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Spin Stabilized Magnetic Levitation of Horizontal Rotors

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Abstract

In this paper we present an analysis of a new configuration for achieving spin stabilized magnetic levitation. In the classical configuration, the rotor spins about a vertical axis; and the spin stabilizes the lateral instability of the top in the magnetic field. In this new configuration the rotor spins about a horizontal axis; and the spin stabilizes the axial instability of the top in the magnetic field.

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1 Introduction

Earnshaw's theorem [9] implies that it is impossible to achieve stable static magnetic levitation in a static magnetic field., However, the discovery of the Levitron^{TM} [7] has shown that it is in fact possible for a spinning top to be in stable equilibrium in a static magnetic field. We refer to this as spin stabilized magnetic levitation. There have been numerous papers analyzing spin stabilized magnetic levitation ,[1], [2],[6], [10], [5]. In this paper we extend these results by considering the case of a rotor that spins about a horizontal axis. Although no such device has yet been built, a program is currently under way to build one. A sketch of what such a device might look like is given in figures 1) and 2). As with the classical Levitron^{TM}, we anticipate that there will be a high degree of sensitivity in such a device, so that it may take an adept experimentalist to build one. For this reason we believe that it is worth presenting the theory even though there is as yet no experimental justification.

Classically, spin stabilized magnetic levitation devices are axisymmetric. In principle we could achieve a horizontally spinning device using systems of magnets that have no symmetry properties at all. However, we choose to consider systems that have enough symmetry so that equilibrium of forces and torques is guaranteed in all directions except for the vertical. One such situation (depicted in figure 1) is the following:

- The base magnets have reflectional symmetry about the planes $y = 0$ and $x = 0$.
- The rotor is axisymmetric and has reflectional symmetry about its midplane.
- The rotor is placed with its center of mass at $(0, 0, z_0)$ and its axis of symmetry pointing in the direction $(1, 0, 0)$.

We will show that due to the symmetry of this configuration, there are no forces in the y and x direction when the rotor is placed symmetrically in the field. Similarly, there are no torques in any direction. Equilibrium in the z direction can be obtained by adjusting the height or weight of the rotor.

A similar situation (depicted in figure 2) exists when the base magnets are anti-symmetric about the plane $x = 0$, and the magnets on the rotor are anti-symmetric with respect to reflections about the midplane of the rotor.

Earnshaw's theorem implies that this equilibrium position must be unstable if the rotor is not spinning. When we analyze the stability of a spinning rotor in such a configuration, we find that the equations for perturbations in the y and z directions decouple from the perturbations in the axial (x) direction, and from the angular perturbations. This implies that it is not possible for spinning to stabilize the perturbations in the y and z directions. If we are going to stabilize this configuration by spinning the rotor, the rotor must be unstable to perturbations in the axial direction (in the absence of spin). In certain situations we can stabilize the perturbations in the axial direction by spinning the rotor. As with the vertically spinning systems, there is an upper and lower spin rate for stable equilibrium.

We would like to emphasize that for spin stabilized magnetic levitation of a vertically spinning rotor in an axisymmetric field it is not possible to stabilize the axial direction by spinning. This means that in the absence of spin, the system is stable axially, and unstable laterally. This is exactly the opposite of horizontal spin stabilized systems that we discuss in this paper.

We now give an outline of the rest of this paper. In section 2) we discuss the symmetry properties of these configurations. In section 3) we show that these properties imply that when the rotor is placed symmetrically in the field, all of the forces and torques vanish except for the force in the vertical direction. In section 4) we derive the equations governing the linear stability of the equilibrium. In section 5) we give simple necessary conditions for stability, a simple stability condition similar to the adiabatic approximation made in [1], and a quartic equation that can be solved to determine the upper and lower spin rates. In section 6) we discuss how to compute the dynamical parameters in the linear stability equations for a given configuration of magnets. In section 7) we discuss how to find configurations of magnets that have the desired stability properties. We give our conclusions in section 8).

2 Symmetry Properties

We assume that the rotor and its magnets are axisymmetric, and that in equilibrium it is aligned with its axis of symmetry in the x direction, and that it spins about the x axis. In equilibrim its center of mass is at $\underline{x} = (0, 0, z_0)$. We consider two different situations.

- Systems where the supporting magnets produce a potential that is anti-symmetric with respect to a reflection about the plane $x = 0$, and symmetric with respect to a reflection about the plane $y = 0$. In this case we assume that the magnets on the rotor are anti-symmetric with respect to reflections about the mid-plane.

- Systems where the supporting magnets produce a potential that is symmetric with respect to reflections about the planes $x = 0$ and $y = 0$. In this case we assume that the magnets on the rotor are symmetric with respect to a reflection about the mid-plane.

We will show that in both of these situations when the center of mass of the rotor is at $y = 0$,$x = 0$, and the axis of symmetry of the rotor is aligned in the x direction, we are guaranteed of having no forces in the y or x direction, and no torques on the rotor. By suitably adjusting the weight of the rotor, or the strengths of the magnets, we can make it so that the force in the z direction balances the force of gravity, which we assume points in the z direction .

The first of these symmetries can be constructed by building a rotor with two dipoles on the axis of symmetry, symmetrically located about the midplane, and both pointing in the same direction along the axis of symmetry. In this case a system of supporting magnets having the proper symmetry could consist of magnets in a plane $z =$ *constant* all pointing in the z direction. In this case any supporting magnet at (x_0, y_0, z_0) would have companion magnets at $(\pm x_0, \pm y_0, z_0)$. The dipole at $(x_0, -y_0, z_0)$ would be in the same direction as the first dipole, and the dipoles at $(-x_0, \pm y_0, z_0)$ would be in the opposite direction. This is just one example of how to achieve this symmetry. More generally we could have the magnets in the base have the dipoles pointing in arbitrary directions as long as their companion magnets have been appropriately reflected.

