SAND REPORT

SAND2002-2595
Unlimited Release
Printed August 2002

Analysis of the Modified Great Lake Equations

David L. Ropp

Prepared by
Sandia National Laboratories
Albuquerque, New Mexico 87185 and Livermore, California 94550

Sandia is a multiprogram laboratory operated by Sandia Corporation, a Lockheed Martin Company, for the United States Department of Energy under Contract DE-AC04-94AL85000.

Sandia National Laboratories

Issued by Sandia National Laboratories, operated for the United States Department of Energy by Sandia Corporation.

Printed in the United States of America. This report has been reproduced directly from the best available copy.

Available to DOE and DOE contractors from
U.S. Department of Energy
Office of Scientific and Technical Information
P.O. Box 62
Oak Ridge, TN 37831

Telephone: (865)576-8401
Facsimile: (865)576-5728
E-Mail: reports@adonis.osti.gov
Online ordering: http://www.doe.gov/bridge

Available to the public from
U.S. Department of Commerce
National Technical Information Service
5285 Port Royal Rd
Springfield, VA 22161

Telephone: (800)553-6847
Facsimile: (703)605-6900
E-Mail: orders@ntis.fedworld.gov
Online order: http://www.ntis.gov/ordering.htm

SAND2002-2595
Unlimited Release
Printed August 2002

Analysis of the Modified Great Lake Equations

David L. Ropp
Computational Mathematics and Algorithms
Sandia National Laboratories
P.O. Box 5800
Albuquerque, NM 87185-1110
dlropp@sandia.gov

Abstract

The great lake equations of Camassa, Holm, and Levermore are considered. Additional terms arising from physical considerations are incorporated into the momentum equation. The resulting equations are then posed in a weak formulation. Solutions of this modified set of equations are shown to exist and, under a certain condition, to be unique. A similar result is shown if the problem has non-homogeneous Dirichlet boundary conditions.

Contents

Figures

Analysis of the Modified Great Lake Equations

1 Introduction

In this report we show the well-posedness of weak solutions of a modified form of the time-independent great lake equations. Naturally, this requires casting the model equations in a weak form. We do this by reformulating the problem in a weak sense, choosing suitable forms, and looking for solutions in the appropriate Sobolev function spaces.

We then need to show that solutions of this weak formulation of the problem are well-posed, *i.e.*, that solutions exist and are unique. We present the theorem which gives well-posedness of solutions of the weak problem. This theorem, due to the work of Leray, was first presented in a complete manner in Ladyshenskaya [9], but we follow the presentation found in Girault and Raviart [6]. We show that our weak formulation satisfies the conditions of this theorem. The criteria required for existence of solutions are nearly identical to those for the two-dimensional Navier-Stokes equations, with the addition of the dispersive great lake terms and the Coriolis and bottom drag terms.

Well-posedness of the great lake equations has already been shown in [11] and [10]. There, however, the equations were cast in a vorticity formulation, with the solutions being the inviscid limit of solutions of a system with an artificial viscosity. These results cannot be easily used here, though, because we are not using the vorticity formulation, which is somewhat more difficult to use when additional physical terms are added.

Finally we show that the conditions for existence of solutions are still satisfied when non-homogeneous boundary conditions are incorporated. The condition for uniqueness is modified accordingly. This proof also follows from a similar proof in [6] for the two-dimensional Navier-Stokes equations, with the addition of the great lake, Coriolis, and bottom drag terms.

We introduce the great lake equations in section 2, and add various physics-based terms to arrive at the modified great lake equations. The weak formulation is presented in section 3, with an explanation of the various function spaces. In section 4 we show that this system satisfies the criteria for existence of solutions, with an extra condition for uniqueness. The same is true when non-homogeneous boundary conditions are added, as in section 5.

2 The Modified Great Lake Equations

We start with the great lake equations of Camassa, Holm, and Levermore ([4], [5]). These equations consider the two-dimensional velocity $\boldsymbol{u}$ and surface disturbance η (measured from the undisturbed surface height) in a lake-sized body of water; see figure 2. The domain Ω may be multiply connected and has a Lipschitz-continuous boundary $\partial\Omega$. In non-dimensional form, the equations are

$$\partial_t \boldsymbol{u}^G + \boldsymbol{u}\cdot\nabla\boldsymbol{u}^G + (\nabla\boldsymbol{u})\boldsymbol{u}^G + \nabla\left(\eta - \frac{1}{2}|\boldsymbol{u}|^2\right) = 0, \tag{1a}$$

$$\nabla\cdot(b\boldsymbol{u}) = 0, \tag{1b}$$

$$\text{on } \partial\Omega, \quad \boldsymbol{u}\cdot\boldsymbol{n} = 0, \tag{1c}$$

where

$$\boldsymbol{u}^G = \boldsymbol{u} + \frac{1}{6}\delta^2 \mathrm{B}^2 \nabla\nabla\cdot\boldsymbol{u}\,.$$

In the above B is the depth of the lake from the undisturbed surface level with constants B_M and B_m such that $\mathrm{B}_M \leq \mathrm{B} \leq \mathrm{B}_m < 0$, and δ is the ratio of the mean depth to the mean horizontal length of disturbances. We assume that δ is small, so that the lake is shallow; the derivation also assumes that $\boldsymbol{u}$ is small compared to the gravity wave speed, that $\boldsymbol{u}$ varies little with depth, and that surface waves are small compared to the depth. Additionally, we assume that $\nabla\mathrm{B}$ is bounded by δ.

These equations describe disturbances with long wavelength and slow wave speed. They have a structure nearly identical to that of the two-dimensional incompressible Euler equations. In particular, these equations have a conserved energy, an advected potential vorticity, and a Poisson-like equation for the height.

These equations lack important physics, however: the Coriolis force, wind stress, bottom drag, and viscosity. We can get an idea of the importance of these terms by computing their relative sizes using the scales for Lake Erie, a prototypical lake. Typical scales are a horizontal velocity scale U of 5 cm/s, a wind speed U_w of 5 m/s, horizontal and vertical length scales X and D of 40 km and 19 m, a Coriolis parameter f of 9.76×10^{-5} s^{-1}, a wind shear coefficient C_t of 3.03×10^{-6}, a bottom drag coefficient C_b of 2.0×10^{-3}, and a viscosity ν of $100\,\mathrm{m}^2$/s. Using these values, we have that relative to the inertial terms, the size of the Coriolis term is $fX/U = 78.08$; that of the wind shear term is $C_t \frac{X}{D}\frac{U_w^2}{U^2} = 64.21$; that of the bottom drag term is

