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Extreme Inputs/Outputs for Multiple Input Miltiple Output Linear Systems

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Abstract

A linear structure is excited at multiple points with a stationary normal random process. The response of the structure is measured at multiple outputs. If the auto spectral densities of the inputs are specified, the phase relationships between the inputs are derived that will minimize or maximize the trace of the auto spectral density matrix of the outputs. If the autospectral densities of the outputs are specified, the phase relationships between the outputs that will minimize or maximize the trace of the input auto spectral density matrix are derived. It is shown that other phase relationships and ordinary coherence less than one will result in a trace intermediate between these extremes. Least favorable response and some classes of critical response are special cases of the development. It is shown that the derivation for stationary random waveforms can also be applied to nonstationary random, transients, and deterministic waveforms.

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1. Introduction

The application of multiple inputs, in both test and analysis, to excite a single structure is becoming more common. In general a stationary random excitation of this nature requires the specification of the complete cross spectral density matrix of the inputs. If the inputs are transients the specification of the complex Fourier spectra is required. However, many times little is known about the exact nature of the inputs. Often, the only known information is the desired autospectra (or Fourier magnitude for transients) at the inputs or at the response points.

For example, the response of a structure at one or several points has been measured in field experiments and the desire is to replicate the response in a laboratory test. Or in the case of analysis, to excite a model of the system that will replicate the field measured response. In general there are multiple solutions to the problem. This paper will explore a specific class of solutions.

In other cases the autospectral densities at several inputs are specified. The inputs can be measured experimental inputs or envelopes derived from a variety of sources. Inconsistencies in these specifications can arise from a variety of sources. The source could be noisy. The source could be a composite of several experiments. The test item or model may differ from the system from which the source was derived resulting in shifting of modal frequencies and mode shapes. In any of these cases the specification of the phase relationships between the inputs is difficult or impossible. Certain phase relations may result in unrealistic inputs or responses. These inputs in essence impose boundary conditions on the model or experiment to force the desired response.

A simple example is a rod excited in the axial direction with forces at the two ends. Intuition tells us that the bar must move as a rigid body at very low frequencies. If the required motion at two response points is not identical, large input forces can result, because higher modes must also be excited to achieve the desired response. As the frequency increases excitation of the higher modes becomes easier and the achievement of different motion at two points becomes easier as a combination of the rigid body translation and higher modes. At low frequencies the rigid body mode is excited most efficiently with the forces at the end in phase. Near the first longitudinal mode the bar is excited most efficiently with forces at the two ends out of phase. The forces required to achieve a given response is influenced by the crossover frequency where the excitation is changed from in-phase to out-of-phase.

This paper attempts to find a rational solution to minimize the forces required to achieve a specified response. For the case of specified response auto spectra density, the solution suggested is the solution that minimizes the trace of the input auto spectral density matrix. The solution that maximizes the trace of the input auto spectra is also derived. The minimum trace solution is a reasonable excitation because it excites the structure in a manner consistent with the dynamic properties of the structure and does not require unreasonable boundary conditions for the structure under test or analysis. It is shown that

any other solution will result in a trace between the minimum and maximum solution. The problem has been solved for a square system (the number of inputs equals the number of response points) [Smallwood, 1986]. This paper will generalize the results and extend the results to rectangular systems.

The case of the transient input with a single input and a single output (SISO) has been solved [Drenick, 1970 and Shinozuka, 1970] and is called the least favorable response. Other cases with single inputs and responses have also been studied are called the critical response [Lopez et al, 2000 and 2001, Baratta et al, 1998, Takewaki, 2001].

In the examples of this paper the input is force and the output is acceleration. But it is important to understand that the method is much more general. For example, the input could be voltage into a power amplifier driving a shaker. The output could be stress or strain. Any pair of quantities for which linear input/output relationships can be derived or measured can be used.

Several related problems involving a multiple-input, multiple-output (MIMO) system are discussed in this paper. In general these problems fall into three classes: 1) the number of inputs exceeds the number of outputs, 2) the number of inputs is the same as the number of inputs, and 3) the number of outputs exceeds the number of inputs. All are discussed.

First class of problems discussed is where the auto spectral densities of the outputs are specified, but the output cross spectral densities are free variables. The problem is to derive a set of inputs that will produce the required output auto spectral densities. The solution is not unique. The solution derived for the output cross spectral densities minimizes (or maximizes) the sum of the input auto spectral densities (the trace of the input cross spectral density matrix). Other solutions are shown to be intermediate to these two extremes. A geometric interpretation of the solution is given.

The second related class of problems where the input auto spectral densities are specified, but where the input cross spectral densities are free variables. Similar to the first problem, the solution derived generates the input cross spectral densities that will minimize (or maximize) the sum of the output auto spectral densities.

It is then shown how the random solution can be applied to other related problems: 1) a transient input, 2) the method of least favorable (or critical response), 3) a pseudo random input, 4) a deterministic input.

Two special cases of a multiple input single output (MISO) systems are then discussed. The first is the case where the single output is a quadratic combination of linear outputs that can be decomposed into a linear MIMO system using Cholesky decomposition. Von Mises stress is an example of such a single output. The other case discussed is a method to evaluate the equivalence between three single axis vibration tests and a single multiple input vibration test.

The last section includes many examples to illustrate the solutions derived.

2. Derivation of the Minimum/Maximum Input Trace

As a practical matter most of the derivations are in the frequency domain. It is assumed that the inputs and responses are sampled at equal time intervals and the random spectra are specified at finite number of evenly spaced frequency lines between zero and half the sampling frequency (the Nyquist frequency). For transients, the spectra are specified using the discrete Fourier transform as an approximation of the continuous transform.

A general multiple input multiple output (MIMO) linear system is represented in Figure 2.1.

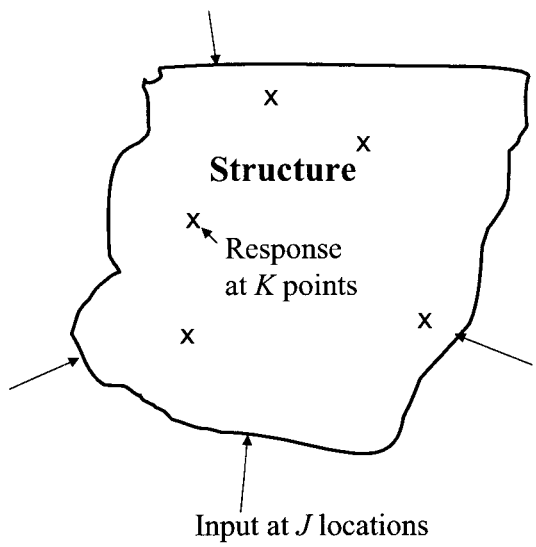

Figure 2.1 General multiple-input, multiple-output (MIMO) linear system

The inputs can be represented as a vector and the outputs as another vector. Each element in the vectors is time history. The elements in the vectors can also be represented in the frequency domain by the Fourier transform.

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-i2\pi ft} df \tag{2.1}$$

This integral does not strictly exist for a stationary random signal and a limiting operation must be defined. To simplify the notation let

$$\mathbf{S}_{XX} = \mathrm{E}[\mathbf{XX}'] \tag{2.4}$$

where **X** is a column vector of the discrete Fourier transforms of a realization of a column vector of the time histories, **x**. Justification for this simplification is contained in [Bendat and Piersol, 2000]. Let **H** represent a matrix of frequency response functions relating the inputs to the outputs of a MIMO system. The cross spectral density matrix of the outputs, $\mathbf{S}_{YY}$, can be stated in terms of the cross spectral density matrix of the inputs, $\mathbf{S}_{XX}$, as

$$\mathbf{S}_{YY} = \mathbf{H}\mathbf{S}_{XX}\mathbf{H}' \tag{2.5}$$

Define **G** as the Moore-Penrose pseudo inverse of **H**

$$\mathbf{G} = \text{pinv}(\mathbf{H}) \tag{2.6}$$

If **H** is square and non-singular the pseudo inverse is the same as the matrix inverse. For the cases where **H** is rectangular or square and singular the Moore-Penrose pseudo inverse produces the result

$$\begin{aligned} &\mathbf{HGH} = \mathbf{H} \\ &\mathbf{GHG} = \mathbf{G} \\ &\mathbf{GH} \text{ and } \mathbf{HG} \text{ are Hermitian.} \end{aligned} \tag{2.7}$$

Let norm(**H**) be defined as the largest singular value of **H**. If **H** has more rows than columns (more outputs than inputs) or is square and singular then the inverse of **H** does not exist. In these cases pinv(**H**) has some, but not all, of the properties of the inverse of **H**. If **H** has more rows than columns, then the overdetermined least squares problem

$$\text{minimize norm}(\mathbf{HX}-\mathbf{Y}) \tag{2.8}$$

does not have a unique solution. The solution

$$\mathbf{X} = \text{pinv}(\mathbf{H})\,\mathbf{Y} \tag{2.9}$$

is the solution for which the norm(**X**) is the smallest.

If **H** has more columns than rows (the number of inputs exceeds the number of outputs) there are many solutions to the problem

$$\mathbf{HX}-\mathbf{Y} = 0 \tag{2.10}$$

The solution of the off-diagonal terms of $\mathbf{S}_{YY}$ that minimizes the trace of $\mathbf{S}_{XX}$ when the diagonal elements of $\mathbf{S}_{YY}$ are specified will be derived. The case where **H** is square and full rank is a special case of this problem. In this case **G** is just the inverse of **H**.

The spectral density matrix of the input can now be expressed in terms of the spectral density matrix of the output as

$$\mathbf{S}_{XX} = \mathbf{G}\mathbf{S}_{YY}\mathbf{G}' \tag{2.11}$$

As a side note: If the complete cross spectral density matrix of the output is specified (not just the diagonal autospectra), Equation 2.11 is used to compute the required input cross spectral density matrix.

Define a measure of the spectral input to the system as the sum of the autospectra of the inputs

$$P = \text{trace}\ (\mathbf{S}_{XX}) \tag{2.12}$$

This will be called the input power. It is not physical power except in special cases. One of the special cases comes from transmission theory using constant real impedance transmission lines, and hence the name). Using Equations (2.11 and 2.12)

$$P = \text{trace}(\mathbf{G}\mathbf{S}_{YY}\mathbf{G}')$$

$$P = \sum_{j=1}^{J}\sum_{k=1}^{K}\sum_{m=1}^{K} G_{jk}\ {}_{YY}S_{km}G_{jm}^{*} \tag{2.13}$$

where J is the number of inputs, K is the number of outputs. It is important to note that ${}_{YY}S_{km}$ appears in the above equation only as a linear term. Cross products of ${}_{YY}S_{km}$ do not appear. This implies that the surface of P above each complex ${}_{YY}S_{km}$ is a plane with a constant gradient. If we specify the autospectra of the outputs, ${}_{YY}S_{kk}$, the range of ${}_{YY}S_{km}$ is restricted to

$$0 \le \left|{}_{YY}S_{km}\right|^2 \le {}_{YY}S_{kk}\ {}_{YY}S_{mm} \tag{2.14}$$

The gradient of P is constant, therefore the minimum and maximum will occur on the boundary of the range of the cross-spectrum. Zero is a trivial result, therefore set

$${}_{YY}\overline{S}_{km} = \left|{}_{YY}S_{km}\right| = \sqrt{{}_{YY}S_{kk}\ {}_{YY}S_{mm}} \tag{2.15}$$

Interchanging the order of summation in Eq. (2.13)

$$P = \sum_{k=1}^{K}\sum_{m=1}^{K} {}_{YY}S_{km}\sum_{j=1}^{J} G_{jm}^{*}G_{jk} \tag{2.16}$$

Let

$$\mathbf{F} = \mathbf{G}'\mathbf{G}$$

or (2.17)

$$F_{mk} = \sum_{j=1}^{J} G_{jm}^{*} G_{jk} \qquad m = 1 \ldots K, \quad k = 1 \ldots K$$

Substituting (2.17) into (2.16) gives

$$P = \sum_{k=1}^{K} \sum_{m=1}^{K} {}_{YY}S_{km} F_{mk} \tag{2.18}$$

Using the fact that both $\mathbf{S_{YY}}$ and $\mathbf{F}$ are Hermitian (2.18) can be expressed as

$$P = \sum_{k=1}^{K} {}_{YY}S_{kk} F_{kk} + 2\sum_{k=1}^{K-1} \sum_{j=k+1}^{K} \mathrm{Re}({}_{YY}S_{kj} F_{kj}^{*}) \tag{2.19}$$

This expression can be written as

$$P = \sum_{k=1}^{K} {}_{YY}S_{kk} F_{kk} + 2\sum_{k=1}^{K-1} \sum_{j=k+1}^{K} {}_{YY}\overline{S}_{kj} \overline{F}_{kj} \cos(a_{kj} - b_{kj}) \tag{2.20}$$

where

$$\begin{aligned} {}_{YY}S_{kj} &= {}_{YY}\overline{S}_{kj} e^{ia_{kj}} \\ F_{kj} &= \overline{F}_{kj} e^{ib_{kj}} \end{aligned} \qquad k = 1 \cdots K, \quad j = 1 \cdots K \tag{2.21}$$

P will be a minimum when all the terms in the second double sum of (2.20) are negative. The maximum will occur when all the terms are positive. The minimum will occur when

$$a_{jk} - b_{jk} = \pi \qquad k = 1 \cdots K, \quad j = 1 \cdots K \tag{2.22}$$

or

$$a_{jik} = \pi + b_{ik} \qquad k = 1 \cdots K, \quad j = 1 \cdots K \tag{2.23}$$

$${}_{YY}\widehat{S}_{jk} = {}_{YY}\overline{S}_{jk} e^{i(b_{jk} + \pi)} \qquad k = 1 \cdots K, \quad j = 1 \cdots K \tag{2.24}$$

The maximum will occur when

$$a_{ik} = b_{ik} \tag{2.25}$$

The minimum P is given by

$$P = \sum_{k=1}^{K} {}_{YY}S_{jj} F_{jj} - 2\sum_{k=1}^{K-1} \sum_{j=k+1}^{K} {}_{YY}\overline{S}_{kj} \overline{F}_{kj} \tag{2.26}$$

The maximum is given by

$$P = \sum_{k=1}^{K} {}_{YY}S_{kk} F_{kk} + 2\sum_{k=1}^{K-1} \sum_{j=k+1}^{K} {}_{YY}\bar{S}_{kj}\bar{F}_{kj} \tag{2.27}$$

To summarize, when the output autospectra are specified, the input with the minimum trace is given by

$$\mathbf{S}_{XX} = \mathbf{G}\hat{\mathbf{S}}_{YY}\mathbf{G}' \tag{2.28}$$

where $\hat{\mathbf{S}}_{YY}$ is specified by Equations (2.21) and (2.23). The response to this input is given by

$$\mathbf{S}_{YY} = \mathbf{H}\mathbf{S}_{XX}\mathbf{H}' = \mathbf{H}\mathbf{G}\hat{\mathbf{S}}_{YY}\mathbf{G}'\mathbf{H}' \tag{2.29}$$

The individual input pairs are fully coherent, which means the vector input can be specified with a single random noise source

$$\mathbf{S}_{XX} = \mathbf{X}\mathbf{W}\mathbf{X}' \tag{2.30}$$

where **X** is now a deterministic complex column vector that is a function of frequency. **W** is the auto spectrum of a white noise source with a unity spectral density. **W** is a real scalar function of frequency (i.e. **W** = 1), which could be factored out. However to keep Eq. (2.30) as the same form as Eq. (2.28) this will not be done.

When the number of inputs exceeds the number of outputs the maximum input has a special meaning. It is not the maximum input that could be derived, which is speculated to be unbounded. It is the maximum input that can be achieved within the constraints of the pseudo inverse, which minimizes the trace of the input spectral density matrix.

For the special case of a single output, **F** is a scalar and the maximum and minimum inputs are the same. The pseudo inverse will result in the input spectral density matrix with the minimum trace.

2.1 Geometric interpretation of input requirement

For simplicity, first consider the case of two inputs and two outputs at one frequency. The auto spectral densities of the two outputs are specified and we want to find the input auto spectral densities for all possible values of the cross spectrum between the outputs. In this example, the magnitude of the auto spectral densities of the two outputs is assumed to be the same. We will construct a three-dimensional plot where the first axis represents the real part of the cross spectral density, the second axis represents the imaginary part of the cross spectral density, and the third axis represents the power (P, the sum of the two input auto (power) spectral densities, the trace of the input spectral density matrix). The allowable output spectra are represented by the interior of a circle on the x-y plane. Points

on the circle represent unity coherence of the outputs with the phase of the cross spectral density between varying. Points in the interior of the circle will represent cross spectral densities with less than unity coherence. A point at the center of the circle will represent zero coherence between the two outputs. The gradient of P is constant, and is therefore a tilted plane above the x-y plane. The circle of allowable outputs is projected as an ellipse on the power plane. Figure 2.1.1 illustrates the plot. Points on the power plane within the ellipse represent the input power for a specific complex cross spectral density between the outputs. One phase angle of the output cross spectral density will represent the lowest part of the ellipse on the power plane. This is the minimum power. Add pi radians to establish the point of maximum power. Because in Eq. (2.13) P is linear in terms of the response cross spectral density matrix, the case for more than two inputs can be interpreted as a series of plots similar to Figure 2.1.1. Each figure represents one of the terms in the summation. The total power is the sum of the individual terms.