The second of these symmetries can be constructed by building a rotor with two dipoles on the axis of symmetry, symmetrically placed about the midplane, and pointing in opposite directions along the axis of symmetry. In this case a system of supporting magnets having the proper symmetry could consist of magnets in a plane $z =$ *constant* all pointing in the z direction. In this case any supporting magnet at (x_0, y_0, z_0) would have companion magnets at $(\pm x_0, \pm y_0, z_0)$. All of the magnets would have their dipoles pointing in the same direction. Once again, this is just one way of achievng systems with this symmetry.

Since the rotor is axisymmetric, the energy of the rotor in an arbitrary magnetic field can be written as

$$Energy = U(\underline{x}, \underline{d})$$

where $\underline{x} = (x, y, z)$ is the center of mass of the rotor, and $\underline{d} = (d_x, d_y, d_z)$ is a unit vector pointing in the direction of the axis of symmetry. The energy satisfies

$$\nabla_x^2 U = 0$$

where ∇_x^2 is the Laplacian with respect to the variable $\underline{x}$.

The energy of systems where the potential is anti-symmetric with respect to reflections about the x axis satisfy the following symmetry properties.

$$U(x, y, z, d_x, d_y, d_z) = U(-x, y, z, d_x, -d_y, -d_z) \tag{1a}$$

$$U(x, y, z, d_x, d_y, d_z) = U(x, -y, z, d_x, -d_y, d_z) \tag{1b}$$

$$U(x, y, z, d_x, d_y, d_z) = -U(x, y, z, -d_x, -d_y, -d_z) \tag{1c}$$

Systems where the potential is symmetric with respect to reflections about the x axis satisfy the identical symmetry properties.

2.1 Examples Illustrating the Symmetry Properties

These symmetry properties become clearer if we consider special cases of such systems. Suppose we have a rotor that has two equal dipoles on the axis of symmetry, each pointing in the direction of the axis of symmetry. We suppose that the magnets are placed symmetrically a distance $\delta/2$ from the center of mass When the rotor gets displaced and rotated, one of the dipoles will be located at $\underline{x}_+ = \underline{x} + \delta/2\underline{d}$, and the other one at $\underline{x}_- = \underline{x} - \delta/2\underline{d}$. The dipole moment of the magnet at $\underline{x}_+$ will be $\underline{m}_+ = m_0\underline{d}$, and the moment at $\underline{x}_-$ will be $\underline{m}_- = m_0\underline{d}$. The total magnetic energy of the rotor will be

$$U(\underline{x}, \underline{d}) = m_0 \left(\underline{d} \cdot \nabla\phi(\underline{x}_+) + \underline{d} \cdot \nabla\phi(\underline{x}_-)\right)$$

It can be verified that assuming that $\phi(x, y, z)$ is symmetric in y and anti-symmetric in x, the energy $U(\underline{x}, \underline{d})$ satisfies the symmetry properties stated in 1). Note that these symmetry properties would hold for more complicated systems, such as rotors having more than one pair of symmetrically placed dipoles, or symmetrically placed rings.

An example illustrating the second sort of symmetry comes from a rotor that once again has symmetrically placed dipoles, but in this case the dipoles are equal and opposite to each other. In this case the energy can be written as

$$U(\underline{x}, \underline{d}) = m_0 \left(\underline{d} \cdot \nabla\phi(\underline{x}_+) - \underline{d}\nabla\phi(\underline{x}_-)\right)$$

Once again it can be verified that if $\phi(x, y, z)$ is symmetric in x and y, then the energy satisfies the symmetry properties 1).

3 Equilibrium

We will now show that assuming our system of magnets and the rotor satisfy the symmetry properties of the last section, we can easily find equilibrium configurations. In particular, we will show that if we place the rotor so that its center of mass is at $(0, 0, z_0)$, and its axis of symmetry is pointing in the direction $(1, 0, 0)$, then there is no torque on the rotor, and the only component of force is in the z direction. By appropriately adjusting the weight or the strengths of the magnets, we can make it so that the force of gravity balances this magnetic force.

The force and torque on the rotor can be computed using

$$\underline{F} = -\nabla_x U(\underline{x}, \underline{d})$$
$$\underline{\tau} = -\underline{d} \times \nabla_d U(\underline{x}, \underline{d})$$

Here, ∇_x is the gradient with respect to $\underline{x}$, and ∇_d is the gradient with respect to $\underline{d}$.

We can derive these formulas using generalizations of the derivations for the force and torque on a point dipole [3]. The principle of virtual work tells us that the change in energy when we move the center of mass without rotating it is given by

$$\delta U = -\underline{F} \cdot \delta \underline{r}$$

where $\underline{F}$ is the force on the top, and $\delta \underline{r}$ is the change in the center of mass of the top. Since we can write $\delta U = \nabla_x U \cdot \delta \underline{r}$, we see that

$$\nabla_x U \cdot \delta \underline{r} = -\underline{F} \cdot \delta \underline{r}$$

Since this must hold for all values of $\delta \underline{r}$ we see that

$$\underline{F} = -\nabla_x U.$$

On the other hand, if we rotate the body about the axis $\underline{e}$ by an angle $\delta\theta$,then the principle of virtual work requires that the change in energy is given by

$$\delta U = -\tau \cdot \underline{e}\delta\theta$$

When we rotate the body about $\underline{e}$ by $\delta\theta$, the change in the unit vector $\underline{d}$ is given by $\delta \underline{d} = \underline{e} \times \underline{d}\delta\theta$. We see that

$$\delta U = \nabla_d U \cdot \delta \underline{d} = \nabla_d U \cdot (\underline{e} \times \underline{d})\delta\theta = (\underline{d} \times \nabla_d U) \cdot \underline{e}\delta\theta$$