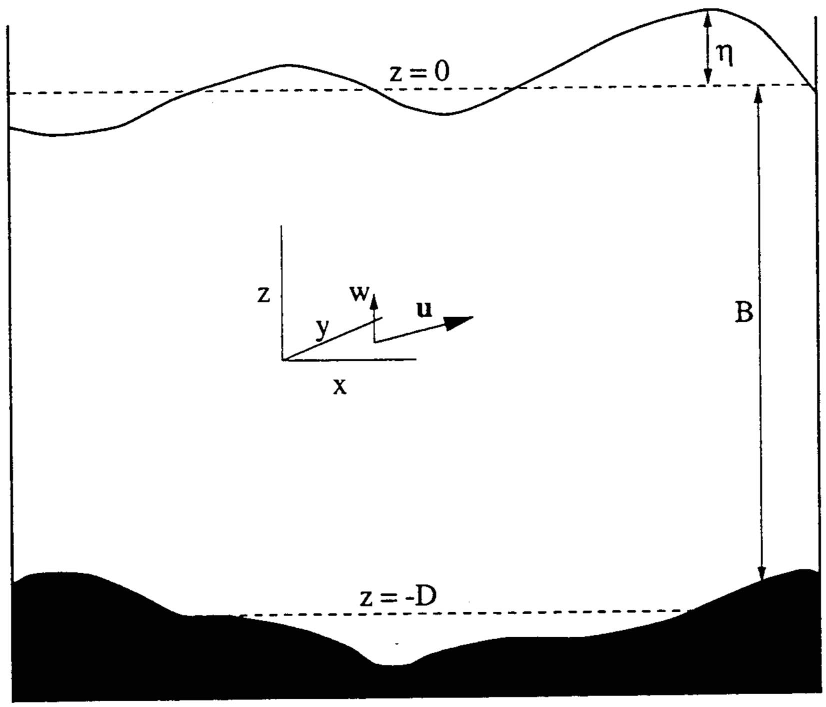

Figure 1. Side view of the basin. The amplitude η of the wave is exaggerated. The lateral boundaries are always assumed to be vertical.

$C_b\frac{X}{D} = 4.21$; and that of viscosity is $\frac{\nu}{UX} = 0.05$. The viscosity term will actually be larger in regions with a large velocity gradient. Thus, we add these terms to the great lake equations. We also add non-homogeneous boundary conditions so that we can incorporate inflow and outflow.

For the Coriolis force we add the term $\frac{1}{\epsilon_R}\boldsymbol{u}^\perp$, where the Rossby number ϵ_R is given by $\epsilon_R = \frac{U}{fL}$, where U and L are the horizontal velocity and length scales, f is the Coriolis parameter, and $(u_1, u_2)^\perp = (u_2, -u_1)$. The value of f depends on the rotation of the Earth and the latitude, but we will assume it to be constant; see [13].

For bottom drag and wind stress we use $\frac{C_b}{\delta}\frac{|\boldsymbol{u}|\boldsymbol{u}}{\mathrm{B}}$ and $\frac{C_t}{\delta}\frac{|\boldsymbol{u}_w|\boldsymbol{u}_w}{\mathrm{B}}$; see [8]. The coefficients C_b and C_t depend on the physics of the the bottom and top surfaces, which we will take to be constant, and $\boldsymbol{u}_w$ is the velocity of the wind. Typical values are $C_t = 3.02 \times 10^{-6}$, $C_b = 2.0 \times 10^{-3}$, and $\boldsymbol{u}_w = 5$ cm/s. Since the wind stress does not depend on the dependent variables, we will often write it simply as $\boldsymbol{f}$.

The viscosity term we use is

$$\frac{1}{\mathrm{B}}\nabla\cdot\left[\nu\big(\mathrm{B}\nabla\boldsymbol{u} + \mathrm{B}(\nabla\boldsymbol{u})^\top - \mathrm{B}\nabla\cdot\boldsymbol{u}I_2\big)\right].$$

Here, I_2 is the 2×2 identity matrix. This form of viscosity is appropriate for the analysis which follows and is derived in [12]. This viscosity is not a molecular viscosity but an eddy viscosity, allowing energy to dissipate at a suitable rate.

Because this viscosity has second-order derivatives, we need to modify the boundary conditions for $\boldsymbol{u}$. We will do so by specifying, in addition to the normal component of $\boldsymbol{u}$, the tangential component as well. Generally we will will have the condition

$$\boldsymbol{u}|_{\partial\Omega} = \boldsymbol{g}(\boldsymbol{x}).$$

In practice, $\boldsymbol{g}$ will be 0 along sidewalls and nonzero at points of inflow or outflow. A compatibility condition, due to incompressibility, is

$$\oint_{\partial\Omega} \boldsymbol{g}\cdot\boldsymbol{n}\,\mathrm{B}ds = 0.$$

Incorporating all the additional terms to the great lake equations gives

$$\begin{aligned}\partial_t\boldsymbol{u}^G + \boldsymbol{u}\cdot\nabla\boldsymbol{u}^G + (\nabla\boldsymbol{u})\boldsymbol{u}^G + \nabla\left(\eta - \frac{1}{2}|\boldsymbol{u}|^2\right) - \frac{1}{\epsilon_R}\boldsymbol{u}^\perp \\ = \frac{C_t}{\delta}\frac{U_w^2}{U^2}\frac{|\boldsymbol{u}_w|\boldsymbol{u}_w}{\mathrm{B}} - \frac{C_b}{\delta}\frac{|\boldsymbol{u}|\boldsymbol{u}}{\mathrm{B}} + \frac{1}{\mathrm{B}}\nabla\cdot\left[\nu\big(\mathrm{B}\nabla\boldsymbol{u} + \mathrm{B}(\nabla\boldsymbol{u})^\top - \mathrm{B}\nabla\cdot\boldsymbol{u}I_2\big)\right],\end{aligned} \tag{2a}$$

$$\nabla\cdot(\boldsymbol{u}) = 0, \tag{2b}$$

where

$$\boldsymbol{u}^G = \boldsymbol{u} + \frac{1}{6}\delta^2 \mathrm{B}^2 \nabla\nabla\cdot \boldsymbol{u}\,, \tag{2c}$$

with boundary conditions

$$\boldsymbol{u}|_{\partial\Omega} = \boldsymbol{g}\,, \quad \text{where } \oint_{\partial\Omega} \boldsymbol{g}\cdot\boldsymbol{n}\, ds = 0\,. \tag{2d}$$

3 Weak Formulation

In this section we will develop the weak formulation of the problem. We will restrict ourselves to the time-independent great lake equations with homogeneous boundary conditions.

The only difference between equations 2a - 2d and the time-independent equations is that the acceleration terms are dropped. Thus, equation 2a becomes

$$\begin{aligned} &\boldsymbol{u}\cdot\nabla\boldsymbol{u}^G + (\nabla\boldsymbol{u})\boldsymbol{u}^G + \nabla\left(\eta - \frac{1}{2}|\boldsymbol{u}|^2\right) - \frac{1}{\epsilon_R}\boldsymbol{u}^\perp \\ &\quad = \frac{C_t}{\delta}\frac{U_w^2}{U^2}\frac{|\boldsymbol{u}_w|\boldsymbol{u}_w}{\mathrm{B}} - \frac{C_b}{\delta}\frac{|\boldsymbol{u}|\boldsymbol{u}}{\mathrm{B}} + \frac{1}{\mathrm{B}}\nabla\cdot\left[\nu\big(\mathrm{B}\nabla\boldsymbol{u} + \mathrm{B}(\nabla\boldsymbol{u})^\top - \mathrm{B}\nabla\cdot\boldsymbol{u}I_2\big)\right], \end{aligned} \tag{3}$$

and the other equations remain the same.

In order to develop a weak formulation we need to define the appropriate function spaces, their norms, and functionals on these spaces. First, we define the familiar $L^p(\Omega)$ spaces:

$$L^p(\Omega) = \left\{v : \textstyle\int_\Omega |v|^p d\Omega < \infty\right\}.$$

These spaces have the norm

$$||v||_{L^p} = \left(\int_\Omega |v|^p \, d\Omega\right)^{1/p}.$$

As usual, the notation $|v|$ denotes the absolute value for a scalar v and the Euclidean norm $(\sum_i |v_i|^2)^{1/2}$ for a vector or tensor $\boldsymbol{v}$.