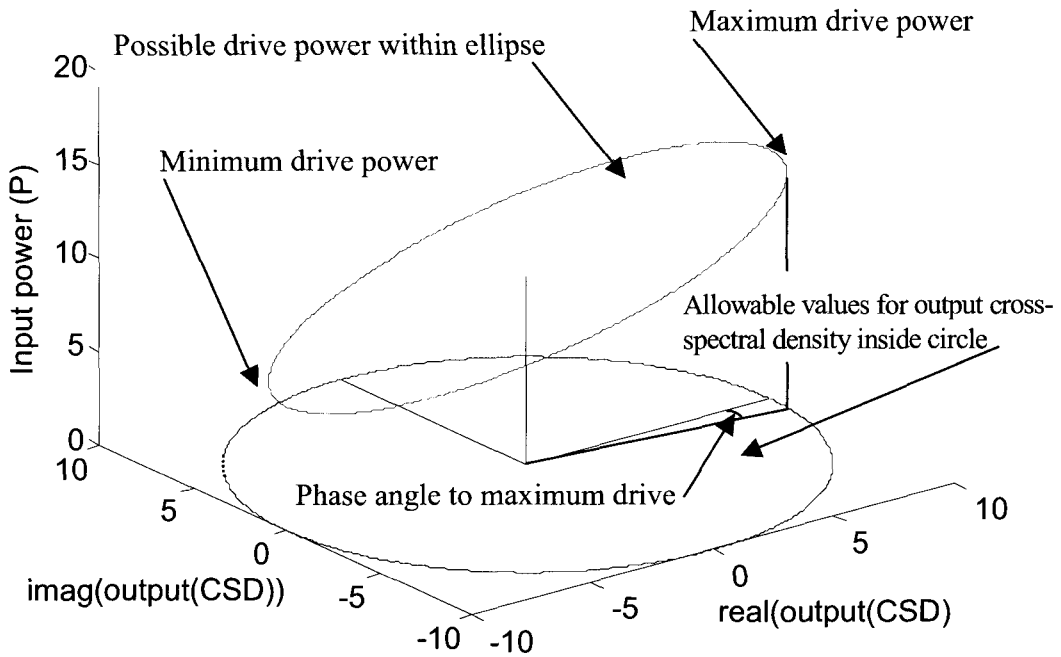

Figure 2.1.1 Geometric interpretation of input requirement

2.2 Summary given the auto spectra of the outputs

- Find the phase of the output spectra that will maximize or minimize the trace of the input spectra.
- Find the cross spectral density matrix of the outputs by combining the derived phase with the given output auto spectra for a fully coherent input.
- Find the input spectral density matrix from $\mathbf{S}_{XX} = \mathbf{G}\mathbf{S}_{YY}\mathbf{G}'$.

3. Derivation of the Minimum/Maximum Output Trace

In this section we assume the auto spectral densities of the inputs are specified and we want to find the input cross spectral densities that minimizes or maximizes the trace of the output auto spectral densities. The development is essentially the same as in the previous section. Equation 2.12 is the starting point. The trace of the matrix of output spectral densities is given by

$$P_y = \text{trace}(\mathbf{S_{YY}}) \tag{3.1}$$

Using Equation (2.5)

$$P_y = \text{trace}(\mathbf{HS_{XX}H'})$$

$$P_y = \sum_{k=1}^{K}\sum_{j=1}^{J}\sum_{m=1}^{J} H_{kj}\ {}_{XX}S_{jm} H^*_{km} \tag{3.2}$$

As before the gradient of P_y is constant, therefore the minimum will occur on the boundary of the range of allowable values of the input cross spectral density matrix. Zero is a trivial result, therefore set

$${}_{XX}\overline{S}_{jm} = \left|{}_{XX}S_{jm}\right| = \sqrt{{}_{XX}S_{jj}\ {}_{XX}S_{mm}} \tag{3.3}$$

Interchanging the order of summation yields

$$P = \sum_{j=1}^{J}\sum_{m=1}^{J} {}_{XX}S_{jm} \sum_{k=1}^{K} H^*_{km} H_{kj} \tag{3.4}$$

Let

$$\mathbf{F_H} = \mathbf{H'H}$$

or (3.5)

$${}_{H}F_{mj} = \sum_{k=1}^{K} H^*_{km} H_{kj} \qquad m = 1\cdots J, \quad j = 1\cdots J$$

As in the previous section let

$$\begin{aligned} {}_{XX}S_{mj} &= {}_{XX}\overline{S}_{mj} e^{ia_{mj}} \\ {}_{H}F_{mj} &= {}_{H}\overline{F}_{mj} e^{ib_{mj}} \end{aligned} \qquad m = 1\cdots J, \quad j = 1\cdots J \tag{3.6}$$

The minimum trace of the output spectral density matrix is given by the input spectral density matrix where

$$a_{mj} = \pi + b_{mj} \tag{3.7}$$

$$_{XX}\hat{S}_{mj} = {}_{XX}\bar{S}_{mj} e^{i(b_{mj}+\pi)} \tag{3.8}$$

The maximum occurs when

$$a_{mj} = b_{mj} \tag{3.9}$$

3.1 Summary given the autospectra of the inputs

- Find the phase of the input spectra that will maximize or minimize the trace of the output spectra.
- Find the cross spectral density matrix of the inputs by combining the derived phase with the given input auto spectra for a fully coherent input.
- Find the output spectral density matrix from $\mathbf{S}_{YY} = \mathbf{H}\mathbf{S}_{XX}\mathbf{H}'$.

4. Application to Transient and Sine Data

If the input is a deterministic transient or steady state sine the response can be written in the frequency domain as

$$\mathbf{Y} = \mathbf{HX} \tag{4.1}$$

Y is now a complex quantity that gives the amplitude and phase of the outputs. **H** is the same as before. **X** is a complex quantity giving the amplitude and phase of the inputs. For a transient **X** and **Y** are the Fourier transforms of the transient time histories. For a sine the phase is relative to some arbitrary reference. Some matrix spectral relations are analogous to the definition of cross spectral density (Eq. 2.4).

$$\begin{aligned} \mathbf{S}_{XX} &= \mathbf{XX}' \\ \mathbf{S}_{YY} &= \mathbf{YY}' \end{aligned} \tag{4.2}$$

These equations do not require the use of the expectation operator because the functions are deterministic. The diagonal terms of the spectral matrix are the amplitudes squared of the sine (or Fourier) components. The off diagonal terms define the phase relationships between pairs of signals. Equations 2.5 and 2.11 still apply. The development for the deterministic case is the same as for the random waveforms. When the output amplitudes are specified Equations 2.23-2.27 define the relative phase between the outputs. When the input spectral amplitudes are specified, Equations 3.6-3.8 define the relative phase between the inputs. The phase of one of the inputs or outputs must still be defined. If the inputs are specified the outputs are determined from Equation 2.5. If the outputs are specified the inputs are determined from Equation 2.11.

4.1 Least favorable transient response

If the system is a single input single output (SISO) system, a input transient that forces all the output frequency domain components to have zero phase results in the least favorable response [Drenick,1970, Shinozuka, 1970]. This is the largest possible peak output for any input with a spectral envelope equal to or less than a specified envelope. Other authors [Baratta et al, 1998, Lopez et al, 2000 and 2001, Takewaki, 2001] call this the critical response. In summary the least favorable response is given by

$$y_{\max}(t) = y(0) = \int_{-\infty}^{\infty} |X(f)||H(f)|e^{i(-\beta(f)+\beta(f))}df = \int_{-\infty}^{\infty} |X(f)||H(f)|df \tag{4.3}$$

where

$$H = |H|e^{i\beta(f)}$$

and

$X(f)$ the Fourier transform of the input time history, $x(t)$, and $|X(f)|$ is the magnitude of the input Fourier spectrum amplitude.

If the system is a multiple input single output (MISO) system the results derived in the previous section are a generalization of the least favorable response.

Consider the contribution of an input X_1 to the output. Just as for the single input least favorable response set

$$X_1(f) = |X_1|e^{i(-\beta_1(f))} \tag{4.4}$$

where $H_i(f) = |H_i(f)|e^{i\beta_i(f)}$ (4.5)

which results in

$$Y_1(f) = |X_1(f)||H_1(f)| \tag{4.6}$$

The contribution of $x_1(t)$ to the total response will be the same as given by equation 4.3. The relative phases between the inputs are given by $\mathbf{F_H}$. For the single output case, the relative phase between X_n and X_1 is given by $(-\beta_n + \beta_1)$. The absolute phase of the X_n input is then given by $(-\beta_n + \beta_1 - \beta_1 = -\beta_n)$.

This results in the contribution from the X_n input to the output as

$$Y_n(f) = |H_n|e^{i\beta_n(f)}|X_n|e^{i(-\beta_n(f)} = |H_n||X_n| \tag{4.7}$$

The responses caused by the various inputs are all maximum and positive at $t = 0$. Therefore, superposition will occur and the total response will be the maximum possible response, the least favorable response generalized for the MISO case.

For the transient case with a single output the minimization of the Fourier magnitude of the output (Section 2) does not seem to be particularly useful as a limiting case.

If the system has multiple inputs and outputs the phase of one of the inputs or outputs must be specified. A single set of inputs that maximizes the response of all the outputs does not exist. If the phase of one of the inputs is known, a single set of inputs that maximizes the trace of the output spectra does exist. If the phase of one of the outputs can be specified (as for example zero phase that results in maximizing the response of that output at $t = 0$) a set of inputs can be found that produces that response and also maximize the trace of the spectra of all the responses.

4.2 Pseudo random input

Whenever we seek to define a pseudo-random input, the phase of the first input can be randomized. If we want to define a pseudo-random signal with a specified temporal modulated envelope, we can generate a pseudo-random signal, transform to the time domain, multiply by a suitable temporal window, and transform back to the frequency domain. This is frequently referred to as a product model. The phases of the remaining inputs are then defined relative to the first windowed signal.

4.3 A deterministic time history for the first input or output

We can also define the first response (or input) as a specified time history. The FFT of the time history will define the amplitude and phase of the first input (or response) time history. The Fourier magnitude of the remaining inputs (or responses) can be defined independently or can be a defined as a function of the Fourier magnitude of the first signal. The algorithm to minimize the input (or output) defines amplitude and phase of the inputs (or responses) and the phase of the remaining responses (or inputs).

5.0 The special cases of multiple inputs and a single output

If there is only one output the problem reduces to the superposition of the response to J inputs. If the maximum or minimum response is desired the inputs will be fully correlated. If the auto spectral densities of the inputs are specified, Section 3 is used to determine the phase angles between the inputs for the minimum and maximum output spectral density. If the output auto spectrum is specified, Section 2, Equation 2.28 is used to derive the inputs to minimize or maximize the trace of the input spectral density matrix.

If the input is a transient, Section 4.1 has already discussed the solution as a generalization of least favorable response.

5.1 Von Mises stress as the output variable

Consider the von Mises stress that is a quadratic combination of six stress components

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x & \sigma_y & \sigma_z & \sigma_{xy} & \sigma_{xz} & \sigma_{yz} \end{bmatrix}^T \tag{5.1}$$

For the square of the von Mises stress the quadratic combination is

$$p(t)^2 = \sigma_x^2 + \sigma_y^2 + \sigma_z^2 - (\sigma_x\sigma_y + \sigma_x\sigma_z + \sigma_y\sigma_z) + 3(\sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{xz}^2) \tag{5.2}$$

Consider quadratic functions of stress, written in the following form

$$p^2 = \boldsymbol{\sigma}'\mathbf{A}\boldsymbol{\sigma} \tag{5.3}$$

For von Mises stress, **A** is [Segalman et al, 2000]

$$\mathbf{A} = \begin{bmatrix} 1 & -1/2 & -1/2 & 0 & 0 & 0 \\ -1/2 & 1 & -1/2 & 0 & 0 & 0 \\ -1/2 & -1/2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix} \tag{5.4}$$

The matrix **A** can be factored using a modified Cholesky decomposition [Smallwood and Paez, 1993] into

$$\mathbf{A} = \mathbf{B}'\mathbf{B} \tag{5.5}$$

$$\mathbf{B} = \begin{bmatrix} 1 & -1/2 & -1/2 & 0 & 0 & 0 \\ 0 & \sqrt{3}/2 & -\sqrt{3}/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{3} \end{bmatrix} \tag{5.6}$$

B has a row of zeros because **A** is singular and has a rank of 5.

Let

$$\mathbf{q} = \mathbf{B}\boldsymbol{\sigma} \tag{5.7}$$

Then

$$p^2 = \mathbf{q}'\mathbf{q} = \boldsymbol{\sigma}'\mathbf{B}'\mathbf{B}\boldsymbol{\sigma} = \boldsymbol{\sigma}'\mathbf{A}\boldsymbol{\sigma} \tag{5.8}$$

Because **B** is not a function of time, but just a matrix of real constants, transforming to the frequency domain gives

$$\mathbf{Q} = \mathbf{BS} \tag{5.9}$$

$$\mathbf{S} = \mathbf{HX} \tag{5.10}$$

which gives

$$\mathbf{Q} = \mathbf{BHX} \tag{5.11}$$

where **Q** is the Fourier transform of **q**, and **S** is the Fourier transform of $\boldsymbol{\sigma}$. The stress response is a linear function of the excitation.

$$P^2 = \mathbf{Q}'\mathbf{Q} = \sum_{j=1}^{6} Q_j^* Q_j \tag{5.12}$$

where P^2 is a spectral measure of the von Mises stress.

Consider the multiple input multiple output (MIMO) system, $\mathbf{QQ}'$. Noting that for any column vector **Q**, $\mathbf{Q}'\mathbf{Q} = trace(\mathbf{QQ}')$, the trace of this system is the same as the above equation

$$P^2 = \text{trace}(\mathbf{QQ}') = \sum_{j=1}^{6} Q_j Q_j^* \tag{5.13}$$

Thus **Q** defines a system with six responses and J inputs that is equivalent to the statement of the von Mises stress.

Combining **B** and **H** reduces the problem to the MIMO problem already studied. For random vibration, the spectrum of the von Mises stress can be found for inputs with any cross spectral density matrix. If only the input auto spectral densities are given, cross spectral densities of the inputs can be found that will minimize or maximize the root-mean-square (rms) von Mises stress. Once the cross spectral density matrix of the inputs are known the individual stresses can be derived from Equation 2.5, $\mathbf{S}_{ss} = \mathbf{HS}_{xx}\mathbf{H}'$.

5.2 Multiple axis vibration tests

An issue of interest in random vibration or sine testing is the equivalence or severity of a multiple axis vibration test (all three axes excited simultaneously) relative to three single axis tests. The response to each of the three single axis tests can be found from a standard single input single (or multiple) output analysis. The response to the multiple axis test can be handled as a three input single (or multiple) output system. Standard methods allow the determination of the response to any specified input cross spectral density matrix. The methods in this paper also allow the determination of the specific input cross spectral densities that will result in the maximum and minimum spectrum of the response (or the trace of the output spectra for the multiple output case). To determine the severity of the tests a damage criteria must be used which relates the combination of response spectra and duration of the test to damage. If such a model exists the tests can be compared.

6. Examples of Stationary Random

To illustrate the method a simple free-free bar is excited axially under various conditions (Examples 1-8). The bar has its first resonance at 100 Hz. The damping is set at 1% for all modes. The input points are various combinations of force excitations at 0, .6, and 1 times the length of the bar. The acceleration responses are measured at the same locations. The mass of the bar is 1000 lb. This bar could represent a slip table excited at various points. The 3 by 3 matrix of frequency response functions (g/pound) is plotted at Figures 6.0.1 (magnitude) and 6.0.2 (phase). Reciprocity holds for this system, therefore the frequency response function matrix is symmetric. For the various examples the input and response locations are stated in terms of their location as a fraction of the total bar length. The desired response is a flat spectrum of 0.01 g^2/Hz from 10 to 1000 Hz.