When we equate this expression to the expression from the principle of virtual work, and require that it hold for all values of $\underline{e}$ and θ we get

$$\underline{\tau} = -\underline{d} \times \nabla_d U.$$

The symmetry properties of the energy show that for both the anti-symmetric and symmetric cases we have

$$U(x,0,z,1,0,0) = U(-x,0,z,1,0,0)$$
$$U(0,y,z,1,0,0) = U(0,-y,z,1,0,0)$$

When the rotor is placed symmetrically in the field, the forces F_x and F_y in the x and y directions satisfy

$$F_x(0,0,z,1,0,0) = -\frac{\partial U(0,0,z,1,0,0)}{\partial x} = 0$$
$$F_y(0,0,z,1,0,0) = -\frac{\partial U(0,0,z,1,0,0)}{\partial y} = 0$$

To show that the torques vanish, we substitute $x = 0$, $y = 0$ into the symmetry propertry $U(x,y,z,d_x,d_y,d_z) = U(-x,y,z,d_x,-d_y,-d_z)$ to get

$$U(0,0,z,1,d_y,d_z) = U(0,0,z,1,-d_y,-d_z)$$

This shows that the energy at $x = y = 0$ is an even function of d_y and d_z, and hence the derivatives with respect to d_y and d_z must vanish. Using the fact that $\underline{\tau} = -\underline{d} \times \nabla_d U$ we see that

$$\underline{\tau}(0,0,z,1,0,0) = 0$$

We see that based on the symmetry of our problem, if we put the rotor so that its center of mass is at $x = y = 0$, and so that its axis of symmetry is pointing in the x direction; there will be no forces in the x or y directions, and no torques at all.

4 The Linearized Equations of Motion

We describe the kinematics of the rotor in a manner similar to [6]. In our discussion the coordinates (x, y, z) refer to coordinates fixed in space. We assume that the body is axisymmetric with a moment of inertia of I_3 about the axis of symmetry, and I_1 about the other two principal axes.

We will orient the body by rotating about the z axis by θ, the y axis by ϕ and then the x axis by ψ. If the rotor is spinning about the x axis with angular velocity ω_0, then a small perturbation to this state gives approximate angular momenta L_y and L_z of

$$L_y = I_1\dot{\phi} + I_3\omega_0\theta$$
$$L_z = I_1\dot{\theta} - I_3\omega_0\phi$$

These formulas can be derived rigorously by expressing the angular momenta in terms of the the angular variables and their derivatives, and then assuming that θ and ϕ are small. They also have a simple intuitive interpretation. The expression for L_y consists of two terms. The first term is the angular momentum we would get if ω_0 were zero, and the body were spinning about the y axis. The second term is the angular momentum we would get if the body kept spinning about the axis of symmetry with angular velocity ω_0, but was slowly tilted by an amount θ about the x axis. As a result of this tilting some of the angular momentum that was intially in the x direction gets projected onto the y axis. A similar interpretatiion can be given for the angular momentum in the z direction.

The linearized equations of motion can be written

$$m\ddot{x} = F_x(x, y, z, \theta, \phi)$$
$$m\ddot{y} = F_y(x, y, z, \theta, \phi)$$
$$m\ddot{z} = F_z(x, y, z, \theta, \phi)$$

$$I_1\ddot{\theta} - I_3\omega_0\dot{\phi} = \tau_z(x, y, z, \theta, \phi)$$
$$I_1\ddot{\phi} + I_3\omega_0\dot{\theta} = \tau_y(x, y, z, \theta, \phi)$$

In the linear approximation, the forces and torques are linear functions of (x, y, z, θ, ϕ). In the linear approximation, we have

$$\underline{d} = (d_x, d_y, d_z) = (1, \theta, -\phi)$$

Also, in the linear approximation the forces and torques are derivable from a quadratic potential. The symmetry properties show that many of the terms in the quadratic potential must be missing. For example, the fact that $U(x, y, z, d_x, d_y, d_z) = U(x, -y, z, d_x, -d_y, d_z)$ implies that the Taylor series expansion of the energy cannot have any terms of the form xy, yz, $y\phi$, $x\theta$, $z\theta$, or $\theta\phi$. . The fact that $U(x, y, z, d_x, d_y, d_z) = U(-x, y, z, d_x, -d_y, -d_z)$ implies that we cannot have any terms of the form xy, xz , $y\phi$, $y\theta$, $z\phi$, or $z\theta$. Using these symmetry properties we conclude that the linearized equations of motion are of the form

$$m\ddot{y} + A_1 y = 0$$
$$m\ddot{z} + A_2 z = 0$$
$$m\ddot{x} - Ax - B\phi = 0$$
$$I_1\ddot{\theta} - I_3\omega_0\dot{\phi} - C_1\theta = 0$$
$$I_1\ddot{\phi} + I_3\omega_0\dot{\theta} - C_2\phi - Bx = 0$$

Note that the equations for y and z decouple from the other equations. This means that in order to have stability we must have A_1 and A_2 both be bigger than zero. In other words, the system would have to be stable to lateral perturbations if the rotor were not spinning. The fact that $\nabla_x^2 U = 0$ (or Earnshaw's theorem) implies that $A_1 + A_2 = A$, and hence the system must be unstable to axial perturbations if the rotor is not spinning.