More generally, for $m \geq 0$ and $1 \leq p \leq \infty$, we denote by $W^{m,p}(\Omega)$ the space

$$W^{m,p}(\Omega) = \{v \in L^p(\Omega); \partial^\alpha v \in L^p(\Omega) \quad \forall |\alpha| \leq m\}\,.$$

In the above, we use the multi-index notation $\alpha = (\alpha_1, \alpha_2)$, with the α_i being non-negative integers, $\partial^\alpha = \partial_x^{\alpha_1}\partial_y^{\alpha_2}$, and $|\alpha| = \alpha_1 + \alpha_2$. These spaces have the norm

$$\|v\|_{m,p} = \begin{cases} \left(\sum_{|\alpha|\le m} \|\partial^\alpha v\|_{L^p}^p\right)^{1/p} & \text{for } 1 \le p < \infty\,, \\ \max_{|\alpha|\le m} \left(\operatorname{ess\,sup}_\Omega |\partial^\alpha v|\right) & \text{for } p = \infty\,. \end{cases}$$

When $p = 2$, we use the notation $H^m(\Omega) = W^{m,p}(\Omega)$ and $\|v\|_m = \|v\|_{m,2}$. Extending this shorthand to $m = 0$, we use $\|v\|_0$ to denote the $L^2(\Omega)$ norm of v.

The inner product $\langle u, v\rangle$ is defined as

$$\langle u, v\rangle = \int_\Omega uv \, \mathrm{d}\Omega\,.$$

We will also use an inner product weighted by the topography B. For that we use the notation

$$\langle u, v\rangle_{\mathrm{B}} = \int_\Omega uv \, \mathrm{B} \mathrm{d}\Omega\,.$$

If v is any element of $W^{m,p}(\Omega)$, with $1 \le p < \infty$, then the set of distributions u for which $\langle u, v\rangle$ is finite is called the dual of $W^{m,p}(\Omega)$. This dual space is denoted $W^{-m,p'}(\Omega)$, where $\frac{1}{p} + \frac{1}{p'} = 1$. This space has the norm

$$\|u\|_{-m,p'} = \sup_{v\in W^{m,p}(\Omega)} \frac{\langle u, v\rangle}{\|u\|_{m,p}}\,.$$

More precisely, if $\mathcal{D}'(\Omega)$ is the set of distributions acting on infinitely differentiable functions with compact support, then

$$W^{-m,p'}(\Omega) = \{u \in \mathcal{D}'(\Omega) : \|u\|_{-m,p'} < \infty\}\,.$$

The Sobolev embedding theorems for these spaces can be found in [1].

For real $s > 0$, the space $H^s(\mathbf{R}^2)$ can be defined by

$$H^s(\mathbf{R}^2) = \left\{u \in L^2(\mathbf{R}^2) : (1 + |\boldsymbol{k}|)^{s/2}\hat{u}(\boldsymbol{k}) \in L^2(\mathbf{R}_k^2)\right\},$$

where $\hat{u}(\boldsymbol{k})$ is the Fourier transform of u. This space has the norm

$$\|u\|_{s,\mathbf{R}^2} = \left[\|u\|_0^2 + (1 + |\boldsymbol{k}|^2)^{s/2}\|\hat{u}(\boldsymbol{k})\|_0^2\right]^{1/2}.$$

For an open subset Ω of $\mathbf{R}^2$ we can define $H^s(\Omega)$ by

$$H^s(\Omega) = \left\{u \in L^2(\Omega) : \exists u' \in H^s(\mathbf{R}^2) \text{ such that } u'|_\Omega = u\right\},$$

with the norm

$$\|u\|_{s,\Omega} = \inf_{u' \in H^s(\mathbf{R}^2), u'|_\Omega = u} \|u'\|_{s,\mathbf{R}^2} \,.$$

When s is an integer this norm is equivalent to the norm defined earlier. In our work the value of these spaces is due to a trace theorem (see [1]) which states that, given a bounded domain Ω in $\mathbf{R}^2$ with a Lipschitz-continuous boundary, the range of the mapping that restricts a function u in $H^1(\Omega)$ to the boundary $\partial\Omega$ is in $H^{1/2}(\partial\Omega)$. Moreover, this mapping is onto, so that for any function g in $H^{1/2}(\partial\Omega)$, there exists a function u in $H^1(\Omega)$ such that $u|_{\partial\Omega} = g$.

Since our problem is on a two-dimensional domain, we will often be concerned with vector-valued functions $\boldsymbol{v} = (v_1, v_2)$. Such functions are said to be in the space $W^{m,p}(\Omega)^2$ if and only if both v_1 and v_2 are in $W^{m,p}(\Omega)$, for any of the $W^{m,p}(\Omega)$ defined above. The corresponding norm is

$$\|\boldsymbol{v}\|_{m,p} = \begin{cases} \left(\|v_1\|_{m,p}^p + \|v_2\|_{m,p}^p\right)^{1/p} & \text{for } 1 \le p < \infty \,, \\ \max_{|\alpha| \le m} (\operatorname{ess\,sup}_\Omega(|\partial^\alpha v_1| + |\partial^\alpha v_2|) & \text{for } p = \infty \,. \end{cases}$$

Two spaces that we will use often in this discussion are

$$L_0^2(\Omega) = \left\{h \in L^2(\Omega), \textstyle\int_\Omega h \, \mathrm{d}\Omega = 0\right\}$$

and

$$H_0^1(\Omega)^2 = \left\{\boldsymbol{v} \in H^1(\Omega)^2, \boldsymbol{v}|_{\partial\Omega} = \boldsymbol{0}\right\}.$$

The boundary values of any $v \in H^1(\Omega)^2$ can be specified as in the definition of $H_0^1(\Omega)^2$ if the domain Ω is bounded and has a boundary $\partial\Omega$ that is Lipschitz-continuous. Due to the restriction of $H_0^1(\Omega)$ on the boundary and the Poincaré inequality, the semi-norm

$$|\boldsymbol{v}|_1 = \left(\int_\Omega |\partial_x \boldsymbol{v}|^2 + |\partial_y \boldsymbol{v}|^2 \mathrm{d}\Omega\right)^{1/2}$$

is actually a norm, and is the norm that we will use for this space.

Next, with this notation in place, we can define the following forms:

$$a(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v}) = a_0(\boldsymbol{u}, \boldsymbol{v}) + a_1(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v}) + a_2(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v}) + a_3(\boldsymbol{u}, \boldsymbol{v}) + a_4(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v}) \qquad (4)$$

and

$$b(\boldsymbol{u}, \eta) = \int_\Omega \eta \nabla \cdot (\mathrm{B}\boldsymbol{u}) \, \mathrm{d}\Omega \qquad (5)$$

where

$$a_0(\boldsymbol{u},\boldsymbol{v}) = \frac{\nu}{2}\int_\Omega [\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^\top - \nabla\cdot\boldsymbol{u}I_2] : [\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^\top - \nabla\cdot\boldsymbol{v}I_2]\mathrm{B}d\Omega\,, \tag{6}$$

$$a_1(\boldsymbol{w};\boldsymbol{u},\boldsymbol{v}) = \int_\Omega (\boldsymbol{w}\cdot\nabla\boldsymbol{u})\cdot\boldsymbol{v}\,\mathrm{B}d\Omega\,, \tag{7}$$

$$a_2(\boldsymbol{w};\boldsymbol{u},\boldsymbol{v}) = \int_\Omega (\boldsymbol{w}^G - \boldsymbol{w})(\boldsymbol{v}\cdot\nabla\boldsymbol{u} - \boldsymbol{u}\cdot\nabla\boldsymbol{v})\mathrm{B}d\Omega\,, \tag{8}$$

$$a_3(\boldsymbol{u},\boldsymbol{v}) = -\frac{1}{\epsilon_R}\int_\Omega \boldsymbol{u}^\perp\cdot\boldsymbol{v}\,\mathrm{B}d\Omega\,, \tag{9}$$

$$a_4(\boldsymbol{w};\boldsymbol{u},\boldsymbol{v}) = \frac{C_b}{\delta}\int_\Omega |\boldsymbol{w}|\boldsymbol{u}\cdot\boldsymbol{v}\,d\Omega\,. \tag{10}$$

