to a collection of simple examples and compared with the results by traditional statistical estimators in order to assess the accuracy of the proposed method. These studies demonstrated that the method could be applied for the case of analytical or experimental models for linear or nonlinear systems. The proposed method was then applied to two complex applications of direct interest to Sandia. First, we applied the method to assess design reliability of a MEMS inertial switch. Second, we considered RB component vibration response during normal re-entry, where the objective was to estimate the time-dependent probability of component failure. This last application was directly relevant to re-entry random vibration analysis at Sandia and may provide insights on test-based and/or model-based qualification of weapon components for random vibration environments.

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A Class of non-stationary, non-ergodic output

We consider the special class of non-stationary, non-ergodic output that can be expressed as a collection of non-overlapping segments where, under proper scaling, each scaled segment can be viewed as stationary and ergodic.

Let $X(t)$, $t \in [0, \tau)$, be an output process of interest expressed as a collection of $m \geq 1$ non-overlapping segments, denoted by $\{X^{(k)}(t),\ k = 1, \ldots, m\}$. Segment $X^{(k)}(t)$ is defined on time interval $[\tau_k, \tau_{k+1})$, $k = 1, \ldots, m$, where $0 = \tau_1 < \tau_2 < \cdots < \tau_{m+1} = \tau$ provides a partition of $[0, \tau)$. We assume scaling functions $\mu_k(t)$ and $\sigma_k(t) > 0$ exist such that, for all $k = 1, \ldots, m$,

$$Z^{(k)}(t) = \frac{X^{(k)}(t) - \mu_k(t)}{\sigma_k(t)},\ t \in [\tau_k, \tau_{k+1}), \tag{37}$$

is a stationary and ergodic process, *i.e.*, Eq. (5) is true when $X(t)$ is replaced by $Z^{(k)}(t)$. Let $(X_1^{(k)} = X^{(k)}(t_1), X_2^{(k)} = X^{(k)}(t_2), \ldots, X_{n_k}^{(k)} = X^{(k)}(t_{n_k}))$ denote the available sample of process $X^{(k)}(t)$, where $\tau_k = t_1 < \cdots < t_{n_k} = \tau_{k+1}$ denotes a partition of time interval $[\tau_k, \tau_{k+1})$. Quantities

$$\mu_i^{(k)} = \frac{1}{2w_k} \sum_{j=i-w_k}^{i+w_k} X_j^{(k)} \quad \text{and}$$

$$\sigma_i^{(k)} = \left[\frac{1}{2w_k} \sum_{j=i-w_k}^{i+w_k} \left(X_j^{(k)} - \mu_i^{(k)} \right)^2 \right]^{1/2} \tag{38}$$

where w_k is a positive integer for each $k = 1, \ldots, n$, provide estimates for $\mu_k(t_i)$ and $\sigma_k(t_i)$, respectively, where $t_i \in [\tau_k, \tau_{k+1})$, $i = 1, \ldots, n_k$.

Under the assumption that each $Z^{(k)}(t)$ is a stationary, ergodic process, the approaches assuming stationary, ergodic output presented in Sections 3 and 4 can be used to estimate the mean crossing rates for each $X^{(k)}(t)$ and, therefore, estimates of system performance. For example, our estimate for system reliability as defined by Eq. (2) is

$$p_{\mathrm{F}}(\tau) \approx 1 - \exp\left[- \left(\sum_{k=1}^{m} \nu(\alpha_k(t))\ \frac{s_k}{\tau} \right) \tau \right], \tag{39}$$

where

$$\alpha_k(t) = \frac{a - \mu_k(t)}{\sigma_k(t)},\ k = 1, \ldots, m, \tag{40}$$

$\nu(\alpha_k(t))$ denotes the $\alpha_k(t)$–upcrossing rate of $Z^{(k)}(t)$ at time $t \in [\tau_k, \tau_{k+1})$, and $s_k = \tau_{k+1} - \tau_k$, $k = 1, \ldots, m$.

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