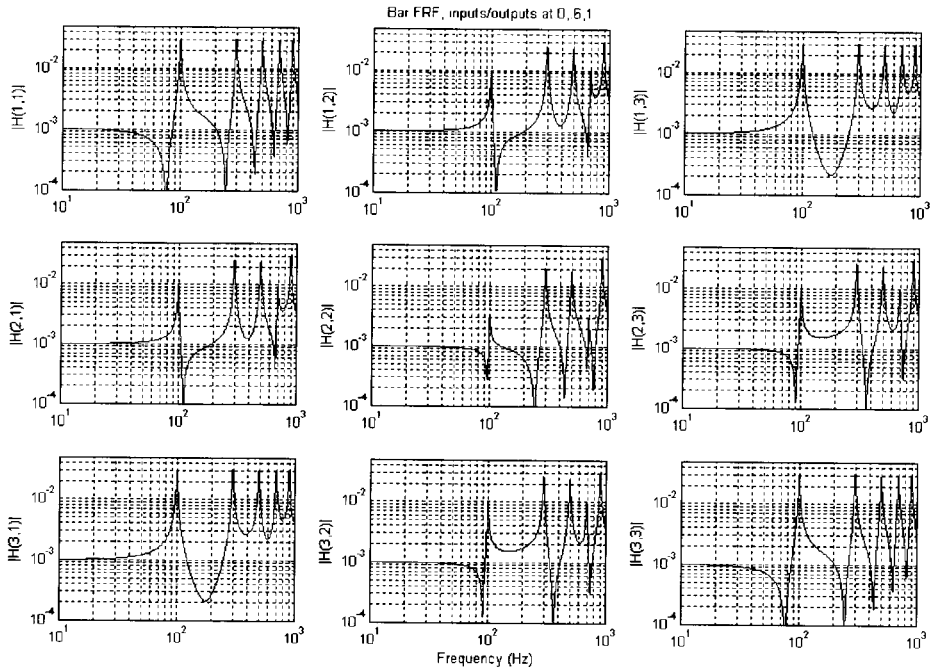

Figure 6.0.1 FRF's, 1000 lb bar, magnitude (g/lb)

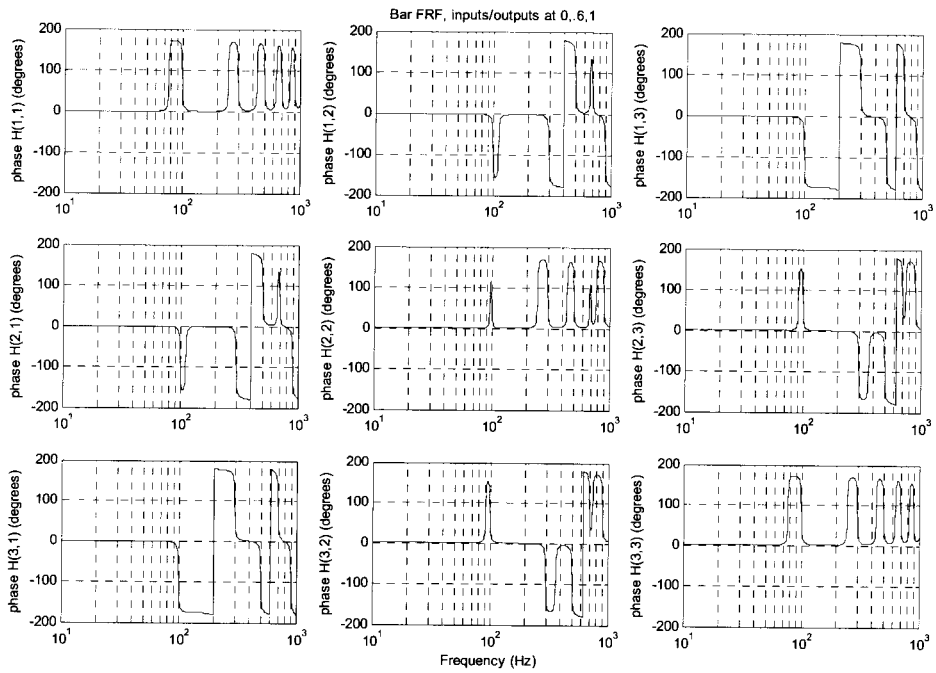

Figure 6.0.2 FRF's, 1000 lb bar, phase (degrees)

6.1 Example 1, Input at point 0, response at location 1

Notice from Figure 6.0.1 ($H(1,3)$) that no antiresonances exist at the response location (1) when the bar is driven at the 0 end. The input required to achieve the desired response is shown in Figure 6.1.1. Notice that no large peaks in the input are required because of the absence of antiresonances.

6.2 Example 2, Input at point 0, response at location .6

The required input is shown in Figure 6.2.1. Notice that now a large input is required at the antiresonances of 112 and 670 Hz.

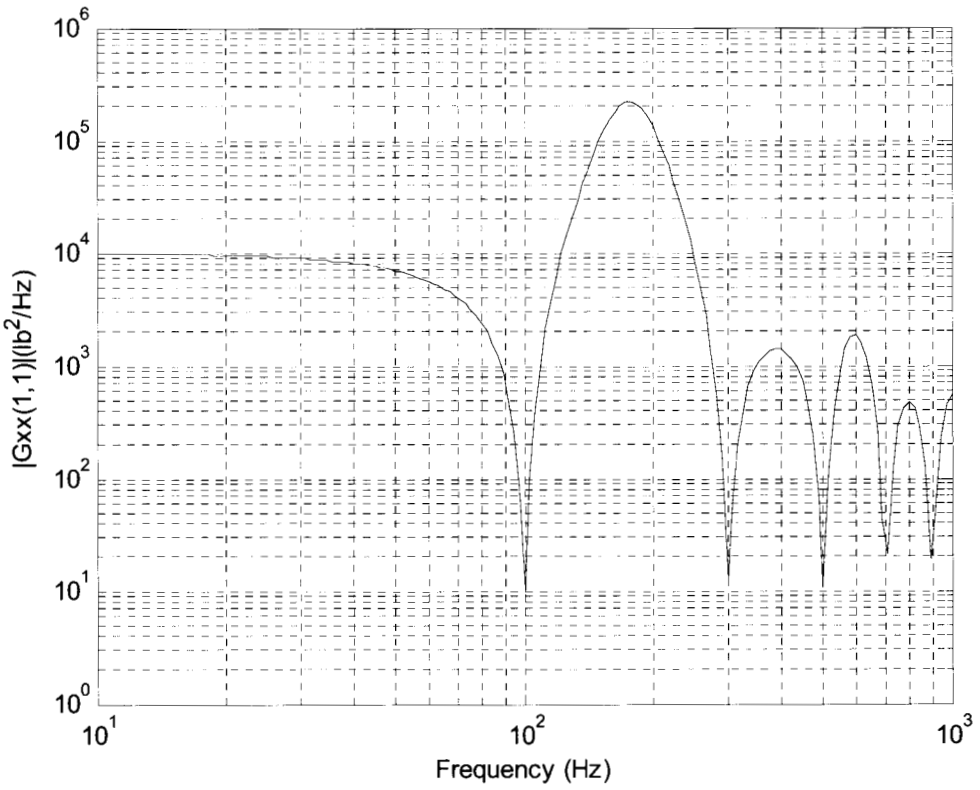

Figure 6.1.1 The required input at point 0 to achieve the desired response at point 1.

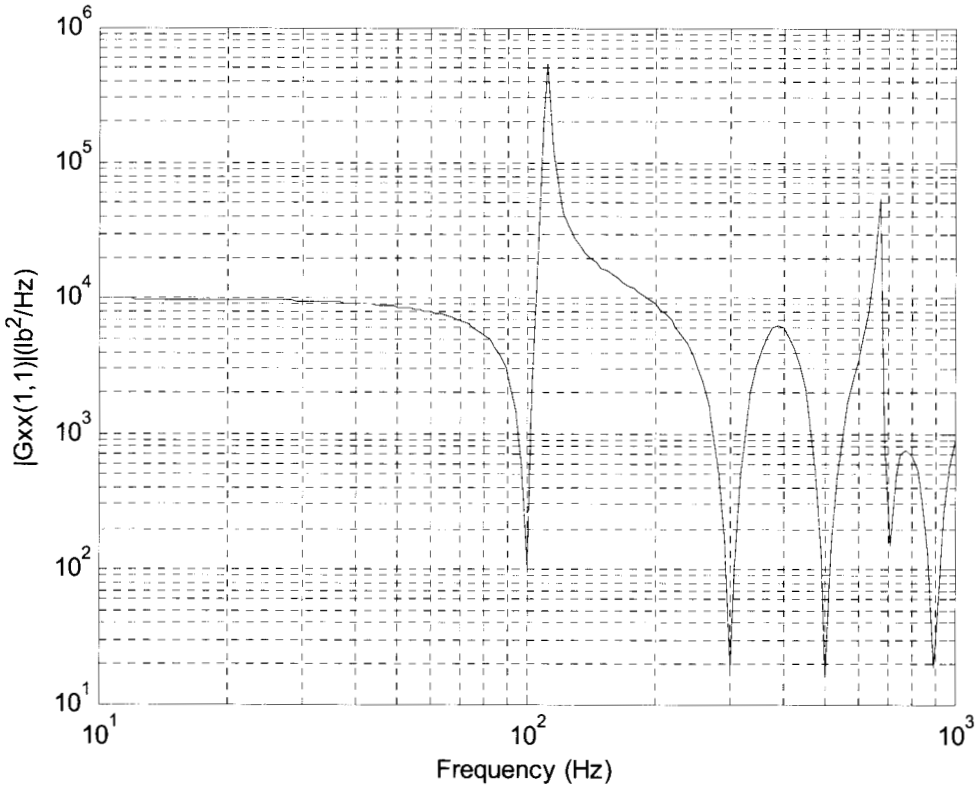

Figure 6.2.1 The required input at point 0 to achieve the desired response at point .6

6.3 Example 3, Input at points 0 and 1, response at location 1

In this example two inputs are used at points 0 and 1. The response is controlled at one point. Figure 6.3.1 shows the required inputs. The methods derived in Section 2 were used to derive the inputs with the minimum trace. The coherence between the inputs is unity. The phase between the two inputs is shown in location (2,1) of the plot. Notice that the phase changes 180° as a resonance is approached. These phase changes provide the minimum input to achieve the desired response.

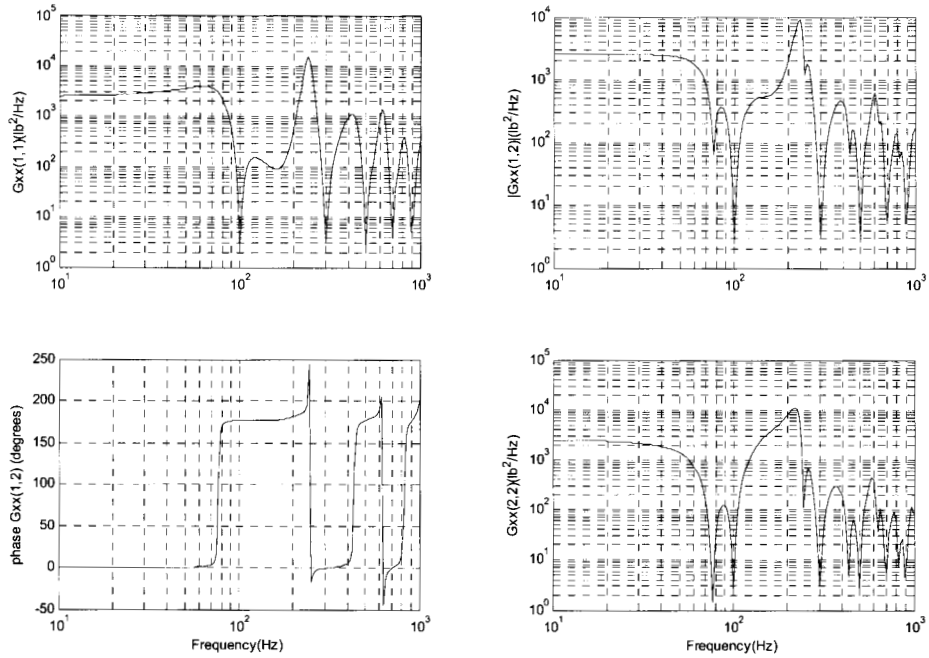

Figure 6.3.1 Inputs at points 0 and 1, response at point 1

6.4 Example 4, Input at points 0 and 1, response at location .6

The required minimum input is shown in Figure 6.4.1. Note that the peak required input has been reduced more than an order of magnitude from the single input case, and the peaks in the input at the antiresonant frequencies have been eliminated. At low frequencies the inputs are in phase and shared equally. Above the first resonant frequency the input phase has been shifted 180°.

Examples 3 and 4 illustrate how multiple inputs can overcome the antiresonance problem of driving a system with a single input. The antiresonance for each of the inputs with respect to the output will generally occur at different frequencies, allowing the system to be more easily driven.

6.5 Example 5, Input at points 0 and 1, response at locations 0 and 1

The results are shown as Figures 6.5.1 and 6.5.2. The desired output has been achieved at both response points at all frequencies. The phase between the outputs changes as a resonance is approached. The phase of the inputs is changed to achieve the minimum input.

Examples 4 and 5 illustrate how to phase the inputs to achieve the desired outputs with a minimum input. Intuition tells us the inputs should be in phase at low frequencies and out of phase when the first resonant frequency dominates. The method provides a logical procedure for choosing the crossover frequencies.

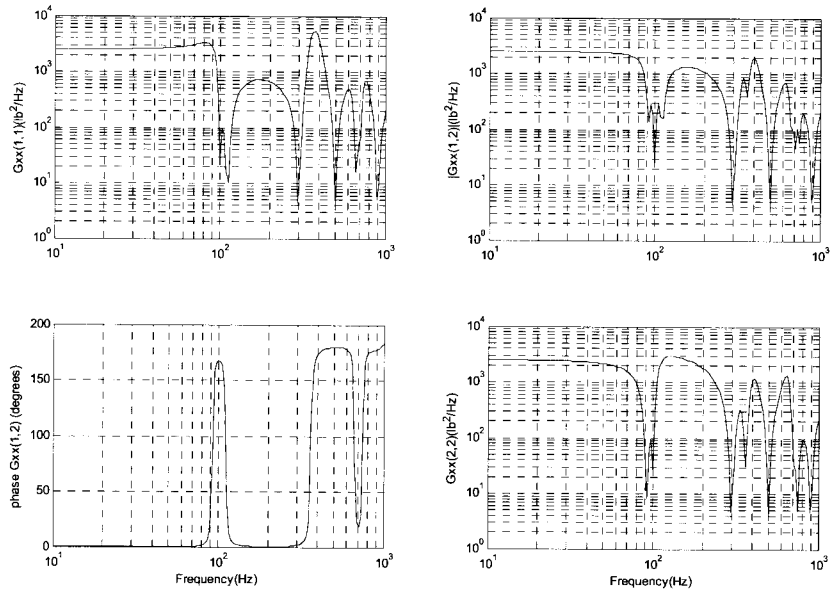

Figure 6.4.1 Inputs at 0 and 1, output at .6

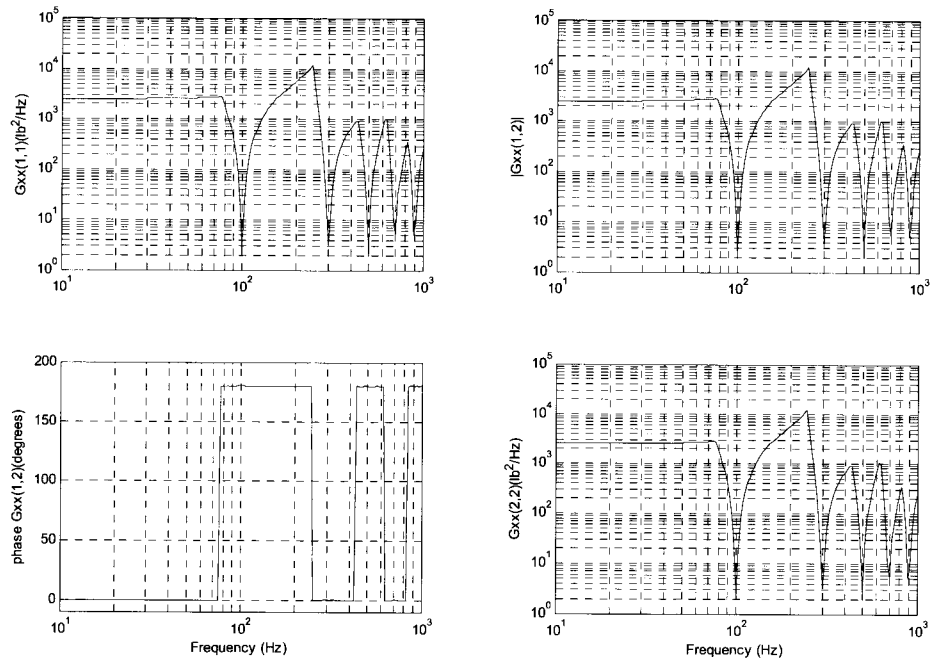

Figure 6.5.1 Input force, inputs at 0 and 1; outputs at 0 and 1 case

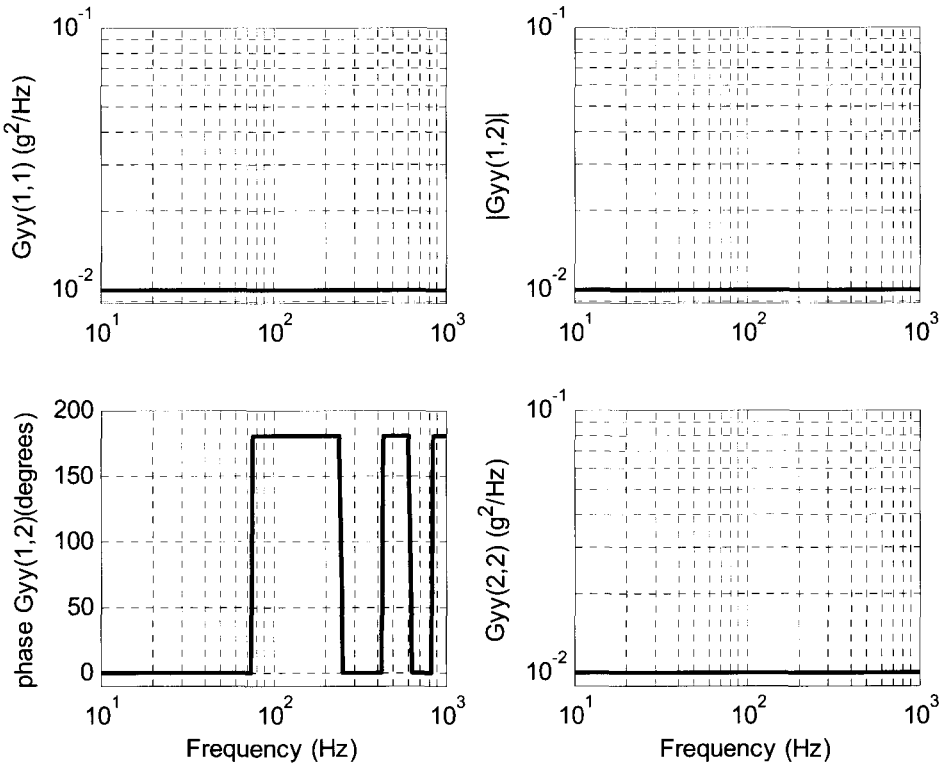

Figure 6.5.2 Output acceleration, inputs at 0 and 1; outputs at 0 and 1

6.6 Example 6, Two inputs, two outputs, different autospectrum

This example is the same as example 5 except the desired spectrum at point 1 will be increased to .011 g^2/Hz. This causes problems at some frequencies dominated by a single mode. Low frequencies are dominated by the rigid body mode is an example. The results are shown as Figures 6.6.1-6.6.3. As can be seen the desired results are achieved at the cost of increased force. Recognizing the problem and increasing the tolerance for rejecting singular values in the pseudo inverse can reduce the force. At frequencies where the frequency response functions are near singular the small singular values are set to zero. The result is smaller force at the expense of an exact solution. The results are shown in Figures 6.6.4-6.6.6. As can be seen the maximum force is reduced particularly at low frequencies. The solution is scaled such that the sum of the squares of the response auto spectral densities are as close to the sum of the squares of the desired solution as possible.