4.1 The Dimensionless Equations of Motion

We now introduce the dimensionless variables

$$x = \sqrt{I_1/m}\,\hat{x}$$
$$t = \sqrt{m/A}\,\hat{t}$$

In terms of these dimensionless variables, we get the dimensionless equations (after dropping the hats for notational convenience)

$$\ddot{x} - x - \sqrt{\Lambda}\phi = 0 \tag{2a}$$
$$\ddot{\theta} - \Omega\dot{\phi} - \Gamma_1\theta = 0 \tag{2b}$$

$$\ddot{\phi} + \Omega\dot{\theta} - \Gamma_2\phi - \sqrt{\Lambda}x = 0 \tag{2c}$$

Here we have introduced the dimensionless parameters

$$\Gamma_1 = \frac{mC_1}{I_1 A} \tag{3}$$

$$\Gamma_2 = \frac{mC_2}{I_1 A} \tag{4}$$

$$\Lambda = \frac{mB^2}{I_1 A^2} \tag{5}$$

$$\Omega^2 = \frac{I_3^2\omega_0^2 m}{I_1^2 A} \tag{6}$$

5 The Stability of the Equilibrium

We now analyze the stability of the system of equations 2). In the first subsection we compute the characteristic equation governing the stability, and give some necessary conditions for stability.. In the next subsection we carry out an analysis assuming that Γ_1, Γ_2, and Λ are all large. This analysis gives very simple criteria for stability, and we believe it is similar to making the adiabatic assumption as in [1] (see the discussion in Appendix B) . In the next subsection we use results from the theory of polynomials that allow us to predict the exact upper and lower spin rates by solving a quartic equation. This is similar to the procedure carried out in [2] in the analysis of the vertically spinning Levitron^{TM}.

5.1 The Characteristic Equation and its Properties

We now assume solutions of the form $e^{i\sigma t}$ in the linearized dynamical equations. This leads to the characteristic polynomial

$$(\sigma^2 + 1)\left((\sigma^2 + \Gamma_1)(\sigma^2 + \Gamma_2) - \Omega^2\sigma^2\right) - \Lambda(\sigma^2 + \Gamma_1) = 0 \tag{7}$$

Expanding this we get

$$G(q,\Omega) = q^3 + q^2\left(1 + \Gamma_1 + \Gamma_2 - \Omega^2\right) + q\left(\Gamma_1 + \Gamma_2 + \Gamma_1\Gamma_2 - \Lambda - \Omega^2\right) + \Gamma_1\Gamma_2 - \Lambda\Gamma_1 = 0 \tag{8}$$

where

$$q = \sigma^2.$$

In order for our system to be stable, all of the roots of equation 8) must be real and positive. Descartes theorem [8] implies that for an equation of the form

$z^3 + px^2 + qz + r = 0$ to have all real and positive roots, it is necessary that $p < 0$, $q > 0$, and $r < 0$. Furthermore, if all of the roots are real, then these conditions are both necessary and sufficient conditions for all of the roots to be positive. This, along with the condition that $A > 0$ gives us several necessary conditions for stability

$$\Omega^2 > 1 + \Gamma_1 + \Gamma_2 \tag{9a}$$

$$\Gamma_1 + \Gamma_2 + \Gamma_1\Gamma_2 - \Lambda > \Omega^2 \tag{9b}$$

$$\Lambda\Gamma_1 > \Gamma_1\Gamma_2 \tag{9c}$$

$$\Lambda > 0 \tag{9d}$$

The last of these conditions is the requirement that $A > 0$ in order to have lateral stability. As with the vertically spinning spin stabilized magnetic levitation, we see that there is both an upper and a lower value of Ω for stability.

5.2 Asymptotic Stability Analysis

We can gain considerable insight into these equations by analyzing their behavior when Γ_1, Γ_2 and Λ are all large. We claim that this is similar to making the adiabatic approximation as in [1]. We elaborate on the connection between our asymptotic stability criterion and the adiabatic approximation in appendix B.

To be precise, we assume that

$$\Lambda = \lambda/\epsilon^2$$

$$\Gamma_1 = \gamma_1/\epsilon^2$$

$$\Gamma_2 = \gamma_2/\epsilon^2$$

$$\Omega = \omega/\epsilon$$

If we substitute these expression into 7), multiply by ϵ^4, and set $\epsilon = 0$, we get the equation

$$\sigma^2\gamma_2 = \lambda - \gamma_2$$

This gives us two roots of our 6 th order polynomial. We can only have positive solutions to σ^2 if

$$\gamma_2 > 0,$$

and

$$\lambda - \gamma_2 > 0.$$

We get the four other roots by assuming that $\sigma = \hat{\sigma}/\epsilon$ This gives us the equation

$$\hat{\sigma}^2\left((\hat{\sigma}^2+\gamma_1)(\hat{\sigma}^2+\gamma_2)-\Omega^2\hat{\sigma}^2\right)=0$$

After factoring out $\hat{\sigma}^2$ this is the characteristic equation for for a spinning rotor in a harmonic potential.

$$(\hat{\sigma}^2+\gamma_1)(\hat{\sigma}^2+\gamma_2)-\Omega^2\hat{\sigma}^2=0$$

A simple application of the quadratic equation shows that in order for this to have all real roots we must have $\Gamma_1\Gamma_2 > 0$, which along with our previous stability criterion requires that both Γ_1 and Γ_2 be positive. We must also have $\Gamma_1+\Gamma_2-\Omega^2 < 0$, and $(\Gamma_1+\Gamma_2-\Omega^2)^2-4\Gamma_1\Gamma_2 > 0$. By choosing Ω large enough we can satisfy all of these criterion.

We can give a simple interpretation of these stability conditions. If Γ_1, Γ_2 and Λ are large, and the system is not responding too quickly, the second of equations 2) implies that

$$\Gamma_2\phi+\sqrt{\Lambda}x=0$$

This is equivalent to saying that as the rotor moves around, it orients itself so that there is no torque on it. This gives us the expression $\phi=-\sqrt{\Lambda}x/\Gamma_2$. When we substitute this into the first of equations 2) we get

$$\ddot{x}+x(\Lambda/\Gamma_2-1)=0$$

We see that this will be a stable harmonic oscillator provided $\Lambda > \Gamma_2$. This is the first of our asymptotic stability conditions. In order to satisfy this condition we must have $\Gamma_2 > 0$, which implies that the rotor would want to flip over in the absence of spin.