In the expression for $a_0(\boldsymbol{u},\boldsymbol{v})$, we have that $(\nabla\boldsymbol{u})_{ij} = \frac{\partial u_i}{\partial x_j}$, $I_2 = \delta_{ij}$ ($= 1$ iff $i = j$, 0 otherwise), and the tensor product $\boldsymbol{G} : \boldsymbol{H}$ is defined by $\boldsymbol{G} : \boldsymbol{H} = \sum_{i=1}^2\sum_{j=1}^2 G_{ij}H_{ji}$. The weak form of the steady-state great lake equations is then the following:

Problem 1. *Given $\boldsymbol{f} \in H^{-1}(\Omega)^2$, find $\boldsymbol{u} \in H_0^1(\Omega)^2$ and $\eta \in L_0^2(\Omega)$ such that*

$$a(\boldsymbol{u};\boldsymbol{u},\tilde{\boldsymbol{u}}) - b(\tilde{\boldsymbol{u}},\eta) = \langle\boldsymbol{f},\tilde{\boldsymbol{u}}\rangle_{\mathrm{B}} \quad \text{for every } \tilde{\boldsymbol{u}} \in H_0^1(\Omega)^2, \tag{11a}$$

$$b(\boldsymbol{u},\tilde{\eta}) = 0 \quad \text{for every } \tilde{\eta} \in L_0^2(\Omega). \tag{11b}$$

It is clear that any solution of equations 2b - 3 will be a solution of equation 11. Similarly, any solution of 11 with sufficient smoothness so that the appropriate derivatives exist will also be a solution to equations 2b - 3. However, solutions of equation 11 without such sufficient smoothness do not make sense as solutions of 2b - 3. Thus, classical solutions of 2b - 3 are called strong solutions, while solutions of Problem 1 are called weak solutions.

4 Existence and Uniqueness

In this section we examine existence and uniqueness of problem 1. We will state the theorems guaranteeing existence and uniqueness of solutions for a general problem, and then show that our equations satisfy the criteria for existence and state the condition for uniqueness.

We consider two Hilbert spaces X and M, with norms $\|\cdot\|_X$ and $\|\cdot\|_M$, respectively. We also introduce a bilinear continuous form $b(\boldsymbol{v},\eta) : X \times M \to \mathbf{R}$, and a form

$a(\boldsymbol{w};\boldsymbol{u},\boldsymbol{v}) : X \times X \times X \rightarrow \mathbf{R}$ such that for any $\boldsymbol{w} \in X$, $a(\boldsymbol{w};\cdot,\cdot)$ is a bilinear continuous form.

Then we pose the following problem:

Problem 2. *Given $\boldsymbol{f} \in X'$, find $\boldsymbol{u} \in X$ and $\eta \in M$ such that*

$$a(\boldsymbol{u};\boldsymbol{u},\tilde{\boldsymbol{u}}) - b(\tilde{\boldsymbol{u}},\eta) = \langle \boldsymbol{f},\tilde{\boldsymbol{u}}\rangle_{\mathrm{B}} \qquad \forall \tilde{\boldsymbol{u}} \in X\,, \tag{12a}$$

$$b(\boldsymbol{u},\tilde{\eta}) = 0 \qquad \forall \tilde{\eta} \in M\,. \tag{12b}$$

It is useful to introduce the linear operators $A(\boldsymbol{w}) \in L(X;X')$ for $\boldsymbol{w}$ in X, and $B \in L(X;M')$ defined by the following relations:

$$\begin{aligned} \langle A(\boldsymbol{w})\boldsymbol{u},\boldsymbol{v}\rangle_{\mathrm{B}} &= a(\boldsymbol{w};\boldsymbol{u},\boldsymbol{v}) \qquad \forall \boldsymbol{u},\boldsymbol{v} \in X\,,\\ \langle B\boldsymbol{u},\eta\rangle_{\mathrm{B}} &= b(\boldsymbol{u},\eta) \qquad \forall \boldsymbol{u} \in X\,, \eta \in M\,. \end{aligned}$$

Similarly, we define $B' \in L(M;X')$ by

$$\langle \boldsymbol{u}, B'\eta\rangle_{\mathrm{B}} = b(\boldsymbol{u},\eta) \qquad \forall \boldsymbol{u} \in X\,, \eta \in M\,.$$

Then Problem 2 can be reformulated as follows.

Find $(\boldsymbol{u},\eta) \in X \times M$ such that

$$\begin{aligned} A(\boldsymbol{u})\boldsymbol{u} - B'\eta &= f \qquad \textit{in } X'\,,\\ B\boldsymbol{u} &= 0 \qquad \textit{in } M'\,. \end{aligned}$$

Because any solution $\boldsymbol{u}$ of problem 2 must satisfy equation 12b, it is natural to consider the space

$$V = \{\boldsymbol{u} \in X : b(\boldsymbol{u},\tilde{\eta}) = 0 \quad \forall \tilde{\eta} \in M\}\,.$$

We can then associate with problem 2 the following problem:

Problem 3. *Given $\boldsymbol{f} \in X'$, find $\boldsymbol{u} \in V$ such that*

$$a(\boldsymbol{u};\boldsymbol{u},\tilde{\boldsymbol{u}}) = \langle \boldsymbol{f},\tilde{\boldsymbol{u}}\rangle_{\mathrm{B}} \quad \forall \tilde{\boldsymbol{u}} \in V\,. \tag{13}$$

Using operator notation, this problem can be restated as follows.

Find $\boldsymbol{u} \in V$ such that

$$\pi A(\boldsymbol{u})\boldsymbol{u} = \pi \boldsymbol{f} \qquad \text{in } V',$$

where the projection operator $\pi \in L(X'; V')$ is defined by

$$\langle \pi \boldsymbol{f}, \boldsymbol{v} \rangle_{\mathrm{B}} = \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{\mathrm{B}} \qquad \forall \boldsymbol{v} \in V.$$

Since the problem at hand is nonlinear, we use the Brouwer fixed-point theorem; see, *e.g.*, [14]. We state this as it applies to problem 2; a proof can be found in [6].