If the coherence is reduced to near zero Figures 6.6.7-6.6.9 show the results. As can be seen comparing Figures 6.6.1 6.6.4 and 6.6.7 considerable more force is required to keep the response coherence near zero. Notice that with the increased tolerance for rejecting singular values, the coherence at low frequencies is not near zero. The response is dominated by the single rigid body mode and the two response points will be in phase unless very large forces are applied. As the frequency decreases large increases in force

are evident in Figures 6.6.7 and 6.6.9 until a singular value is rejected at about 24Hz. The force is then reduced, but the coherence is no longer near zero.

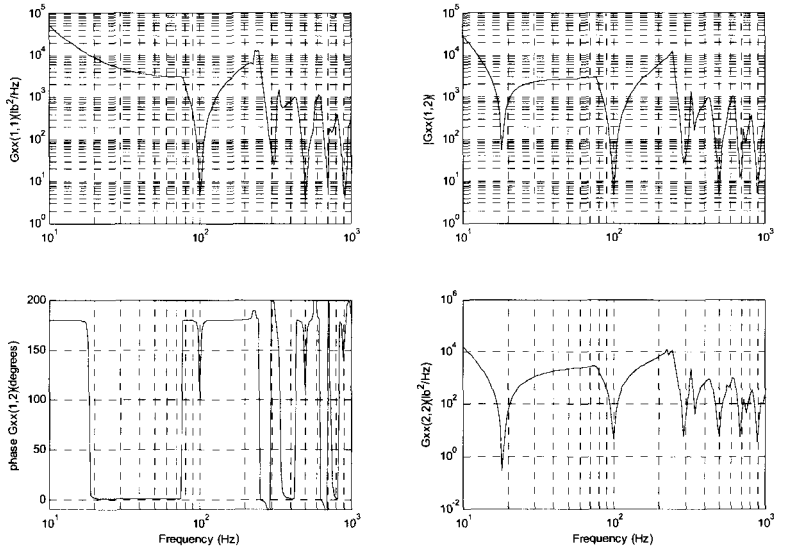

Figure 6.6.1 Input force spectra

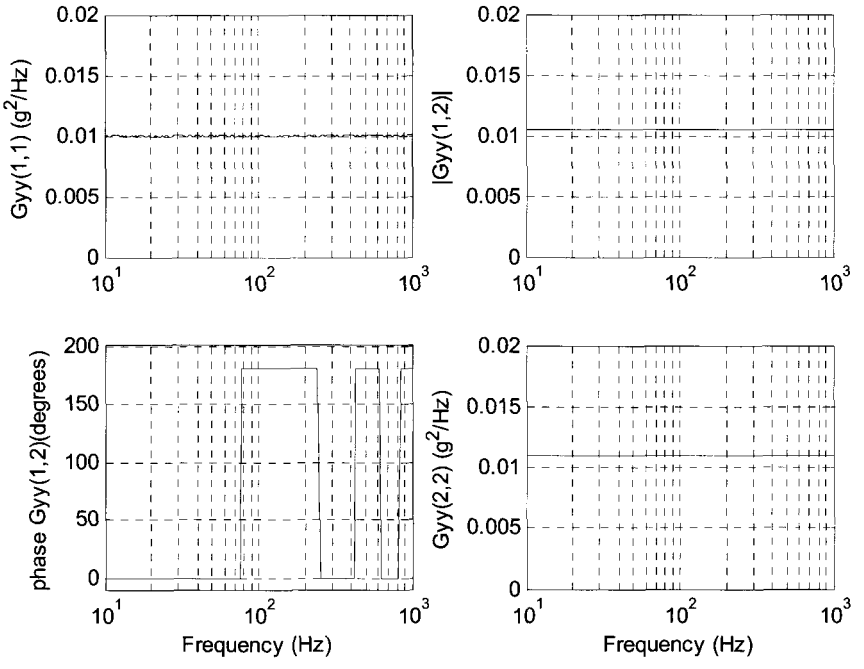

Figure 6.6.2 Response acceleration spectra

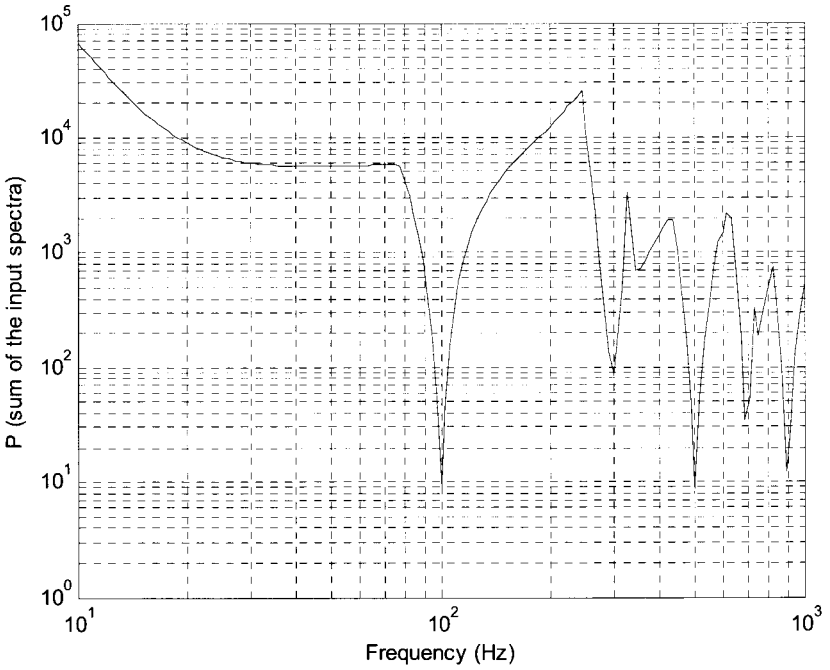

Figure 6.6.3 Sum of input spectra

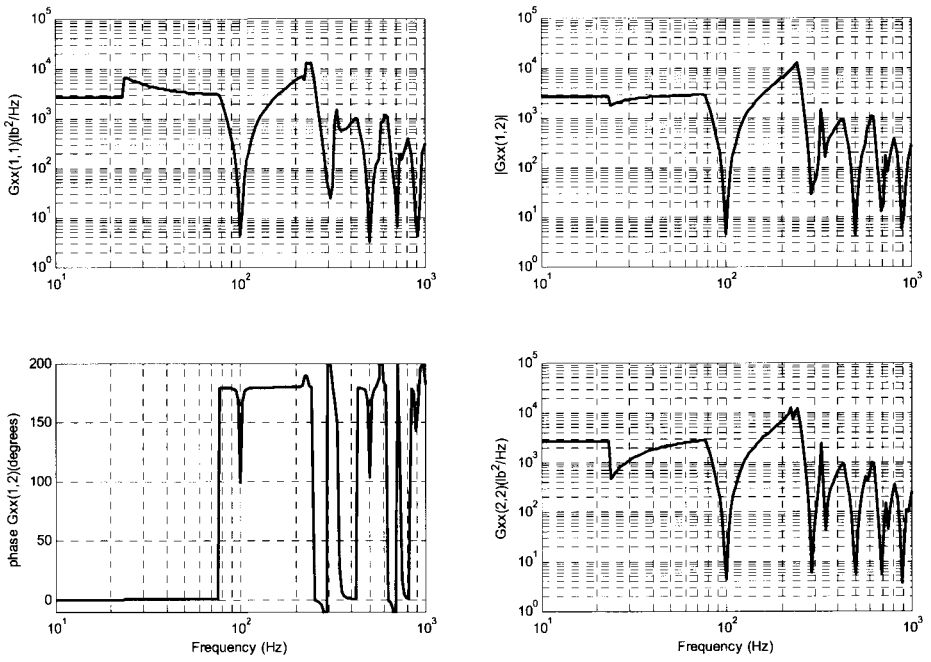

Figure 6.6.4 Input force spectra, increased tolerance

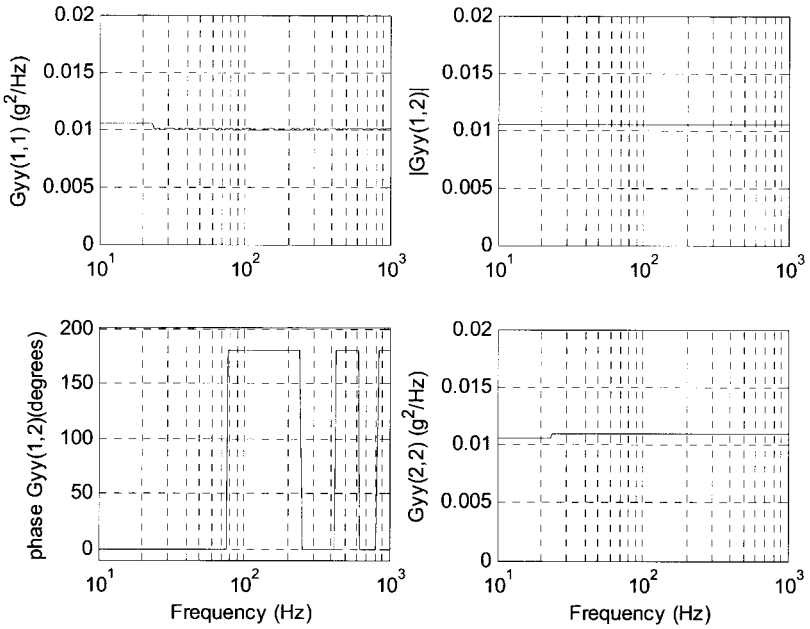

Figure 6.6.5 Response acceleration spectra, increased tolerance

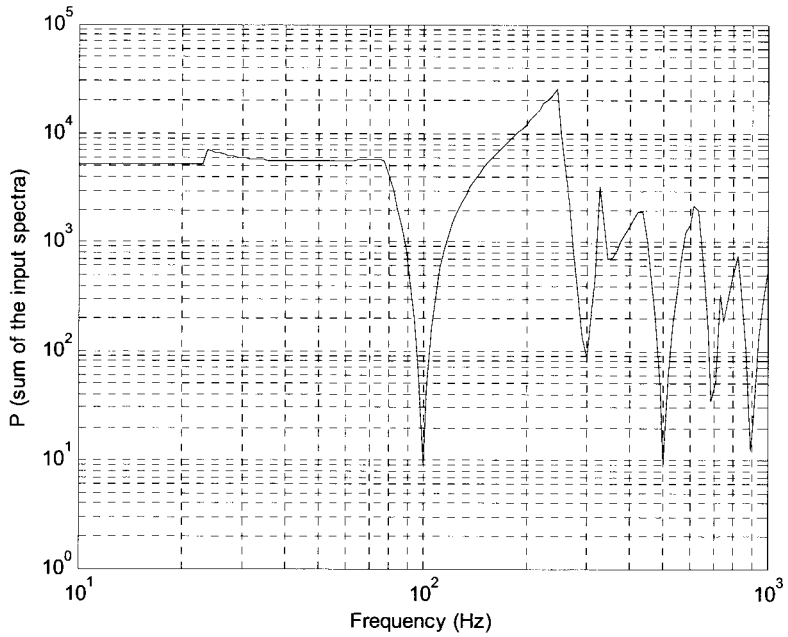

Figure 6.6.6 Sum of input spectra, increased tolerance

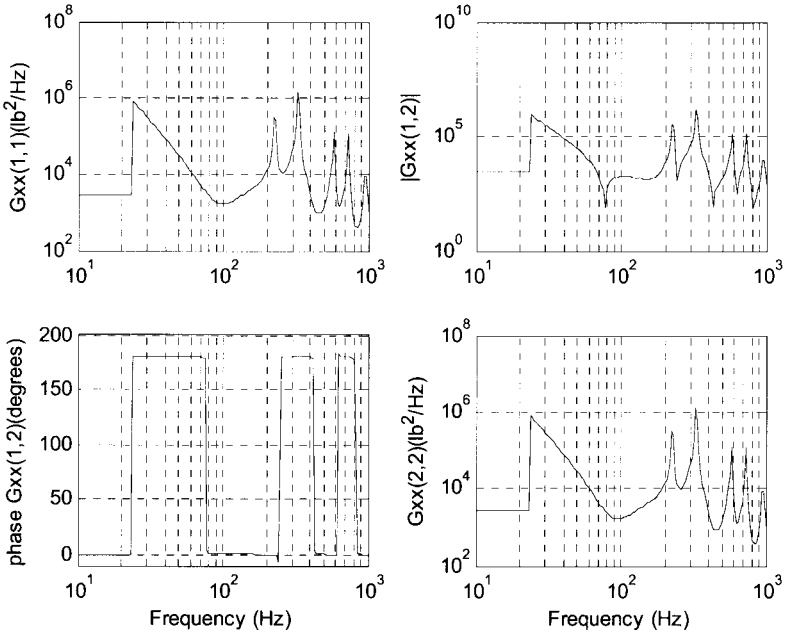

Figure 6.6.7 Input force, coherence near zero

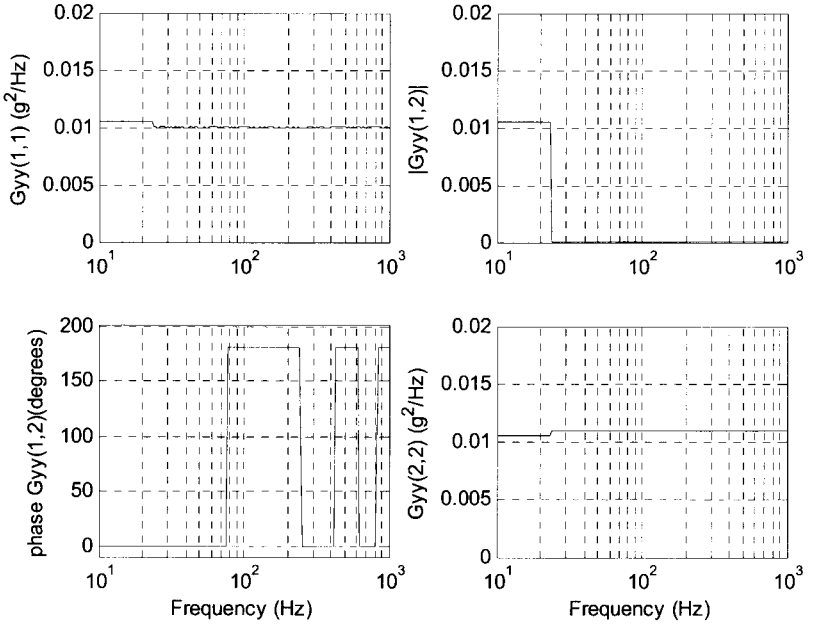

Figure 6.6.8 Response, coherence near zero

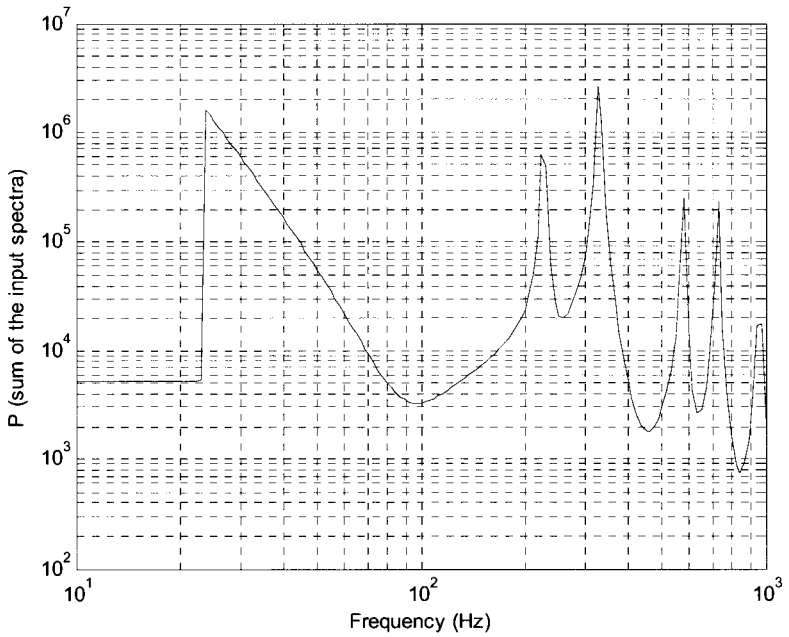

Figure 6.6.9 Sum of input spectra, coherence near zero

6.7 Example 7, Three inputs at 0, .6, 1; Two outputs at 0,1

This example uses three inputs and two response points. An infinite number of solutions provide the correct responses. The procedure proposed here provides the solution with the minimum sum of the input auto spectral densities. The input auto spectral densities are shown in Figures 6.7.1 and 6.7.2. Only the upper triangle part of the matrix is shown because the lower half is the complex conjugate of the upper part. The magnitude is shown in Figure 6.7.1 and the phase in Figure 6.7.2. Note that the inputs are all in phase and about equal magnitude at the low frequencies. This is the desired result. At higher frequencies the amplitudes and phases