Our second criterion is the condition that we are spinning the rotor fast enough that it will not flip over. To analyze this mode we have assumed that σ is order $1/\epsilon$. In this case, the first of equations 2) implies that x is small compared to ϕ. This means that we can solve the second and third equations ignoring x. This is equivalent to considering a rotor spinning in a potential where we ignore the translational energy. This leads to our second stability condition.

The asymptotic analysis we just presented does not predict the existance of an upper spin rate. In order to predict the upper spin rate we once again assume that Γ_1, Γ_2 and Λ are large. We will see that if Ω is too large, the eigenvalues that are order one will eventually go unstable.

Assuming that σ is order unity, and that all of our parameters are large, our eigensystem can be approximated by

$$(\sigma^2+1)x+\sqrt{\Lambda}\phi=0$$
$$-i\Omega\sigma\phi-\Gamma_1\theta=0$$
$$i\sigma\Omega\theta-\Gamma_2\phi-\sqrt{\Lambda}x=0$$

These equations are obtained by ignoring the second derivatives of θ and ϕ in equations 2). They are an extension of the results we have already presented where we ignore all derivatives of these quantities.

These equations imply that

$$(\sigma^2 + 1)\left(\Gamma_1\Gamma_2 - \Omega^2\sigma^2\right) - \Gamma_1\Lambda = 0$$

This is a quadratic equation for σ^2. We need this equation to have positive real roots. In order for this to be so we must have $\Gamma_1\Gamma_2 > \Omega^2$, $\Lambda > \Gamma_2$ and

$$(z - \Gamma_2)^2 - 4z(\Lambda - \Gamma_2) > 0$$

where

$$z = \frac{\Omega^2}{\Gamma_1}$$

This gives a quadratic equation in z whose roots are

$$z_{\pm} = 2\Lambda - \Gamma_2 \pm \sqrt{(2\Lambda - \Gamma_2)^2 - \Gamma_2^2} \tag{10}$$

In order to have real roots we must have $z < z_-$ or $z > z_+$. However, if $z > z_+$, we cannot satisfy the other inequalities necessary to have positive real roots, It follows that we must have $\frac{\Omega^2}{\Gamma_1} < z_-$. This is the asymptotic prediction for the upper spin rate. Note that assuming that Γ_1, Γ_2 and Λ are order $1/\epsilon^2$, this upper limit on the spin rate is also of order $1/\epsilon^2$. On the other hand, the lower spin rate is on the order $1/\epsilon$. It follows that as we make ϵ smaller, the ratio of the upper and lower spin rate can be made very large.

We will now collect all of our results from the asymptoti stability analysis. Assuming that $\Gamma_1 = \gamma_1/\epsilon$, $\Gamma_2 = \gamma_2/\epsilon$, $\Lambda = \lambda/\epsilon^2$, we see that necessary and sufficient conditions for stability are

$$\Gamma_1 > 0 \tag{11a}$$

$$\Gamma_2 > 0 \tag{11b}$$

$$\Lambda > \Gamma_2 \tag{11c}$$

$$\Omega^2 > \Gamma_1 + \Gamma_2 + 2\sqrt{\Gamma_1\Gamma_2} \tag{11d}$$

$$\Omega^2 < \Gamma_1 z_- \tag{11e}$$

Once again we emphasize that if Γ_1, Γ_2 and Λ are order $1/\epsilon^2$, then the lower spin rate is order $1/\epsilon$, and the upper spin rate is order $1/\epsilon^2$. This shows that as we keep the ratios of Γ_1, Γ_2, and Λ fixed but let the quantities get large, the ratio of the upper and lower spin rates also gets large.

In the next section we will show that by finding the roots of a fourth order polynomial we can find exact expressins (that must be computed numerically) for the upper and lower spin rates. Figure 3) shows that our asymptotic estimates for the upper and lower spin rates are in fact quite accurate even for moderate values of Γ_1, Γ_2, and Λ.

5.3 Upper and Lower Bounds on the Spin Rate

We will now find an exact expression for determining the upper and lower spin rates. In order to do this we first note that in a region of stability we must have $\Lambda\Gamma_1 > \Gamma_1\Gamma_2$. This is both one of our asymptotic stability criteria, and one of the conclusions in 9) from Descarte's theorem. This implies that we can never have roots of our characteristic equation $G(q, \Omega) = 0$ (defined in 8)) with $q = 0$. It follows that if Ω_0 is at a boundary of a stability region, then $G(q, \Omega_0)$ must have all real roots, but a small perturbation of Ω will yield complex roots. This implies that on a boundary of a region of stability there must be a root q_0 such that both $G(q_0, \Omega)$ and $G'(q_0, \Omega) = \frac{dG}{dq}$ vanish .

We will write

$$G(q, \Omega) = q^3 + (D - \Omega^2)q^2 + (E - \Omega^2)q + F$$

where

$$D = 1 + \Gamma_1 + \Gamma_2$$
$$E = \Gamma_1 + \Gamma_2 + \Gamma_1\Gamma_2 - \Lambda$$
$$F = \Gamma_1\Gamma_2 - \Lambda\Gamma_1$$

On the boundary of stability G and G' must have a common root, or equivalently, G must have a multiple root. A necessary and sufficient condition that a polynomial have multiple roots is that the discriminant vanishes. This is equivalent to saying that the resultant of G and G' vanishes. Suppose we have two polynomials

$$g(x) = g_0x^3 + g_1x^2 + g_2x + g_3$$

and

$$h(x) = h_0x^2 + h_1x + h_2$$

A necessary and sufficient condition that these two polynomials have roots in common is that the resultant vanish. The resultant is the determinant of the following matrix.

$$R = \begin{pmatrix} g_3 & g_2 & g_1 & g_0 & 0 \\ 0 & g_3 & g_2 & g_1 & g_0 \\ h_2 & h_1 & h_0 & 0 & 0 \\ 0 & h_2 & h_1 & h_0 & 0 \\ 0 & 0 & h_2 & h_1 & h_0 \end{pmatrix}$$