Theorem 1. *Assume the following hypotheses:*

1. *there exists a constant $\alpha > 0$ such that*

$$a(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{u}) \geq \alpha \|\boldsymbol{u}\|_X^2 \quad \forall \boldsymbol{u} \in V; \tag{14}$$

2. *the space V is separable and, for all $\boldsymbol{v} \in V$, the mapping $\boldsymbol{u} \mapsto a(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v})$ is sequentially weakly continuous, i.e., if $\{\boldsymbol{u}_m\}$ is a sequence in V such that $\boldsymbol{u}_m \rightharpoonup \boldsymbol{u} \in V$ weakly in V as $m \to \infty$, then $a(\boldsymbol{u}_m; \boldsymbol{u}_m, \boldsymbol{v}) \to a(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v})$ as $m \to \infty$;*

3. *there exists a constant $\beta > 0$ such that*

$$\inf_{\eta \in M} \sup_{\boldsymbol{u} \in X} \frac{b(\boldsymbol{u}, \eta)}{\|\boldsymbol{u}\|_X \|\eta\|_M} \geq \beta > 0. \tag{15}$$

Then there exists at least one solution $(\boldsymbol{u}, \eta) \in V \times M$ to problem 2.

The first two conditions guarantee that problem 3 will have at least one solution, while the third insures that for each solution $\boldsymbol{u}$ of problem 3 there exists an η such that $(\boldsymbol{u}, \eta)$ is a solution of problem 2. Specifically, condition 15, known as the *inf-sup* condition[1], guarantees that the space V will not be empty, and that for every solution $\boldsymbol{u}$ of problem 3 a unique $\eta \in M$ exists such that $(\boldsymbol{u}, \eta)$ is a solution of problem 2.

It is shown in Lemma I.4.1 of [6] that condition 3 is equivalent to B being isomorphic from $V^{\perp} = \{\boldsymbol{g} \in H_0^1(\Omega)^2; \langle \boldsymbol{g}, \boldsymbol{v} \rangle = 0 \quad \forall \boldsymbol{v} \in V\}$ onto M', and to B' being

[1] Also referred to as the ***div-stability*** condition and as the *LBB* condition, after its co-discoverers Ladyzhenskaya, Brezzi, and Babuška. See [2], [3], or [9]

isomorphic from M onto $V^\circ = \{\boldsymbol{g} \in H^{-1}(\Omega)^2; \langle \boldsymbol{g}, \boldsymbol{v}\rangle = 0 \quad \forall \boldsymbol{v} \in V\}$. Moreover, we have the bounds

$$\|B'\mu\|_{X'} \geq \beta \|\mu\|_M \quad \forall \mu \in M\,,$$

and

$$\|B\boldsymbol{v}\|_{M'} \geq \beta \|\boldsymbol{v}\|_X \quad \forall \boldsymbol{v} \in V^\perp\,,$$

For uniqueness we state the following theorem, whose proof can also be found in [6]. We will need the norm

$$\|\boldsymbol{f}\|_{V'} = \sup_{\boldsymbol{v} \in V} \frac{\langle \boldsymbol{f}, \boldsymbol{v}\rangle}{|\boldsymbol{v}|_1}\,.$$

Theorem 2. *Assume that*

1. *there exits an $\alpha > 0$ such that*

$$a(\boldsymbol{w}; \boldsymbol{v}, \boldsymbol{v}) \geq \alpha \|\boldsymbol{v}\|_X^2 \quad \forall \boldsymbol{v}, \boldsymbol{w} \in V\,, \tag{16}$$

2. *the mapping $\boldsymbol{w} \mapsto \pi A(\boldsymbol{w})$ is locally Lipschitz-continuous, i.e., there exists a monotonically nondecreasing function $L : \mathbf{R}_+ \to \mathbf{R}_+$ such that for all $\mu > 0$,*

$$\begin{gathered} |a(\boldsymbol{w}_1; \boldsymbol{u}, \boldsymbol{v}) - a(\boldsymbol{w}_2; \boldsymbol{u}, \boldsymbol{v})| \leq L(\mu) \|\boldsymbol{u}\|_X \|\boldsymbol{v}\|_X \|\boldsymbol{w}_1 - \boldsymbol{w}_2\|_X \\ \forall \boldsymbol{u}, \boldsymbol{v} \in V, \quad \forall \boldsymbol{w}_1, \boldsymbol{w}_2 \in S_\mu = \{\boldsymbol{w} \in V : \|\boldsymbol{w}\|_X \leq \mu\}\,. \end{gathered} \tag{17}$$

Then if

$$\left(\frac{\|\pi \boldsymbol{f}\|_{V'}}{\alpha^2}\right) L\left(\frac{\|\pi \boldsymbol{f}\|_{V'}}{\alpha}\right) < 1\,, \tag{18}$$

problem 3 has a unique solution in V.

Condition 16 along with the Lax-Milgram theorem guarantees that the operator $\pi A(\boldsymbol{w})$ is invertible for each $\boldsymbol{w} \in V$, while conditions 17 and 18 insure that the map $\boldsymbol{u} \to (\pi A(\boldsymbol{u}))^{-1} \pi \boldsymbol{f}$ is a contraction and hence has a unique solution. Note that in the absence of nonlinear terms, a suitable choice of $L(\mu)$ is $L \equiv 0$, and the last two conditions are automatically satisfied.

Now that we have the criteria for existence and uniqueness, we consider the problem at hand, and set $X = H_0^1(\Omega)$ and $M = L_0^2(\Omega)$. It is useful to modify the forms

$a_1(\boldsymbol{w};\boldsymbol{u},\boldsymbol{v})$ and $a_2(\boldsymbol{w};\boldsymbol{u},\boldsymbol{v})$. Using integration by parts and the definition of $\boldsymbol{w}^G$, we may write them as

$$a_1(\boldsymbol{w};\boldsymbol{u},\boldsymbol{v}) = \frac{1}{2}\int_\Omega \left[(\boldsymbol{w}\cdot\nabla\boldsymbol{u})\cdot\boldsymbol{v} - (\boldsymbol{w}\cdot\nabla\boldsymbol{v})\cdot\boldsymbol{u}\right]\mathrm{B}\,d\Omega\,, \tag{19}$$

$$a_2(\boldsymbol{w};\boldsymbol{u},\boldsymbol{v}) = \frac{\delta^2}{3}\int_\Omega (\boldsymbol{w}\cdot\nabla\mathrm{B})(\nabla\mathrm{B})\cdot(\boldsymbol{v}\cdot\nabla\boldsymbol{u} - \boldsymbol{u}\cdot\nabla\boldsymbol{v})\mathrm{B}\,d\Omega\,. \tag{20}$$

An integration by parts with suitable assumptions of the boundary conditions shows the equivalence of the different forms of $a_1(\boldsymbol{w};\boldsymbol{u},\boldsymbol{v})$ and $a_2(\boldsymbol{w};\boldsymbol{u},\boldsymbol{v})$. As before, we have $a(\boldsymbol{w};\boldsymbol{u},\boldsymbol{v}) = a_0(\boldsymbol{u},\boldsymbol{v}) + a_1(\boldsymbol{w};\boldsymbol{u},\boldsymbol{v}) + a_2(\boldsymbol{w};\boldsymbol{u},\boldsymbol{v}) + a_3(\boldsymbol{u},\boldsymbol{v}) + a_4(\boldsymbol{w};\boldsymbol{u},\boldsymbol{v})$.