between the inputs changes, but are kept within reasonable bounds. The output spectral density matrix is shown in Figures 6.7.3 and 6.7.4. The output spectra are maintained at the desired level at all frequencies. The phase between the outputs varies to keep the input trace a minimum. The trace of the input auto spectral densities is shown as Figure 6.7.5.

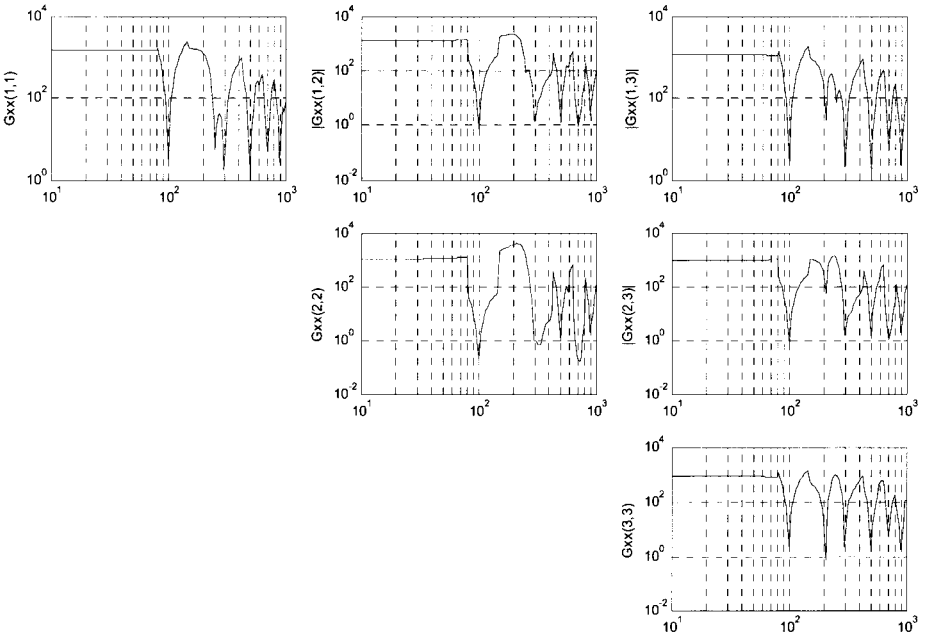

Figure 6.7.1, Three inputs at 0, .6, 1; Two outputs at 0,1, input amplitude

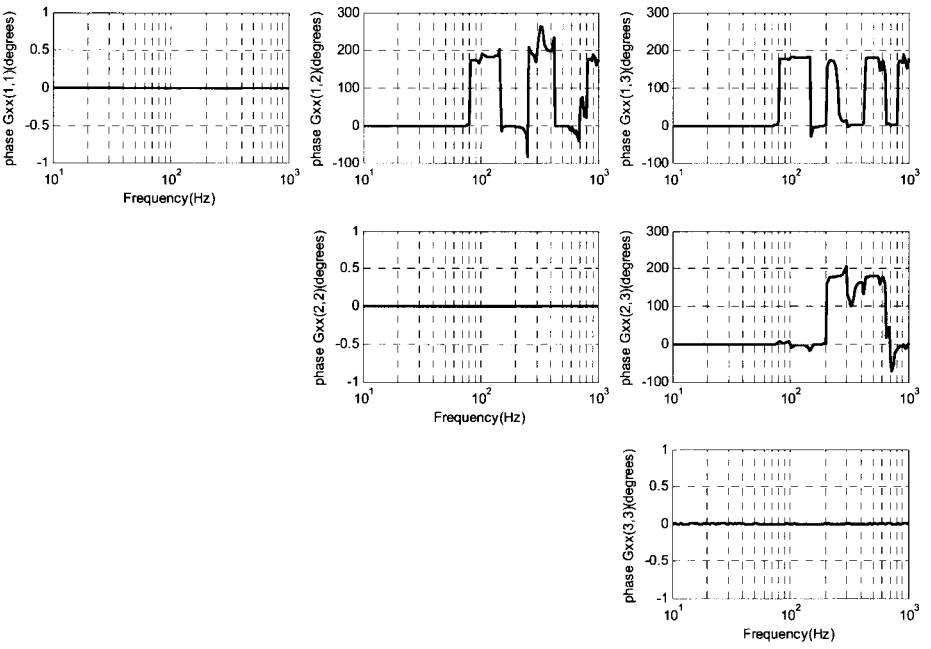

Figure 6.7.2, Three inputs at 0, .6, 1; Two outputs at 0,1; input phase

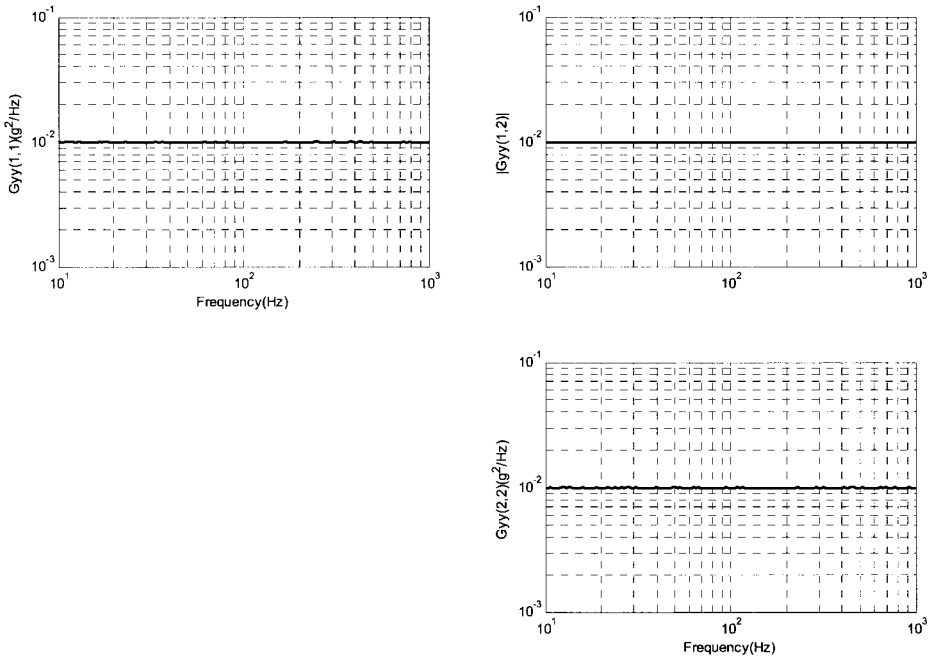

Figure 6.7.3, Three inputs at 0, .6, 1; Two outputs at 0,1; output amplitude

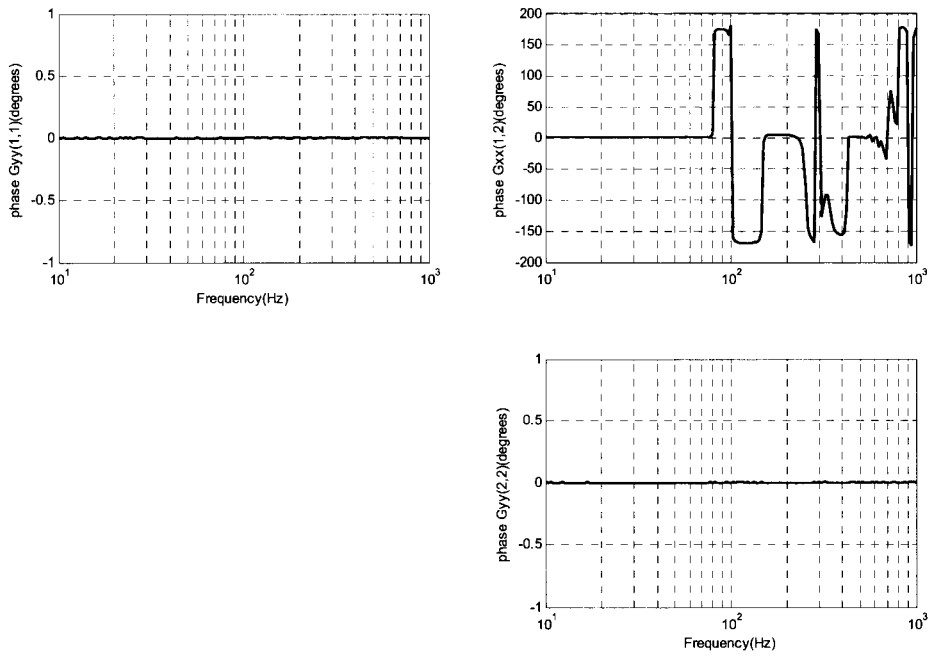

Figure 6.7.4, Three inputs at 0, .6, 1; Two outputs at 0,1; output phase

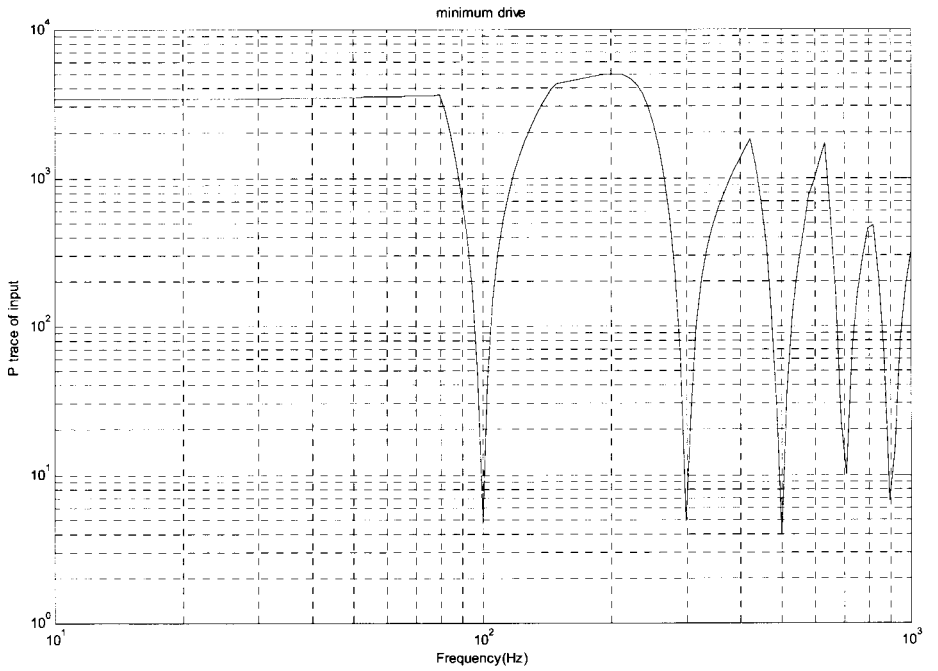

Figure 6.7.5, Three inputs at 0, .6, 1; Two outputs at 0,1; trace of inputs

6.8 Example 8, Two inputs at 0, 1; Three outputs at 0, 0.6, and 1

In this example it is not clear that a solution will exist which can maintain three outputs using only two inputs. A previous paper (Paez, et al, 1987) has shown that it is sometimes possible to control the autospectrum at n^2 points with n inputs. The desired input is 0.01 g^2/Hz at the three output points. Figures 6.8.1-6.8.6 show the result using the minimum input method. In this case it is not possible to achieve the desired results as Figure 6.8.3 illustrates. The responses are near the desired level at low frequencies where the rigid body mode dominates. At higher frequencies the responses deviate significantly from the desired response. However, the sum of the outputs (the trace, Figure 6.8.6) is maintained at the desired level. Notice that the inputs are essentially in phase at the low frequencies (Figure 6.8.1, subplot 1,2), suggesting that the rigid body mode is dominate. In summary, for the case where the number of outputs exceeds the number of inputs, the method gives the minimum input that will control the trace of the outputs. If you want to minimize the error (the difference between the achieved output and the desired output) a different method will have to be used.