When we substitute the polynomials G and G' into this expression we find (with the help of Mathematica) that the resultant can be written as

$$\psi(\Omega) = \Omega^8 + K_6\Omega^6 + K_4\Omega^4 + K_2\Omega^2 + K_0 \tag{12}$$

$$K_6 = (32 - 16D - 16E + 32F)/8$$
$$K_4 = (8D^2 - 96E + 32DE + 8E^2 + 144F - 96DF)/8$$
$$K_2 = (-16D^2E + 96E^2 - 16DE^2 - 144DF + 96D^2F - 144EF)/8$$
$$K_0 = (8D^2E^2 - 32E^3 - 32D^3F + 144DEF - 216F^2)/8$$

This is a quartic polynomial in Ω^2. We have shown that on the boundary of stability $\psi(\Omega)$ must vanish, but we have not shown that any root of this equation will yield a value of Ω that is on the boundary of stability. In appendix A we apply the theory of Hankel matrices [4] to show that the polynomial $G(q, \Omega)$ will have all real roots if and only if $\psi(\Omega) < 0$. We will further show that $G(q, \Omega)$ will have all positive real roots if and only if $\psi(\Omega) < 0$. and all of the inequalities in 9) are satisfied.

If we compute the roots of the polynomial $\psi(\Omega)$, we find that there are roots that do not satisfy the conditions 9). If we limit ourselves to roots that satisfy the conditions 9), we find that the roots of $\psi(\Omega)$ do in fact give the upper and lower limits on the spin rates. Figure 3) shows the numerically computed upper and lower spin rates and compares them to the previously derived asymptotic estimates.

6 Computing the Dynamical Constants

In this section we will explain how for a given configuration of magnets on the rotor and in the base, one can compute the dynamical constants A_1, A_2, A,B, C_1 C_2 and B that are needed in order to compute the stability of the equilibrium. We also show how to compute the lift L.

For simplicity we will assume that the magnets on the rotor can be approximated by dipoles. We could extend this analysis so that the magnets on the rotor were approximated as a combination of axisymmetric dipoles, quadrapoles, and octopoles. However, that would make some of our results quite tedious. We will begin by analyzing the case where the rotor is in an antisymmetric potential. That is , we assume that the potential $f(x, y, z)$ satisfies

$$f(x, y, z) = f(x, -y, z)$$
$$f(x, y, z) = -f(-x, y, z)$$

We will assume that when the rotor is oriented in its equilibrium position, it has dipoles at $(\pm\delta/2, 0, z_0)$, both of magnitude M_R, and both pointing in the direction $(1, 0, 0)$. We will compute the dynamical constants when we have just a single pair of dipoles on the rotor. If we have more than one pair, then the constants can be computed by summing over all the different pairs.

In order to compute the force and torques on the rotor as it gets displaced from its equilibrium, we need to compute the Taylor series (up to the cubic terms) of the magnetic potential about the points $(\pm\delta/2, 0, z_0)$.

$$f(x+\delta/2, y, z_0+z) = \alpha_0 x + \alpha_1 z + \beta_0 xz + \frac{1}{2}\beta_1(2x^2-y^2-z^2) + \frac{1}{2}\beta_2(y^2-z^2) + \Gamma_+(x,y,z)$$

$$\Gamma_+(x,y,z_0+z) = \gamma_0\left(x^3/3 - xy^2/2 - xz^2/2\right) + \gamma_1\left(xy^2/2 - xz^2/2\right) + \gamma_2(z^3/6 - x^2 z/2) + \gamma_3(z^3/6 - y^2 z/2) + ...$$

Around the point $(-\delta/2, 0, z_0)$ we have the Taylor series expansion

$$f(x-\delta/2, y, z_0+z) = \alpha_0 x - \alpha_1 z + \beta_0 xz - \frac{1}{2}\beta_1(2x^2-y^2-z^2) - \frac{1}{2}\beta_2(y^2-z^2) + \Gamma_+(x,y,z)$$

$$\Gamma_-(x,y,z_0+z) = \gamma_0\left(x^3/3 - xy^2/2 - xz^2/2\right) + \gamma_1\left(xy^2/2 - xz^2/2\right) - \gamma_2(z^3/6 - x^2 z/2) - \gamma_3(z^3/6 - y^2 z/2) + ...$$

This is the most general form for the Taylor series (up to cubic terms) of a function $f(x, y, z)$ that is anti-symmetric in x and symmetric in y.

The dynamical constants can be computed with the following procedure which is easily implemented in Mathematica.

- Compute the orientation of the dipole which for small angles is approximated by $\underline{d} = (1 - \theta^2/2 - \phi^2/2, \theta, -\phi)$.
- Set the position of the right dipole to $\underline{x}_+ = \underline{x}_{cm} + \underline{d}\delta/2$, and the dipole moment to $\underline{m}_+ = M_R \underline{d}$.
- Set the position of the left dipole to $\underline{x}_- = \underline{x}_{cm} - \underline{d}\delta/2$, and the dipole moment to $\underline{m}_- = M_R \underline{d}$.
- Compute the magnetic energy $U = \underline{m}_+ \cdot \nabla f(\underline{x}_+) + \underline{m}_- \cdot \nabla f(\underline{x}_-)$.
- Calculate the force $\underline{F} = -\nabla U$
- Compute the torques, which in the linear approximation can be written as $\tau_z = -\frac{\partial U}{\partial \theta}$ and $\tau_y = -\frac{\partial U}{\partial \phi}$.
- Set $\underline{x}_{cm} = \epsilon(\hat{x}, \hat{y}, \hat{z})$, and $\theta = \epsilon\hat{\theta}$, $\phi = \epsilon\hat{\phi}$.
- Expand the forces and torques up to order ϵ.
- Set the lift L equal to the zeroeth order term in the force F_z.
- Set $-A_1$ to the term in F_y that is linearly proportional to y, $-A_2$ to the term in F_z that is linearly proportional to z, and A equal to the term in F_x that is linearly proportional to x. Set B equal to the term in F_x that is linearly proportional to ϕ.