We now show that the conditions of theorem 1 are satisfied. First, note that when $\boldsymbol{w}$, $\boldsymbol{u}$, and $\boldsymbol{v}$ are in V, we have

$$a_i(\boldsymbol{w};\boldsymbol{u},\boldsymbol{v}) = -a_i(\boldsymbol{w};\boldsymbol{v},\boldsymbol{u})\,, \quad i = 1,2\,. \tag{21}$$

Consequently, $a_1(\boldsymbol{w};\boldsymbol{u},\boldsymbol{u}) = a_2(\boldsymbol{w};\boldsymbol{u},\boldsymbol{u}) = 0$. Moreover, $a_3(\boldsymbol{u},\boldsymbol{u}) = 0$ identically, and $a_4(\boldsymbol{w};\boldsymbol{u},\boldsymbol{u}) = \frac{C_b}{\delta}\int_\Omega |\boldsymbol{w}||\boldsymbol{u}|^2 d\Omega > 0$. Thus, for the first condition we have

$$a(\boldsymbol{w};\boldsymbol{u},\boldsymbol{u}) = a_0(\boldsymbol{u},\boldsymbol{u}) + a_4(\boldsymbol{w};\boldsymbol{u},\boldsymbol{u}) \geq \nu\mathrm{B}_m|\boldsymbol{u}|_1^2 + \frac{C_b}{\delta}\int_\Omega |\boldsymbol{w}||\boldsymbol{u}|^2 d\Omega \geq \nu\mathrm{B}_m|\boldsymbol{u}|_1^2\,. \tag{22}$$

Hence, condition 1 of theorem 2 and, setting $\boldsymbol{w} = \boldsymbol{u}$, condition 1 of theorem 1 are both satisfied, with $\alpha = \nu\mathrm{B}_m$.

For the second condition, we show that this condition is satisfied by each a_i. Let $\boldsymbol{u}$ be in V and $\{\boldsymbol{u}_m\}$ be a sequence in V such that

$$\boldsymbol{u}_m \rightharpoonup \boldsymbol{u} \quad \text{weakly in } V \text{ as } m \to \infty\,.$$

Since $a_0(\boldsymbol{u},\boldsymbol{v})$ is topologically equivalent to the $H_0^1(\Omega)$ inner product, we have

$$\lim_{m\to\infty} a_0(\boldsymbol{u}_m,\boldsymbol{v}) = a_0(\boldsymbol{u},\boldsymbol{v})\,. \tag{23}$$

For $a_1(\cdot;\cdot,\cdot)$, note that $H^1(\Omega)^2$ is compactly embedded in $L^2(\Omega)^2$. By the Riesz-Schauder theorem (see, *i.e.*, [14]), any weakly convergent sequence in $H^1(\Omega)^2$ is strongly convergent in $L^2(\Omega)^2$. Thus, we have

$$\lim_{m\to\infty} \|\boldsymbol{u}_m - \boldsymbol{u}\|_0 = 0\,.$$

Now let $\boldsymbol{v}$ be in $\mathcal{V}$, where $\mathcal{V} = \mathcal{D}(\Omega)^2 \cap V$ and $\mathcal{D}(\Omega)$ is the space of all infinitely differentiable functions with compact support in Ω. We want to take the limit of $a(\boldsymbol{u}_m; \boldsymbol{u}_m, \boldsymbol{v})$. An integration by parts gives

$$a_1(\boldsymbol{u}_m; \boldsymbol{u}_m, \boldsymbol{v}) = -\int_\Omega (\boldsymbol{u}_m \cdot \nabla \boldsymbol{v}) \cdot \boldsymbol{u}_m \mathrm{B} d\Omega\,.$$

Note that $\nabla \boldsymbol{v}$ is in $L^\infty(\Omega)$ and that the strong convergence of $\{\boldsymbol{u}_m\}$ in $L^2(\Omega)$ gives $\lim_{m\to\infty} u_m^i u_m^j = u^i u^j$ in $L^1(\Omega)$. Thus, we have

$$\begin{aligned}\lim_{m\to\infty} \big(a_1(\boldsymbol{u}_m; \boldsymbol{u}_m, \boldsymbol{v}) \\ - a_1(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v})\big) &= \lim_{m\to\infty} \left[\int_\Omega \boldsymbol{u} \cdot (\nabla \boldsymbol{v}) \cdot \boldsymbol{u}\, \mathrm{B} d\Omega \right. \\ &\qquad \left. - \int_\Omega \boldsymbol{u}_m \cdot (\nabla \boldsymbol{v}) \cdot \boldsymbol{u}_m \mathrm{B} d\Omega \right] \\ &\le \lim_{m\to\infty} \|\nabla \boldsymbol{v}\|_{L^\infty} \|\mathrm{B}\|_{L^\infty} \sum_{i,j} \|u_m^i u_m^j - u^i u^j\|_{L^1} = 0\,,\end{aligned}$$

so we have

$$\lim_{m\to\infty} a_1(\boldsymbol{u}_m; \boldsymbol{u}_m, \boldsymbol{v}) = a_1(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v})\,. \tag{24}$$

By the density of $\mathcal{V}$ in V, this result holds for all $\boldsymbol{v} \in V$.

For $a_2(\cdot; \cdot, \cdot)$, we again let $\boldsymbol{v}$ be in $\mathcal{V}$ and take the limit of $a_2(\boldsymbol{u}_m; \boldsymbol{u}_m, \boldsymbol{v})$. We consider the two parts of the right side of equation 20 separately. For the first term,

$$\frac{\delta^2}{3} \int_\Omega (\boldsymbol{u}_m \cdot \nabla \mathrm{B})(\nabla \mathrm{B}) \cdot (\boldsymbol{v} \cdot \nabla \boldsymbol{u}_m) \mathrm{B} d\Omega\,,$$

we use the fact that since $\boldsymbol{u}_m \to \boldsymbol{u}$ strongly in L^2 and $\nabla \boldsymbol{u}_m \to \nabla \boldsymbol{u}$ weakly in L^2, then $(\nabla \boldsymbol{u}_m)\boldsymbol{u}_m \to (\nabla \boldsymbol{u})\boldsymbol{u}$ weakly in L^2. For the second term,

$$-\frac{\delta^2}{3} \int_\Omega (\boldsymbol{u}_m \cdot \nabla \mathrm{B})(\nabla \mathrm{B}) \cdot (\boldsymbol{u}_m \cdot \nabla \boldsymbol{v}) \mathrm{B} d\Omega\,,$$

we again use the fact that $\boldsymbol{u}_m \to \boldsymbol{u}$ strongly in L^2 means that $\boldsymbol{u}_m \boldsymbol{u}_m \to \boldsymbol{u}\boldsymbol{u}$ strongly in L^1. Thus, we have

$$\begin{aligned}
\lim_{m\to\infty} \big(a_2(\boldsymbol{u}_m;\boldsymbol{u}_m,\boldsymbol{v}) \\
- a_2(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v})\big) &= \frac{\delta^2}{3}\lim_{m\to\infty}\bigg[\int_\Omega (\boldsymbol{u}_m\cdot\nabla \mathrm{B})(\nabla \mathrm{B})\cdot(\boldsymbol{v}\cdot\nabla\boldsymbol{u}_m - \boldsymbol{u}_m\cdot\nabla\boldsymbol{v})\mathrm{B}\mathrm{d}\Omega \\
&\qquad - \int_\Omega (\boldsymbol{u}\cdot\nabla \mathrm{B})(\nabla \mathrm{B})\cdot(\boldsymbol{v}\cdot\nabla\boldsymbol{u} - \boldsymbol{u}\cdot\nabla\boldsymbol{v})\mathrm{B}\mathrm{d}\Omega\bigg] \\
&\le \frac{\delta^2}{3}\lim_{m\to\infty}\|\nabla \mathrm{B}\|^2_{L^\infty}\|\mathrm{B}\|_{L^\infty} \\
&\quad \times \sum_{i,j}\Big[\big|(\boldsymbol{v},(\nabla u^i_m)u^j_m - (\nabla u^j)u^i)\big| \\
&\qquad + \|\nabla\boldsymbol{v}\|_{L^\infty}\|u^i_m u^j_m - u^i u^j\|_{L^1}\Big] \\
&= 0\,,
\end{aligned}$$

so we have

$$\lim_{m\to\infty} a_2(\boldsymbol{u}_m;\boldsymbol{u}_m,\boldsymbol{v}) = a_2(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v})\,. \tag{25}$$

By the density of $\mathcal{V}$ in V, this result holds for all $\boldsymbol{v}\in V$.