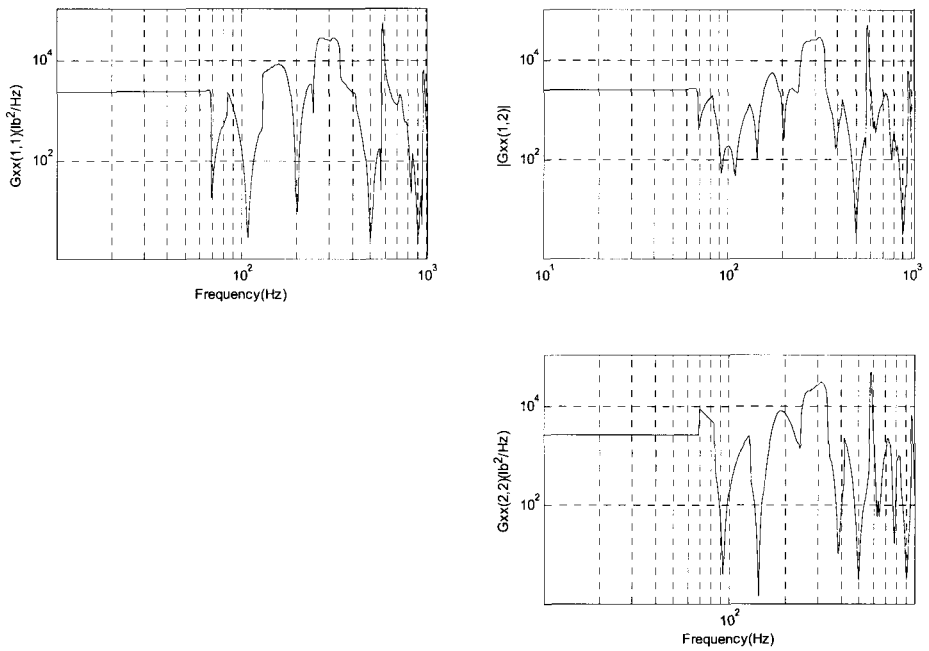

Figure 6.8.1 Two inputs, three outputs; input spectral amplitudes

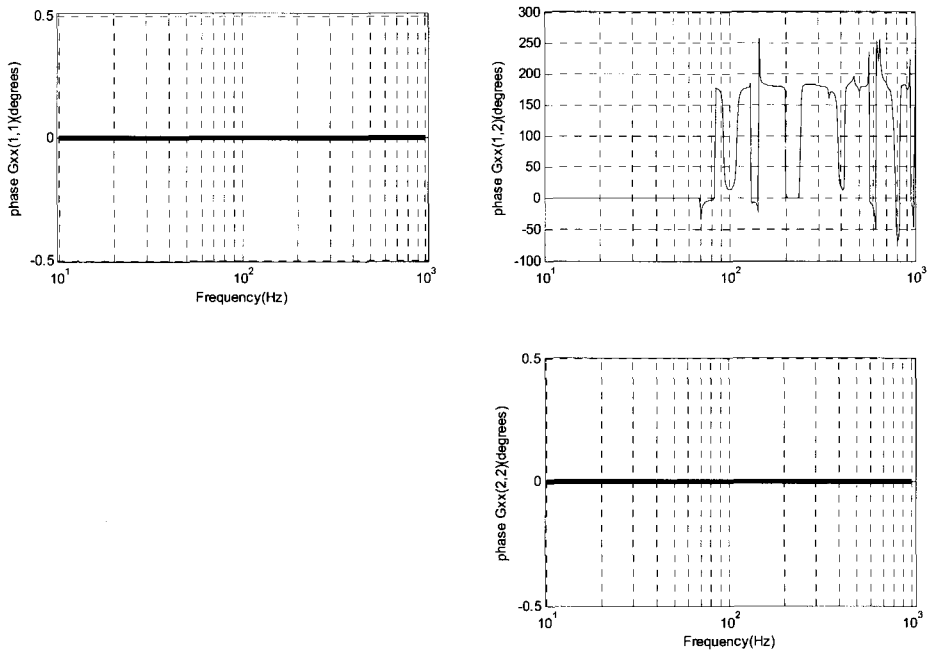

Figure 6.8.2 Two inputs, three outputs; input spectral phase

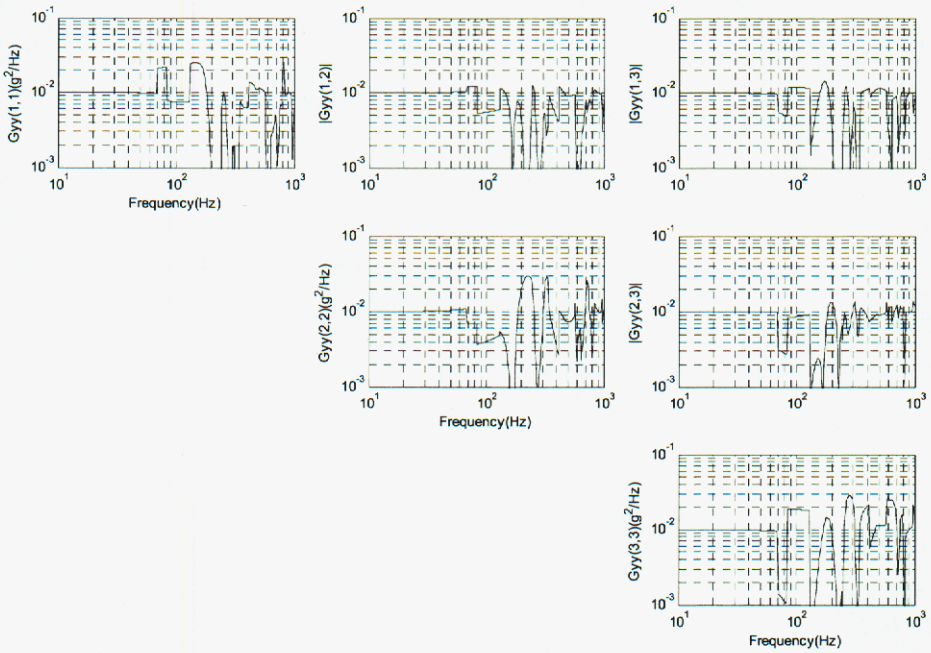

Figure 6.8.3 Two inputs, three outputs; output spectral amplitudes

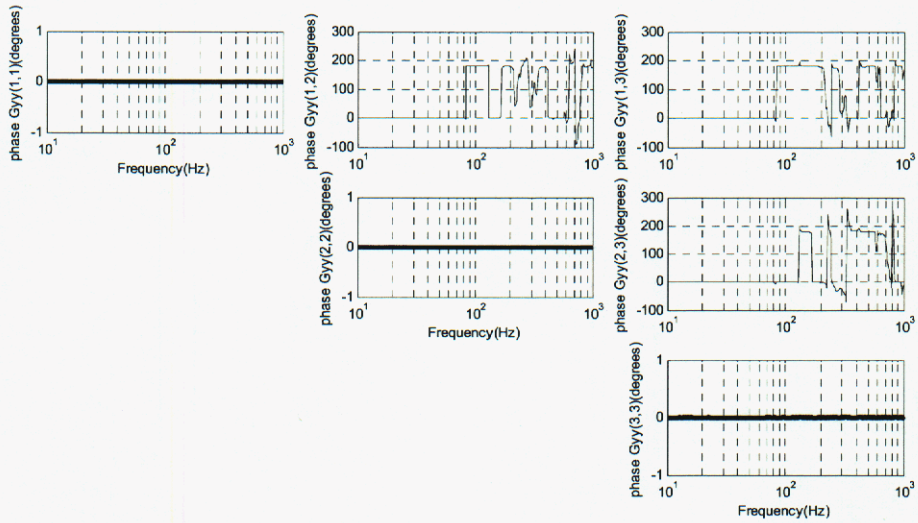

Figure 6.8.4 Two inputs, three outputs; output spectral phase

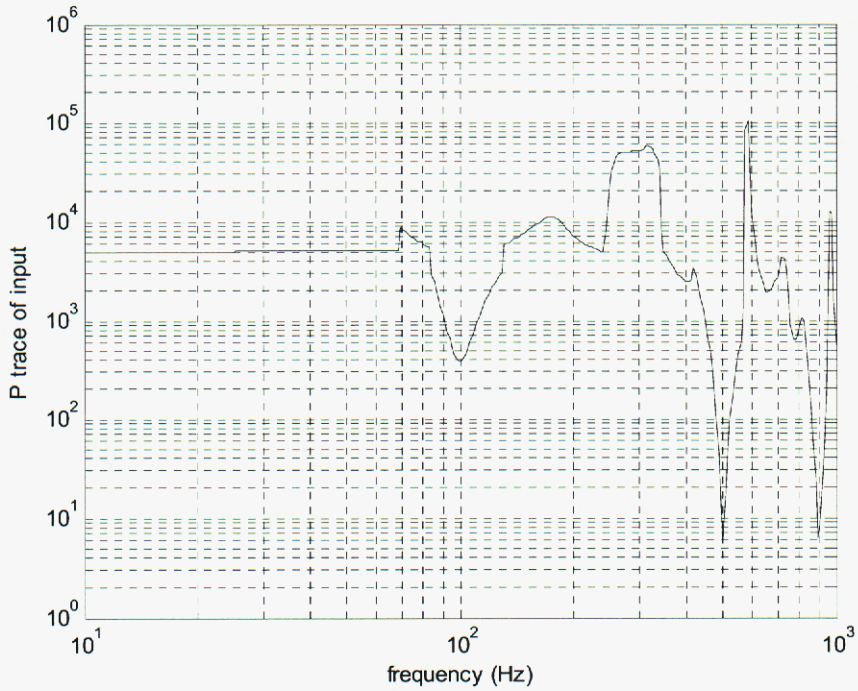

Figure 6.8.5 Two inputs, three outputs; input trace

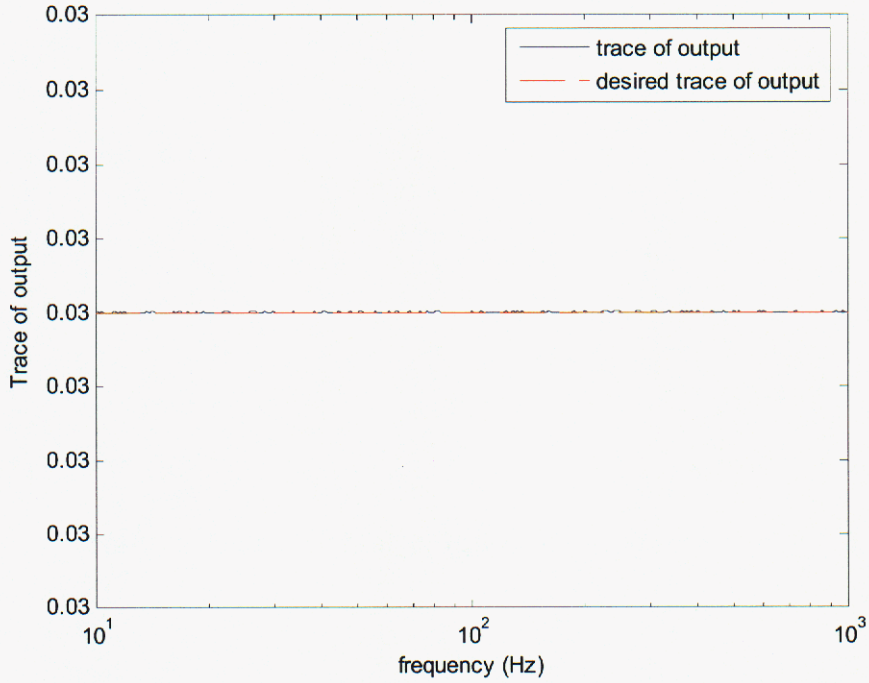

Figure 6.8.6 Two inputs, three outputs, output trace

6.9 Example 9 A beam driven transversely at both ends

In this example a free-free beam is excited in the transverse (lateral) direction. The beam is of unit length. The first free-free bending response is at 300 Hz. The input is force at the two extreme ends (x = 0 and 1). The response is acceleration at the same locations. The beam has a mass of 1000 lb. This system has two rigid body modes at zero frequency (translation and rotation). With two inputs and two responses the matrix of frequency response functions should not be singular at any frequency. The modal damping is set at 1%. The two by two matrix of frequency response functions is shown as Figure 6.9.1

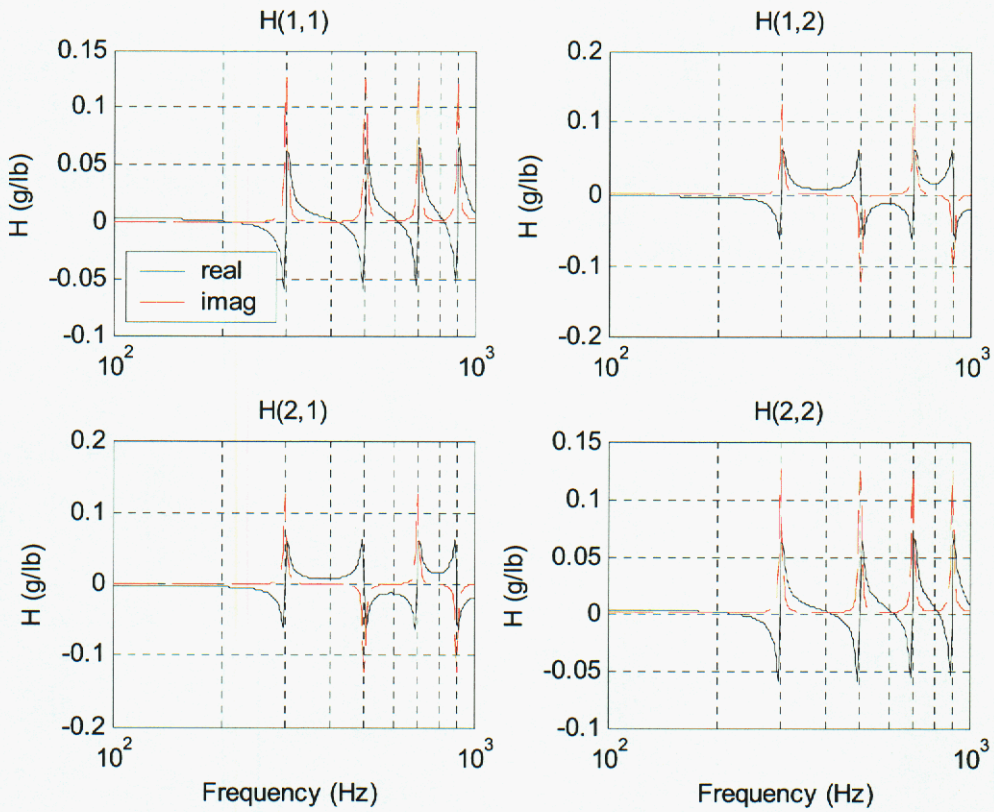

Figure 6.9.1 Beam frequency response functions

The output auto spectral densities are specified as 0.01 g^2/Hz from 100 to 1000 Hz. The minimum and maximum input auto spectral densities to achieve this level were calculated. The input and output spectral density matrices are shown in Figures 6.9.2 and 6.9.3. The trace of the input spectral density matrix (P) is shown as Figure 6.9.4. This curve is hard to trace with a black and white plot as the curve alternately follows the minimum and maximum curves. There are several items of interest in this example. First, the rigid body rotation mode is easier to excite than the rigid body translation at low frequencies, therefore the minimum input results in the input forces with a phase angle of 180° below about 120 Hz. The maximum input at low frequencies occurs with the input

forces in phase. The third curve plotted on Figure 6.9.4 is the trace of the inputs required to keep the outputs in phase over the whole frequency range. Note the input required is minimum at the free-free resonances of the beam as expected. The phase of the inputs is always near 0° or 180° with a fast transition between the two values. This is because of the low damping. The large peak in the maximum input (the input to keep the responses in phase) near 147 Hz deserves an explanation. Figure 6.9.5 plots the operating shape at this frequency when the inputs are in and out of phase. When the inputs are out of phase the shape is primarily rigid body rotation with a little of the first bending mode is excited. The rigid body translation is suppressed. When the inputs are in phase the shape is like the first bending mode of a pinned-pinned beam. Forcing the responses to be in phase has the effect of imposing a pinned-pinned boundary condition on the beam. The input source must look like large impedance to the beam to enforce this condition.

Other input conditions, (other phase angle and/or coherence of less than one) will result in the trace of the inputs between the two extremes of minimum and maximum conditions. The input trace is bounded between these two extremes.

The input auto spectral densities could also be specified and the extremes of the response auto spectra densities calculated. These extremes bound the responses for any inputs with the specified auto spectral densities.