- Set C_1 and C_2 to the terms in τ_z and τ_y that are linearly proportional to θ and ϕ respectively.

After carrying out this procedure, we arrive at the following expressions for the dynamical constants.

$$L = -2m_0\beta_0 \tag{13}$$

$$A_1 = 2m_0(\gamma_1 - \gamma_0) \tag{14}$$

$$A_2 = -2m_0(\gamma_0 + \gamma_1) \tag{15}$$

$$A = -4m_0\gamma_0 \tag{16}$$

$$B = m_0(2\beta_0 - \gamma_2 d) \tag{17}$$

$$C_1 = 2m_0\alpha_0 + m_0 d\,(4\beta_1 - 2\beta_2) + m_0 d^2(\gamma_0 - \gamma_1)/2 \tag{18}$$

$$C_2 = 2m_0\alpha_0 + m_0 d\,(4\beta_1 + 2\beta_2) + m_0 d^2(\gamma_1 + \gamma_0)/2 \tag{19}$$

If we have several systems of dipoles on the rotor, the dynamical constants are the sum of the dynamical constants for each system of magnets.

It should be pointed out that we get the exact same formula for systems with potentials that are symmetric with respect to reflections abut the x axis, and whose rotor magnets are also symmetric with respect to reflections about the midplane. In this case we get the same expansion of the field about the point $(\delta/2, 0, z_0)$, but the expansion about $(-\delta/2, 0, 0)$ is exactly opposite that given for the anti-symmetric case. If we define our fields using the Taylor expansion about $(\delta/2, 0, 0)$, the dynamical constants have the exact same values as those given for the anti-symmetric case.

7 Finding Realizable Confiugrations

So far we have discussed how to compute the dynamical constants assuming that we have a given configuration of magnets. We now discuss how one could in fact find a given configuration of magnets that gives the desired dynamical constants. We will present at least one way of going about this for systems that have potentials with reflectional symmetry about the x axis.

We will suppose that the base magnets consist of $4N$ dipoles all pointing in the z direction. The positions of the dipoles are given by

$$\underline{p}_i = (\pm a_i, \pm b_i, c_i) \;\; i = 1, N$$

and the magnetizations are given by

$$M_i = (0, 0, d_i) \;\; i = 1, N$$

For each value of i (four symmetrically placed magnets in the base), we can compute the dynamical constants $A_1(i)$, $A_2(i)$, $A(i)$, $B(i)$, $C_1(i)$, $C_2(i)$, and

$L(i)$ for $d_i = 1$. The values of the dynamical parameters for the whole system can be obtained by summing over the different sets of magnets multiplied by the strengths of the dipoles. . For example,

$$L = \sum_{i=1}^{N} d_i L(i)$$

If we have 6 or more systems of magnets, we can choose the strengths d_i so that we get any desired values of the parameters that we want. This means that in theory we can specify the desired values of A_1, A_2, L, Γ_1, Γ_2, and Λ that we would like; and thus the values of A, B, C_1, and C_2 that we would like. Once these are known we can determine the dipole strengths of the magnets that give these parameters.

The procedure we have outlined is meant to show that these configurations can be realized in theory. It does not address how to actually find a good configuration. For example, it is possible that the configurations could be very sensitive to small variations in the positions of the magnets, or to their strengths. We have carried out some more elaborate forms of this procedure in order to find possible configurations. We do not feel that it is appropriate to give any specific examples until we have aaalyzed them for their robustness.

8 Conclusions

We have theoretically demonstrated the existance of what is a distinctly different form of spin stabilized magnetic levitation. As with the traditional set up for spin stabilzed magnetic levitation, we expect that most configurations will have a high degree of sensitivity to the placement of the magnets. For this reason we believe that it is necessary to come up with some measure of the robustness of a configuration, and to search over a large class of configurations trying to find robust configurations.

Although nobody has ever used spin stabilized magnetic levitation for anything other than a scientific toy, it is possible that this principle could in fact have practical applications. It is hoped that by showing that the classical vertical configuration is not the only possibility, this paper may contribute to the eventual practical use of this prniciple.

9 Appendix A

We have shown that on the boundary of a region of stability, we must have $\psi(\Omega) = 0$. In this appendix we will show that the condition $\psi < 0$ is a necessary and sufficient condition for $G(q, \Omega)$ to have all real roots. To do this we apply the method of Hankel matrices presented in [4]. In [4], this method is explained for arbitrary polynomials, to simplify the notation , we will limit ourselves to cubic polynomials. Suppose we have a cubic polynomial of the form

$$a_0x^3 - a_1x^2 + a_2x - a_3.$$

The theory we present allows us to determine the number of real roots of this polynomial.