For $a_3(\boldsymbol{u},\boldsymbol{v})$, using the Cauchy-Schwarz inequality and the fact that $\mathrm{B}(\boldsymbol{x}) < \mathrm{B}_M$, we have

$$\begin{aligned}
\lim_{m\to\infty} a_3(\boldsymbol{u}_m - \boldsymbol{u},\boldsymbol{v}) &= -\lim_{m\to\infty}\frac{1}{\epsilon_R}\int_\Omega (\boldsymbol{u}_m-\boldsymbol{u})^\perp\cdot\boldsymbol{v}\,\mathrm{B}\mathrm{d}\Omega \\
&\le \frac{\mathrm{B}_M}{\epsilon_R}\lim_{m\to\infty}\|\boldsymbol{u}_m-\boldsymbol{u}\|_0\|\boldsymbol{v}\|_0 = 0\,,
\end{aligned}$$

since $\boldsymbol{u}_m\to\boldsymbol{u}$ strongly in $L^2(\Omega)$. Thus,

$$\lim_{m\to\infty} a_3(\boldsymbol{u}_m,\boldsymbol{v}) = a_3(\boldsymbol{u},\boldsymbol{v})\,. \tag{26}$$

Finally, for $a_4(\boldsymbol{w};\boldsymbol{u},\boldsymbol{v})$ we have

$$\begin{aligned}
\lim_{m\to\infty}\big(a_4(\boldsymbol{u}_m;\boldsymbol{u}_m,\boldsymbol{v}) - a_4(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v})\big) &= \lim_{m\to\infty}\int_\Omega \big(|\boldsymbol{u}_m|\boldsymbol{u}_m - |\boldsymbol{u}|\boldsymbol{u}\big)\cdot\boldsymbol{v}\,\mathrm{d}\Omega \\
&= \lim_{m\to\infty}\bigg[\int_\Omega \big(|\boldsymbol{u}_m| - |\boldsymbol{u}|\big)\boldsymbol{u}_m\cdot\boldsymbol{v}\,\mathrm{d}\Omega + \int_\Omega |\boldsymbol{u}|(\boldsymbol{u}_m-\boldsymbol{u})\cdot\boldsymbol{v}\,\mathrm{d}\Omega\bigg]\,.
\end{aligned}$$

If we again choose $\boldsymbol{v}\in\mathcal{V}$, then for the first term we have

$$\begin{aligned}
\lim_{m\to\infty}\int_\Omega \big(|\boldsymbol{u}_m| - |\boldsymbol{u}|\big)\boldsymbol{u}_m\cdot\boldsymbol{v}\,\mathrm{d}\Omega &\le \lim_{m\to\infty}\int_\Omega \big||\boldsymbol{u}_m| - |\boldsymbol{u}|\big||\boldsymbol{u}_m\cdot\boldsymbol{v}|\mathrm{d}\Omega \\
\le \lim_{m\to\infty}\int_\Omega |\boldsymbol{u}_m-\boldsymbol{u}||\boldsymbol{u}_m\cdot\boldsymbol{v}|\mathrm{d}\Omega &\le \lim_{m\to\infty}\|\boldsymbol{u}_m-\boldsymbol{u}\|_0\|\boldsymbol{u}_m\|_0\|\boldsymbol{v}\|_{L^\infty} = 0\,,
\end{aligned}$$

since $\boldsymbol{u}_m \to \boldsymbol{u}$ strongly in $L^2(\Omega)$ and $\|\boldsymbol{u}_m\|_0$ and $\|\boldsymbol{v}\|_{L^\infty}$ are finite. For the second term we have

$$\int_\Omega |\boldsymbol{u}|(\boldsymbol{u}_m - \boldsymbol{u})\cdot\boldsymbol{v}\,\mathrm{d}\Omega \le \int_\Omega |\boldsymbol{u}||\boldsymbol{u}_m - \boldsymbol{u}||\boldsymbol{v}|\mathrm{d}\Omega \le \lim_{m\to\infty} \|\boldsymbol{u}_m - \boldsymbol{u}\|_0\|\boldsymbol{u}\|_0\|\boldsymbol{v}\|_{L^\infty} = 0\,,$$

for the same reasons as before. Thus, we have

$$\lim_{m\to\infty} a_4(\boldsymbol{u}_m; \boldsymbol{u}_m, \boldsymbol{v}) = a_4(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v})\,. \tag{27}$$

Finally, combining 23, 24, 25, 26, and 27, we have

$$\lim_{m\to\infty} a(\boldsymbol{u}_m; \boldsymbol{u}_m, \boldsymbol{v}) = a(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v})\,.$$

Hence, condition 2 is satisfied.

Finally, for the third condition we use the fact that this condition holds when the bottom is flat, i.e.,

$$\inf_{\eta\in L^2} \sup_{\boldsymbol{u}\in H^1} \frac{\int \eta\nabla\cdot\boldsymbol{u}\mathrm{d}\Omega}{|\boldsymbol{u}|_1\|\eta\|_0} \ge \beta_{flat} > 0\,.$$

This result can be found in, for example, [9]. For the problem at hand we have the expression

$$\inf_{\eta\in M} \sup_{\boldsymbol{u}\in X} \frac{b(\boldsymbol{u},\eta)}{|\boldsymbol{u}|_1\|\eta\|_0} = \inf_{\eta\in M} \sup_{\boldsymbol{u}\in X} \frac{\int \eta\nabla\cdot(\mathrm{B}\boldsymbol{u})\mathrm{d}\Omega}{|\boldsymbol{u}|_1\|\eta\|_0}\,.$$

Let $\boldsymbol{w} = \mathrm{B}\boldsymbol{u}$. Since $\mathrm{B} \neq 0$, we have that $\boldsymbol{u} = \boldsymbol{w}/\mathrm{B}$ is well defined, and

$$\begin{aligned}
|\boldsymbol{u}|_1^2 = \int_\Omega |\nabla\boldsymbol{u}|^2\mathrm{d}\Omega &= \int_\Omega \left|\nabla\left(\frac{\boldsymbol{w}}{\mathrm{B}}\right)\right|^2\mathrm{d}\Omega \\
&= \int_\Omega \left|\frac{\nabla\boldsymbol{w}}{\mathrm{B}} - \frac{\boldsymbol{w}\nabla\mathrm{B}}{\mathrm{B}^2}\right|^2\mathrm{d}\Omega \\
&\le \frac{C_1}{\mathrm{B}_m^2}\left[\int_\Omega |\nabla\boldsymbol{w}|^2\mathrm{d}\Omega + \frac{\delta^2}{\mathrm{B}_m^2}\int_\Omega |\boldsymbol{w}|^2\mathrm{d}\Omega\right] \\
&\le \frac{C_1}{\mathrm{B}_m^2}\left[1 + \frac{\delta^2 C_2(\Omega)}{\mathrm{B}_m^2}\right]|\boldsymbol{w}|_1^2 = C^2(\mathrm{B},\Omega)|\boldsymbol{w}|_1^2\,,
\end{aligned}$$