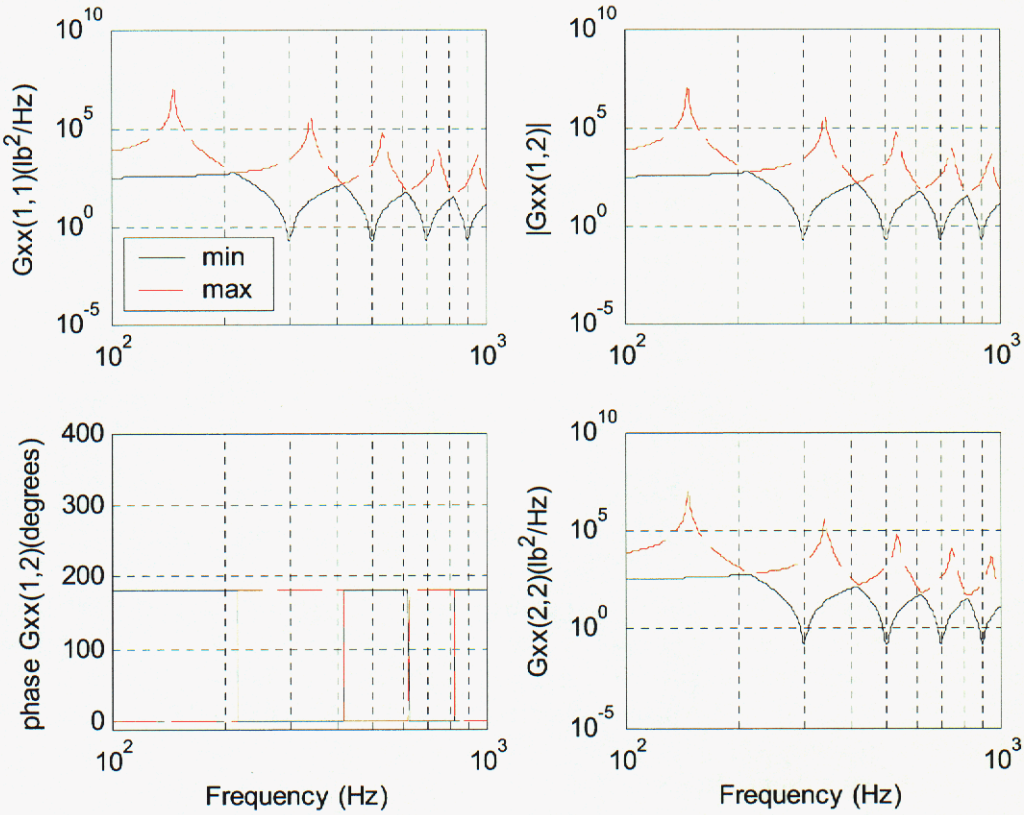

Figure 6.9.2 Input spectral density matrix

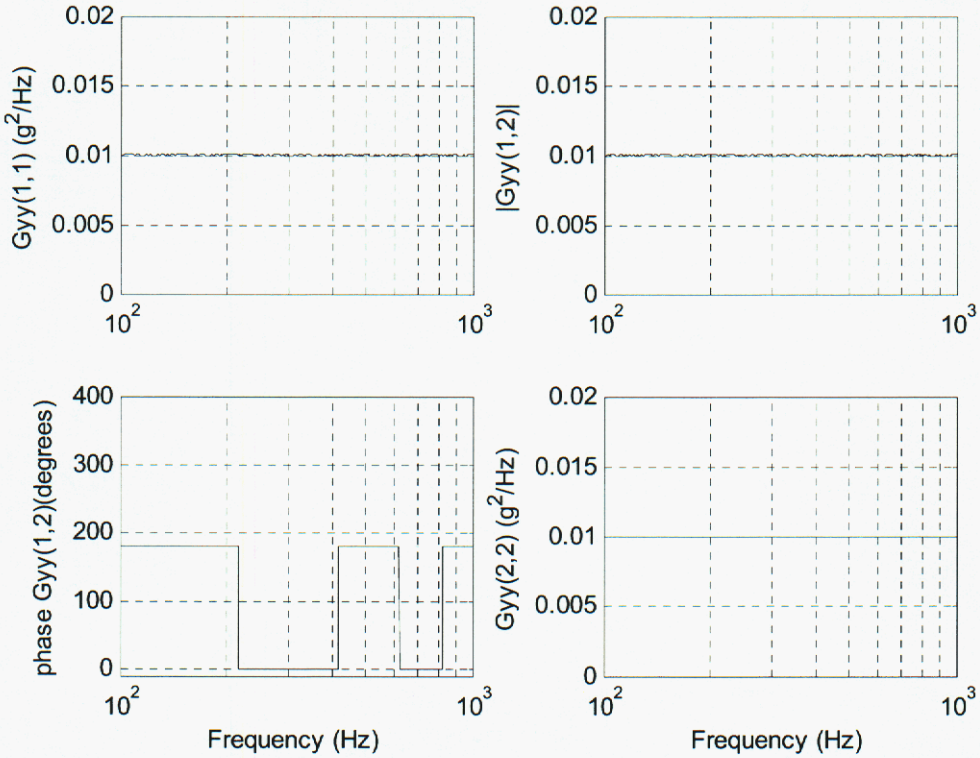

Figure 6.9.3 Output spectral density matrix

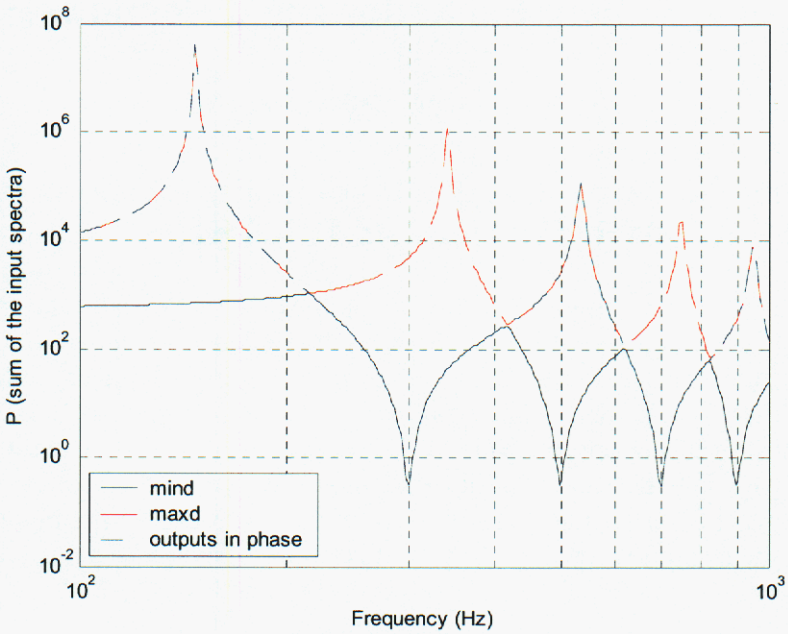

Figure 6.9.4 Trace of the input spectral density matrix

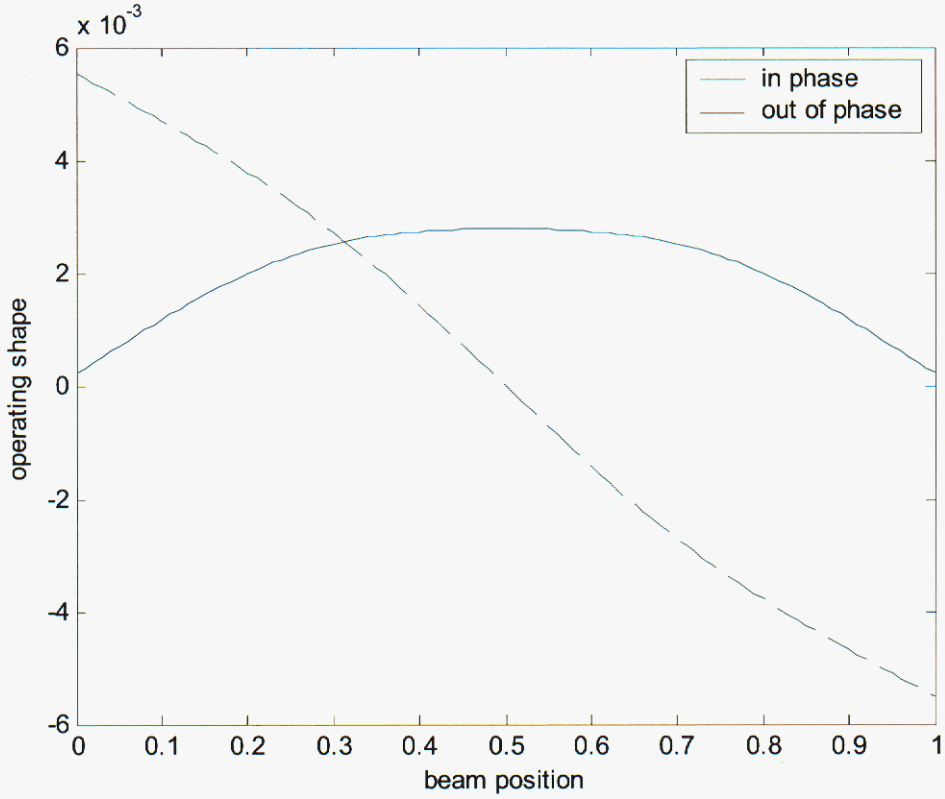

Figure 6.9.5 Operating shape of the beam near 47 Hz.

7. Examples of Transient Excitations and Responses

The same model as was used for the random examples 1-8 will be used.

7.1 Example 1, Input at points 0 and 1, response at locations 0 and 1, Response 1 is a decaying sinusoid at 100 Hz.

The desired response at point one (Figure 7.1.1) is defined as a exponentially decaying sinusoid at a 100 Hz, where a compensating waveform at 10 Hz has been added to force the initial and final, acceleration, velocity, and displacement to zero. The decay rate of the 100 Hz component is 2%. The decay rate of the compensating waveform is 1. This waveform will have most of its energy in a band of frequencies near 100 Hz. The second response is defined as having the same Fourier magnitude as the first response. The system is essentially a rigid body near 100 Hz. Therefore we would expect the inputs to be nearly in phase and the response of point 2 to be similar to the response at point 1. The inputs and responses for minimum input trace are shown in Figures 7.1.2-7.1.3.

7.2 Example 2, Input at points 0 and 1, response at locations 0 and 1, Response 1 is a decaying sinusoid at 400 Hz.

The desired response at point one (Figure 7.2.1) is defined as a exponentially decaying sinusoid at a 400 Hz, where a compensating waveform at 10 Hz has been added to force the initial and final acceleration, velocity, and displacement to zero. The decay rate of the 400 Hz component is 2%. The decay rate of the compensating waveform is 1. This waveform has most of its energy in a band of frequencies near 400 Hz. The second response is defined as having the same Fourier magnitude as the first response. The system is dominated by the first mode near 400 Hz. Therefore we would expect the inputs to be nearly out phase and the response of point 2 to be similar to the response at point 1 but inverted. The inputs and responses for minimum input trace are shown in Figures 7.2.2-7.2.3.