Suppose (x_0, x_1, x_2) are the roots to this polynomial (which of course we do not know). We begin by computing the Newton polynomials

$$\sigma_k = x_0^k + x_1^k + x_2^k$$

Eventhough we do not know the roots to the polynomial, we can compute the Newton polynomials. This follows from the fact that the σ_k's are symmetric polynomials in the variables x_i, and hence can be written as polynomials in the coefficients a_j of our polynomial. The theory of how to do this is explained in [8]. We will need to know σ_k up to $k = 4$. We can compute these recursively using

$$\begin{aligned}
\sigma_0 &= 3 \\
\sigma_1 &= a_1 \\
\sigma_2 &= a_1\sigma_1 - 2a_2 \\
\sigma_3 &= a_1\sigma_2 - a_2\sigma_1 + 3a_3 \\
\sigma_4 &= a_1\sigma_3 - a_2\sigma_2 + a_3\sigma_1
\end{aligned}$$

We now form the Hankel matrix.

$$H = \begin{pmatrix} \sigma_0 & \sigma_1 & \sigma_2 \\ \sigma_1 & \sigma_2 & \sigma_3 \\ \sigma_2 & \sigma_3 & \sigma_4 \end{pmatrix}$$

The number of real roots is equal to $3 - 2V$ where V is the number of sign changes in the sequence D_0, D_1, D_2 where $D_0 = \sigma_0$,

$$D_1 = det \begin{pmatrix} \sigma_0 & \sigma_1 \\ \sigma_1 & \sigma_2 \end{pmatrix}$$

and

$$D_2 = Det(H)$$

In order to have all real roots all of the determinants D_0, D_1 and D_2 must be positive. However, for a cubic polynomial, it is not possible to have D_1 be negative while D_2 is positive. This can be shown algebraically, or by noting that if this were the case, then our formula for the number of real roots would yield a negative number of real roots, which is impossible. It follows that a necessary and sufficient condition for our cubic polynomial to have all real roots is that the determinant D_2 is positive.

When we substitiute the coefficients from the polynomial $G(q, \Omega)$ into the general expression for D_2, this yields the polynomial $-\psi(\Omega)$. It follows that a necessary and sufficient condition for $G(q, \Omega)$ to have all real roots is that $\psi(\Omega) < 0$.

If a polynomial has all real roots, then a necessary and sufficient condition that all of its roots are positive is that its coeffecients alternate in sign. This implies that our system will be stable if and only if both $\psi(\Omega) < 0$, and all of the inequalities n 9) are satisfied.

10 Appendix B

In this appendix we will discuss the relation between the asymptotic stability analysis made in section 5.2 (assuming Γ_1, Γ_2 and Ω are large) and the adiabatic approximation presented for the vertically spinning Levitron^{TM} in [1]. We will show that these two approaches give the same results, and we will show that the conditions that the dimensionless parameters Γ_1, Γ_2 and Ω be large are equivalent to the conditions stated n [1] for the adiabatic approximation to hold.

Since the adiabatic approximation in [1] is worked out for a point dipole, we will now restrict our analysis to that case. That is, we will assume that our rotor only has a single dipole pointing in the direction of the axis of symmetry. This is an example of one of our two symmetries that we discussed in section 2).

We begin by applying the adiabatic approximation to our problem. Following [1], we argue that assuming that the top is fast we can make the approximation

$$\underline{L} = I_3 \omega_0 \underline{d}$$

Here ω_0 is the initial spin of the top, and $\underline{d}$ is the unit vector in the direction of the axis of symmetry. This assumes that we can ignore all components of the angular momentum except for the component about the axis of symmetry of the top. As pointed out in [1] the fast top approximation holds as long as the spin of the top is large compared to the precession rate of the top.

Under this fast top approximation, the equation for the change in angular momentum can be written as

$$\omega_0 I_3 \dot{\underline{d}} = -m_0 \underline{d} \times \underline{B}$$

Here m_0 is the dipole moment of the dipole on the rotor. In the adiabatic approximation this equation implies that the quantity

$$\mu_{ad} = \underline{d} \cdot \underline{B} / \mid \underline{B} \mid$$

stays constant. In order for this approximation to hold it is necessary that rate of change of the vector $\underline{d}$ be large compared to the rate of change of the quantity $\underline{d} \cdot \underline{B} / \mid \underline{B} \mid$.

In the adiabatic approximation, the magnetic energy of the rotor can be written as

$$U_{mag} = -\mu_{ad} \mid \underline{B} \mid$$

This is equivalent to saying that the top is moving in an effective potential that depends only on the center of mass of the top, not on its orientation. This effective potential is computed by using the magnetic energy $U(\underline{x}, \underline{d}) = -m_0 \underline{d} \cdot \underline{B}$ of the top, but using the fact that $\underline{d}$ is always pointing in the direction of the magnetic field. This is clearly equivalent to the approximation made in section 5.2 where we assumed that the rotor always orients itself so that there is no torque on it; and then using this to get an effective simple harmonic oscillator for the x component.

We would now like to show that the criteria that our parameters Γ_1, Γ_2 and Ω be large are equivalent to the criteria given in [1] for the adiabatic approximation to hold. We will discuss the scaling properties using the dimensionless lnearized equations of motion 2. The precession frequency of the top is given by

$$\Omega_{prec} = \sqrt{\frac{\Gamma_1 \Gamma_2}{\Omega^2}} \tag{20}$$

This precession frequency is obtained by ignoring the second derivatives of θ and ϕ and the term $\sqrt{\Lambda}$ in equations 2. The fast top assumption assumes that this precession rate is small compared to the spin rate of the top. This can be written as

$$\Omega >> \Omega_{prec} \tag{21}$$

Another condition stated in [1] for the adiabatic approximation to hold is that the bobbing freqency of the top be much less than the precession rate of the top. Physically this means that as the top moves around it can quickly orient itself so that it is aligned with the direction of the magnetic field. In our case the bobbing frequency of the top is obtained by ignoring the term $\sqrt{\Lambda}\phi$ in equations 2). Since we have made our equations dimensionless by this bobbing frequency, our bobbing frequency is unity. The condition that the precession rate is fast compared to the bobbing frequency can be written as

$$\Omega_{prec} >> 1 \tag{22}$$

In order to satisfy both of the conditions in eqns. 21) and 22), it is clearly necessary that $\Omega >> 1$. The condition that $\Omega_{prec} >> 1$ implies that

$$\Gamma_1 \Gamma_2 >> \Omega^4$$

Since Ω is large, this implies that the product $\Gamma_1 \Gamma_2$ must be large. In the case of an axisymetric top considered in [1] this would imply that $\Gamma_1 = \Gamma_2 >> 1$.

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