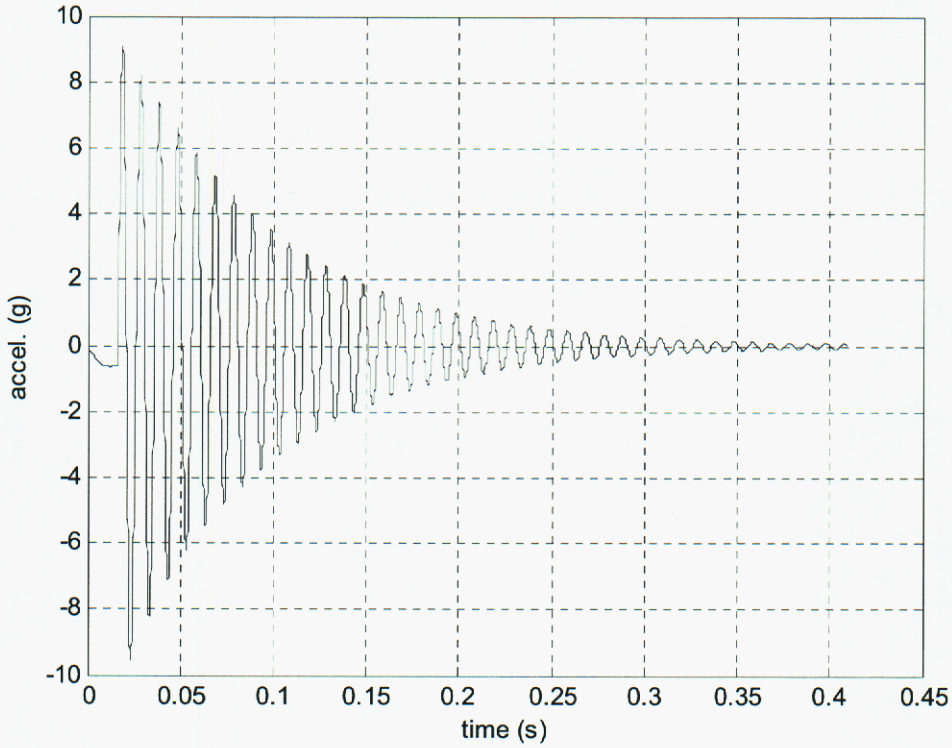

Figure 7.1.1 Desired response 1, 100 Hz exponentially decaying sinusoid

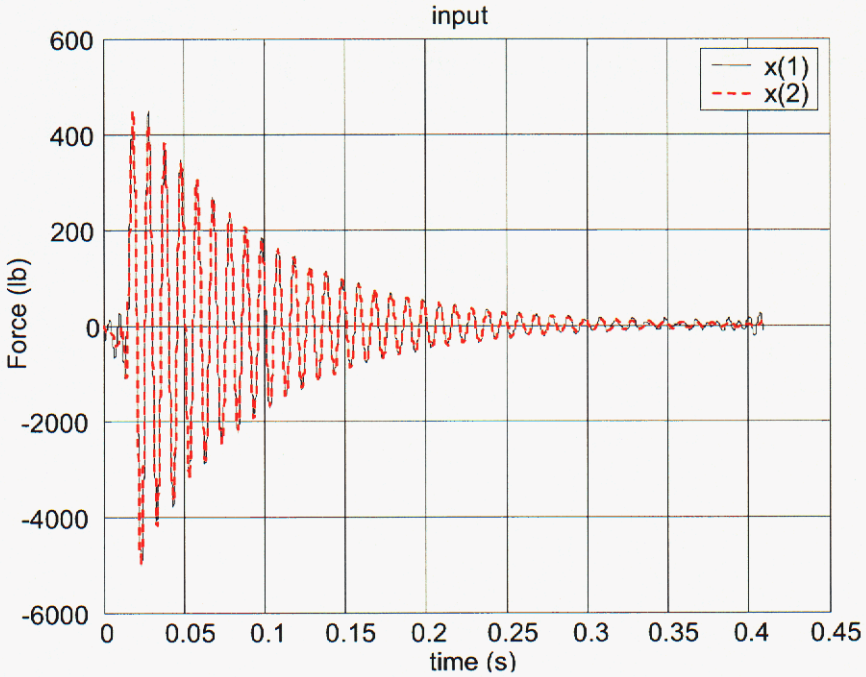

Figure 7.1.2 Input waveforms, 100 Hz exponentially decaying sinusoid

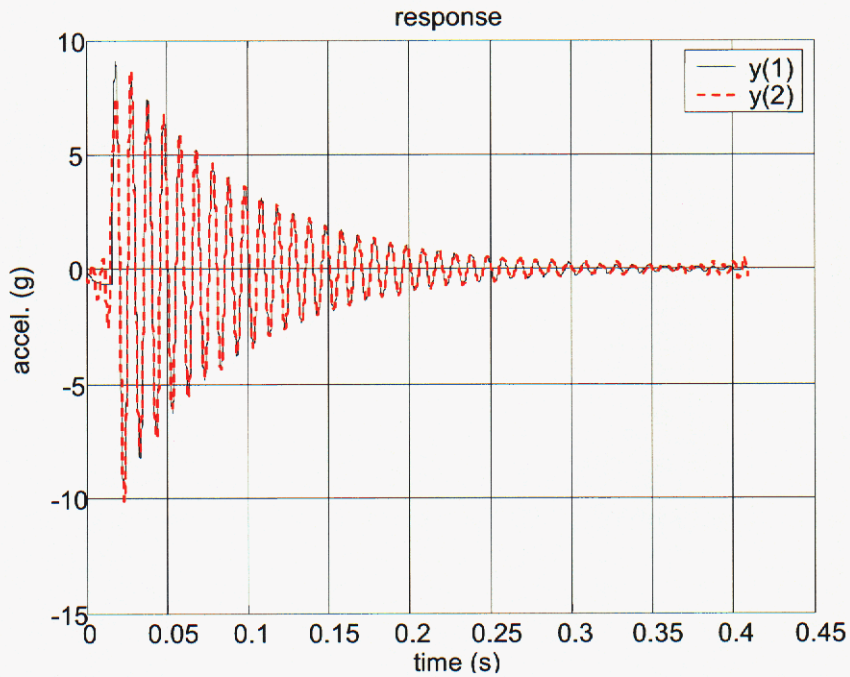

Figure 7.1.3 Response waveforms, 100 Hz exponentially decaying sinusoid

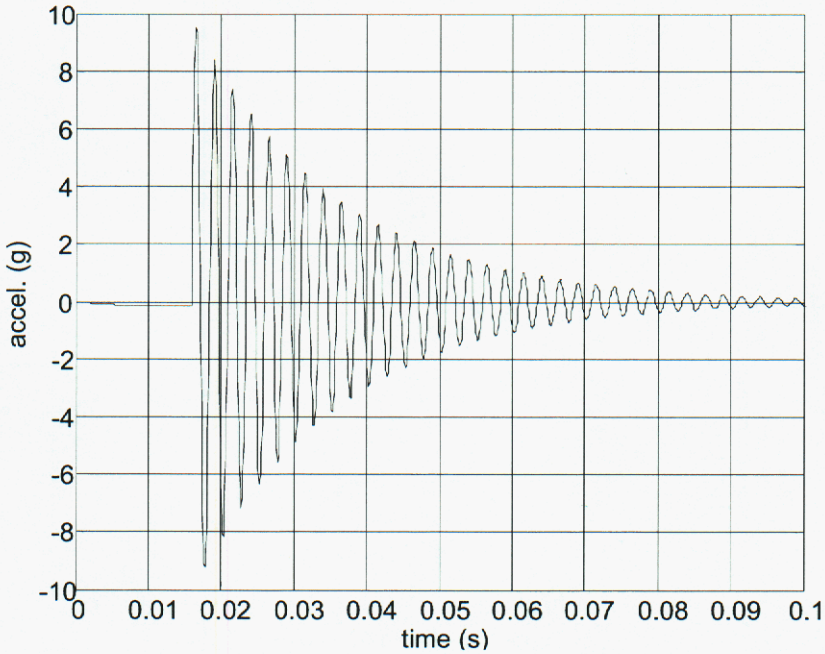

Figure 7.2.1 Desired response 1, 400 Hz exponentially decaying sinusoid

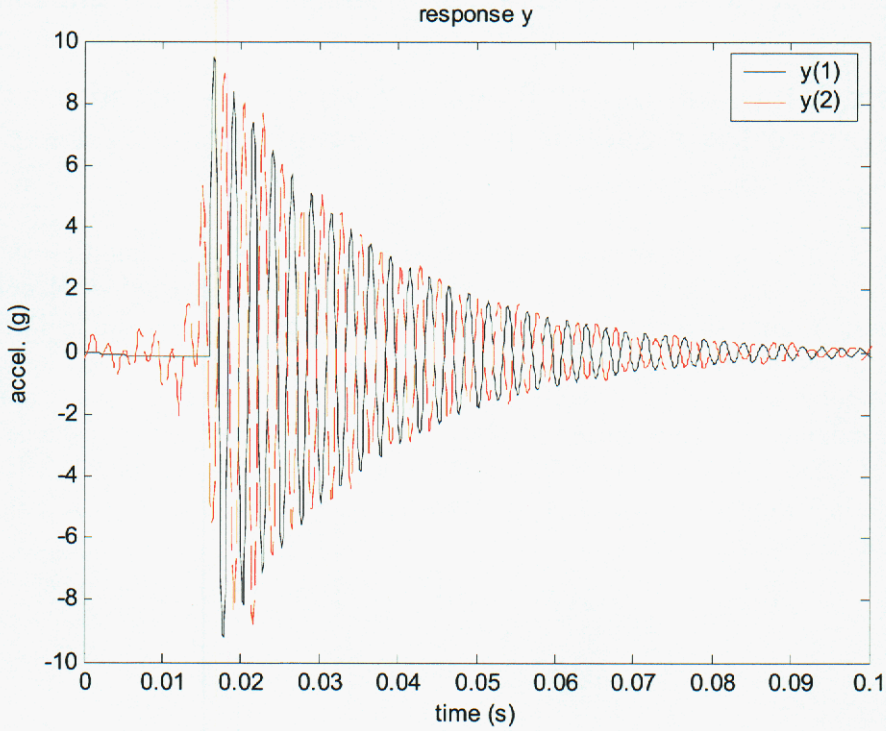

Figure 7.2.2 Response waveforms, 100 Hz exponentially decaying sinusoid

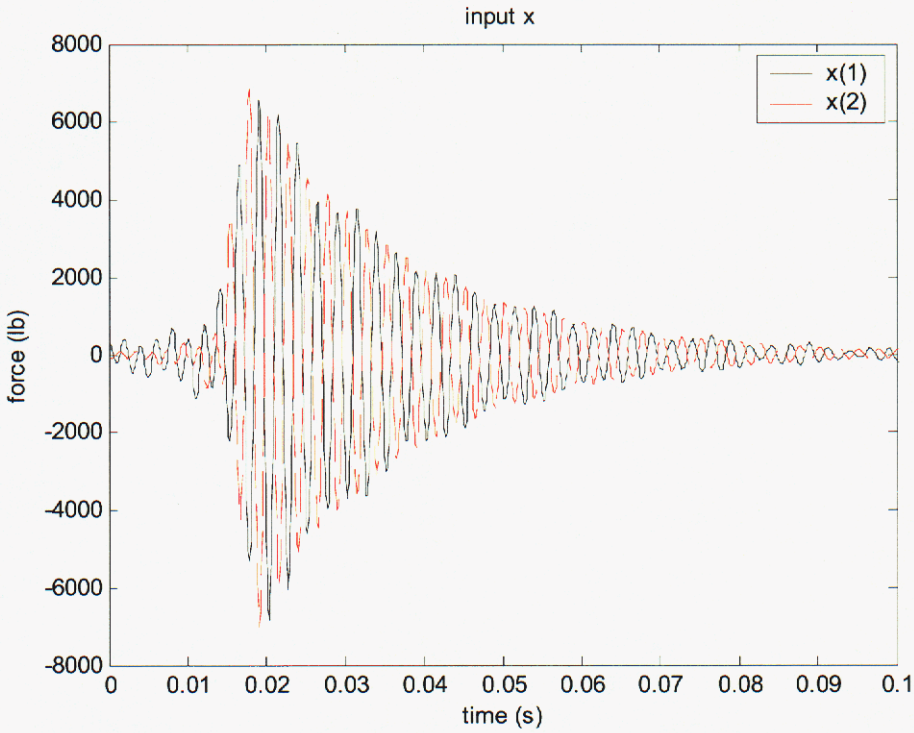

Figure 7.2.3 Input waveforms, 400 Hz exponentially decaying sinusoid

7.3 Example 3, Input at points 0 and 1, response at locations 0 and 1, Response 1 is a pseudo-random waveform with a t exp(-at) window.

For this example a realization of a stationary normal random process multiplied by a window function is used for the desired response at point 0. The window is defined as te^{-at}. The desired auto spectral density is shown in Figure 7.3.2 as a solid line. An estimate of the spectrum for a single realization is shown as a dotted line. A realization of the response at point 0 is shown in Figure 7.3.1. The desired auto spectral density at point 1 is the same as the desired auto spectral density as point 0. The inputs and responses for minimum input are shown in Figures 7.3.4 and 7.3.5. Notice that the response envelope at point 1 (shown in Figure 7.3.3) also has the character as a te^{-at} envelope even though it was not specified.

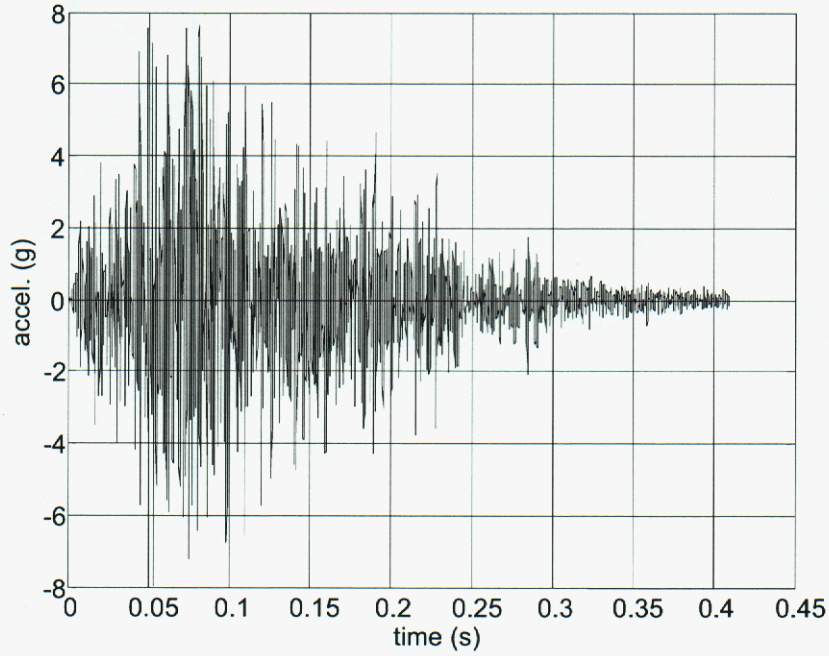

Figure 7.3.1 Desired response at point 0

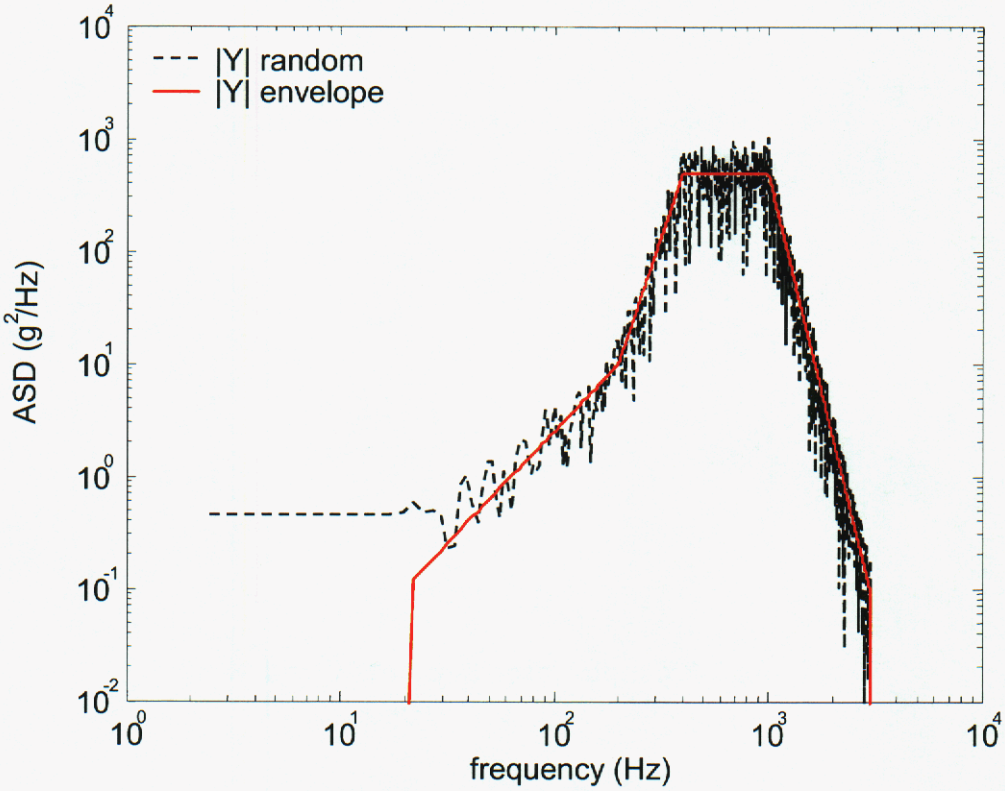

Figure 7.3.2 Desired spectrum envelope at point 0 and 1, and realized spectrum at point 0 for a single realization

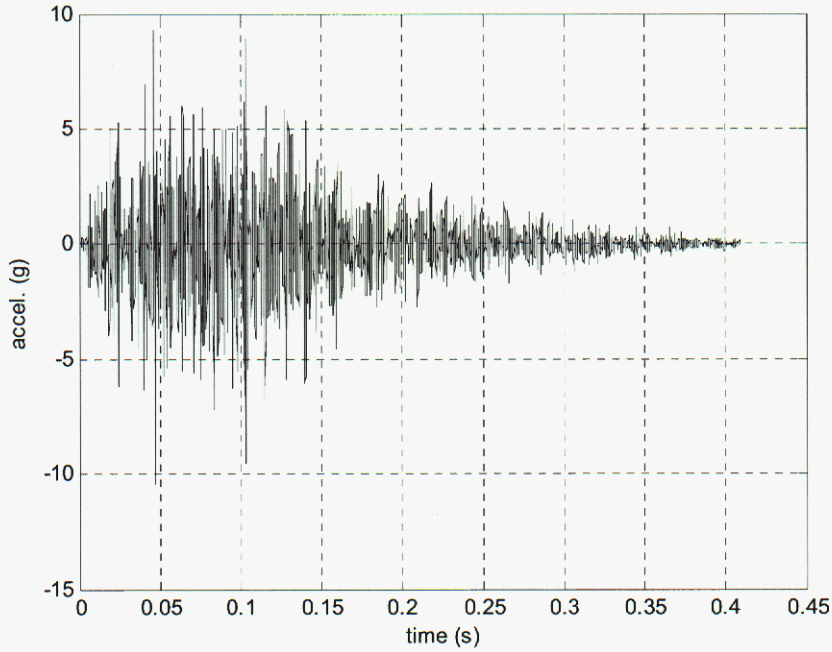

Figure 7.3.3 Response at point 1

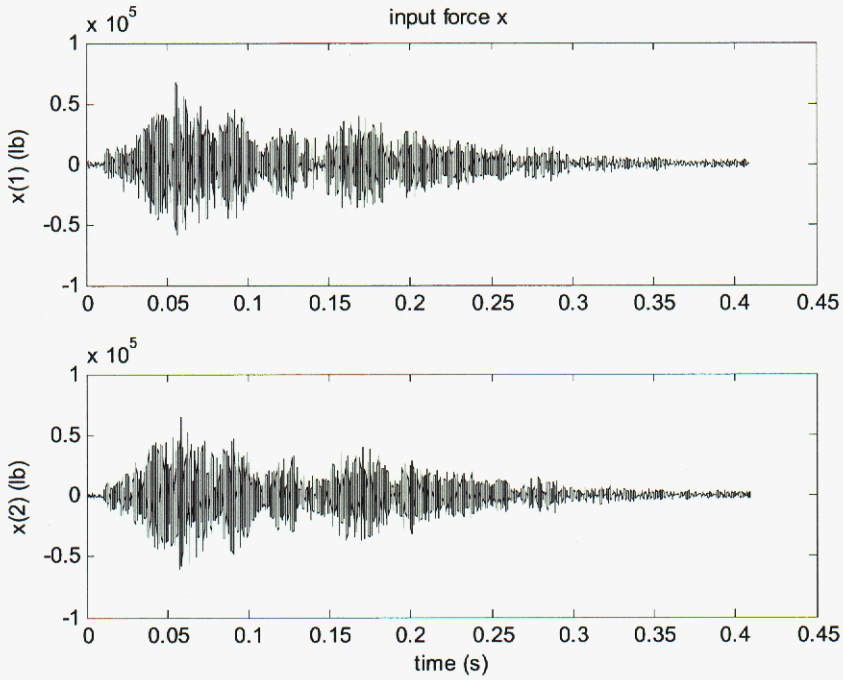

Figure 7.3.4 Input force at points 0 and 1.

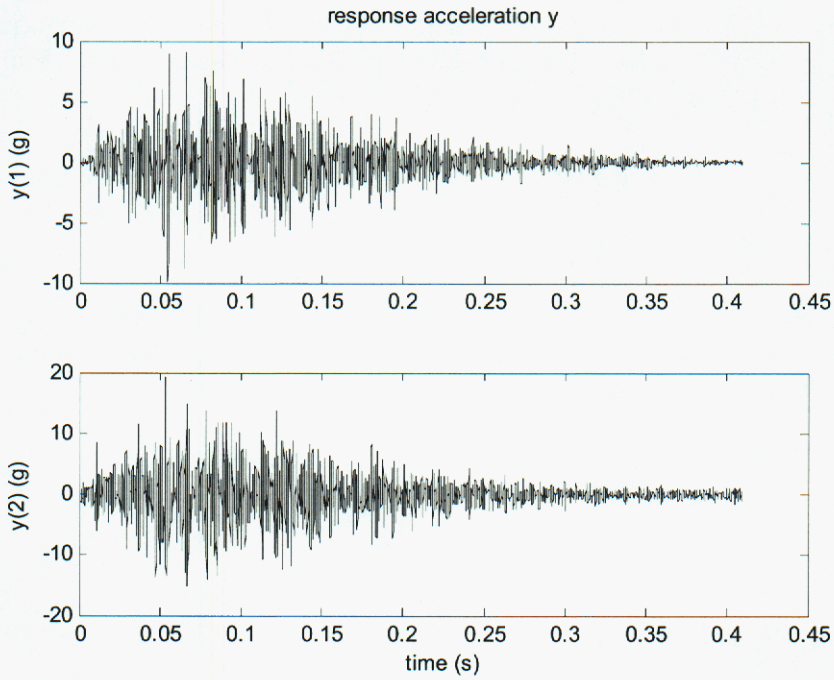

Figure 7.3.5 Response at points 0 and 1 for a second realization

Conclusions

A method has been derived for computing the minimum and maximum input (or the minimum and maximum output) for the multiple input, multiple output problem. The method has wide application for problems involving stationary random, nonstationary random, sine, and transient inputs. The method can be used in dynamic testing situations or dynamic simulations.

References

Baratta, A; Elishakoff, I; Zaccaro, G; Shinozuka, M., "A generalization of the Drenick-Shinozuka model for bounds on the seismic response of a single-degree-of-freedom system," Earthquake Engineering & Structural Dynamics; (1998) v.27, p.423-437.

Bendat, J.S. and Piersol, A.G., *Random Data, Analysis and Measurement Procedures*, 3rd edition, Wiley, 2000.

Drenick, Rudolf F., 1970, "Model-Free Design of Aseismic Structures," J. of Engr. Mech. Division, Proc. of ASCE, pp 483-493, Aug. 1970.

Lopez, OA; Chopra, AK; Hernandez, JJ, "Critical response of structures to multicomponent earthquake excitation," Earthquake Engineering & Structural Dynamics; Dec 2000; v.29, no.12, p.1759-1778.

Lopez, OA; Chopra, AK; Hernandez, JJ, "Evaluation of combination rules for maximum response calculation in multicomponent seismic analysis," Earthquake Engineering & Structural Dynamics; Sep 2001; v.30, no.9, p.1379-1398.

Paez, T. L., Smallwood, D. O., and Buksa, E. J., "Random Control at n^2 Points Using n Shakers,", Proc. of the IES (now the Institute of Environmental Sciences and Technology, IEST), pp 271-275, May 1987.

Segalman, D.J., Fulcher, C.W.G., Reese, G.M., and Field, R.V. Jr., "An Efficient Method for Calculating RMS von Mises Stress in a Random Environment," Journal of Sound and Vibration; 17 Feb. 2000; vol.230, no.2, p.393-410.

Shinozuka, Masanobu, 1970, "Maximum Structural Response to Seismic Excitations," J. of Engr. Mech. Division, Proc. of ASCE, pp 729-738, October 1970.

Smallwood, D. O., 1972, "An Extension of a Transient Vibration Technique Using Least Favorable Response," Sandia National Laboratories Report, SC-RR-720735, Nov. 1972.

Smallwood, D. O., 1973, "A Transient Vibration Test Technique Using Least Favorable Responses," Shock and Vibration Bulletin, No. 43, Part 1, pp 151-164, June 1973.

Smallwood, DO, Paez, TL, "A frequency domain method for the generation of partially coherent normal stationary time domain signals," Shock and Vibration, Vol.1, No. 1, pp45-53 (1993).

Smallwood, D. O., Woodall, T.D., Buksa, E.J., "Minimum drive requirements for a multiple input multiple output linear system," Proc. of the Institute of Environmental Sciences (now the Institute of Environmental Sciences and Technology) 1986, pp295-302.

Takewaki, I, "A new method for non-stationary random critical excitation," Earthquake Engineering & Structural Dynamics; April 2001; v.20; no.4, p.519-536.

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