using the Poincaré inequality and the fact that $\nabla \mathrm{B} = \mathcal{O}(\delta)$. Thus, we have that

$$
\begin{aligned}
\inf_{\eta \in M} \sup_{\boldsymbol{u} \in X} \frac{\int \eta \nabla \cdot (\mathrm{B}\boldsymbol{u}) \mathrm{d}\Omega}{|\boldsymbol{u}|_1 \|\eta\|_0} &= \inf_{\eta \in M} \sup_{\boldsymbol{w} \in X} \frac{\int \eta \nabla \cdot (\boldsymbol{w}) \mathrm{d}\Omega}{|\boldsymbol{w}/\mathrm{B}|_1 \|\eta\|_0} \\
&\geq \frac{1}{C(\mathrm{B}, \Omega)} \inf_{\eta \in M} \sup_{\boldsymbol{w} \in X} \frac{\int \eta \nabla \cdot (\boldsymbol{w}) \mathrm{d}\Omega}{|\boldsymbol{w}|_1 \|\eta\|_0} \\
&\geq \frac{\beta_{flat}}{C(\mathrm{B}, \Omega)} = \beta > 0 \,.
\end{aligned}
$$

The above steps show that Problem 1 satisfies the conditions of theorem 1. Thus, we have proved

Theorem 3. *Problem 1 has a solution* $(\boldsymbol{u}, \eta) \in H_0^1(\Omega)^2 \times L_0^2(\Omega)$.

Next we consider the uniqueness of solutions of problem 1. To apply theorem 2, we introduce

$$
\mathcal{N} = \sup_{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V} \frac{|a_1(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v})| + |a_2(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v})| + |a_4^*(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v})|}{|\boldsymbol{w}|_1 |\boldsymbol{u}|_1 |\boldsymbol{v}|_1},
$$

where

$$
a_4^*(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v}) = \frac{C_b}{\delta} \int_\Omega |\boldsymbol{w}| |\boldsymbol{u} \cdot \boldsymbol{v}| \mathrm{d}\Omega \,.
$$

With this we can state the following theorem.

Theorem 4. *If*

$$
\nu > \nu_0 = \sqrt{\mathcal{N}} \|\boldsymbol{f}\|_{V'}\,, \tag{28}
$$

then problem 1 has a unique solution in $V \times L_0^2(\Omega)$.

Proof: We need to check the conditions of theorem 2. We have already shown that condition 1 is satisfied. Next we let $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}_1$, and $\boldsymbol{w}_2$ be in V. Because a_0 and a_3 do not depend on the first argument of a and a_1 and a_2 are trilinear, we have

$$
\begin{aligned}
|a(\boldsymbol{w}_1; \boldsymbol{u}, \boldsymbol{v}) - a(\boldsymbol{w}_2; \boldsymbol{u}, \boldsymbol{v})| \leq{}& |a_1(\boldsymbol{w}_1 - \boldsymbol{w}_2; \boldsymbol{u}, \boldsymbol{v})| + |a_2(\boldsymbol{w}_1 - \boldsymbol{w}_2; \boldsymbol{u}, \boldsymbol{v})| \\
&+ |a_4(\boldsymbol{w}_1; \boldsymbol{u}, \boldsymbol{v}) - a_4(\boldsymbol{w}_2; \boldsymbol{u}, \boldsymbol{v})| \,.
\end{aligned} \tag{29}
$$

For this last term we have

$$
\begin{aligned}
|a_4(\boldsymbol{w}_1;\boldsymbol{u},\boldsymbol{v}) - a_4(\boldsymbol{w}_2;\boldsymbol{u},\boldsymbol{v})| &= \left|\frac{C_b}{\delta}\int_\Omega (|\boldsymbol{w}_1| - |\boldsymbol{w}_2|)\boldsymbol{u}\cdot\boldsymbol{v}\mathrm{d}\Omega\right| \\
&\le \frac{C_b}{\delta}\int_\Omega \big||\boldsymbol{w}_1| - |\boldsymbol{w}_2|\big|\,|\boldsymbol{u}\cdot\boldsymbol{v}|\mathrm{d}\Omega \\
&\le \frac{C_b}{\delta}\int_\Omega |\boldsymbol{w}_1 - \boldsymbol{w}_2|\,|\boldsymbol{u}\cdot\boldsymbol{v}|\mathrm{d}\Omega = a_4^*(\boldsymbol{w}_1 - \boldsymbol{w}_2;\boldsymbol{u},\boldsymbol{v})\,.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
|a(\boldsymbol{w}_1;\boldsymbol{u},\boldsymbol{v}) - a(\boldsymbol{w}_2;\boldsymbol{u},\boldsymbol{v})| &\le |a_1(\boldsymbol{w}_1 - \boldsymbol{w}_2;\boldsymbol{u},\boldsymbol{v})| + |a_2(\boldsymbol{w}_1 - \boldsymbol{w}_2;\boldsymbol{u},\boldsymbol{v})| \\
&\quad + |a_4^*(\boldsymbol{w}_1 - \boldsymbol{w}_2;\boldsymbol{u},\boldsymbol{v})| \\
&\le \mathcal{N}|\boldsymbol{u}|_1|\boldsymbol{v}|_1|\boldsymbol{w}_1 - \boldsymbol{w}_2|_1\,.
\end{aligned}
$$

Thus, in theorem 2 if we set $L \equiv \mathcal{N}$ condition 2 is satisfied, and the inequality 28 is equivalent to the inequality 18. Hence, subject to condition 28, solutions of problem 1 are unique. □

5 Existence and Uniqueness with Non-homogeneous Boundary Conditions

Here we show that the existence result of the previous section will still hold when $\boldsymbol{u}$ has non-homogeneous boundary conditions. The idea is to show that in this case the solution can be written as a sum of two functions: one which satisfies the boundary condition and contributes little in the interior, and another which has a homogeneous boundary condition and solves an equation on the interior that satisfies the conditions for existence from section 4. We also present a condition for uniqueness.

For non-homogeneous boundary conditions we consider

$$
\boldsymbol{u} = \boldsymbol{g}(\boldsymbol{x}) \quad \text{on } \partial\Omega\,, \tag{30}
$$

where $\boldsymbol{g}$ is not necessarily identically 0. As the domain may be multiply connected, we denote by $\partial\Omega_i$, $i = 1, 2, \ldots, p$ the separate components of the boundary $\partial\Omega$, as in figure 2. Using this notation, we have a compatibility condition for $\boldsymbol{g}$:

$$
\oint_{\partial\Omega_i} \boldsymbol{g}\cdot\boldsymbol{n}\,\mathrm{B}\mathrm{d}s = 0\,, \quad i = 1, 2, \ldots, p. \tag{31}
$$

The compatibility condition is required for solutions of problem 1 because, due to equation 11b, we have

$$\int_{\Omega} \nabla \cdot (\mathrm{B}\boldsymbol{u}) \mathrm{d}\Omega = \oint_{\partial\Omega} \boldsymbol{u} \cdot \boldsymbol{n} \, \mathrm{B} \mathrm{d}s = \oint_{\partial\Omega} \boldsymbol{g} \cdot \boldsymbol{n} \, \mathrm{B} \mathrm{d}s = 0 \, .$$

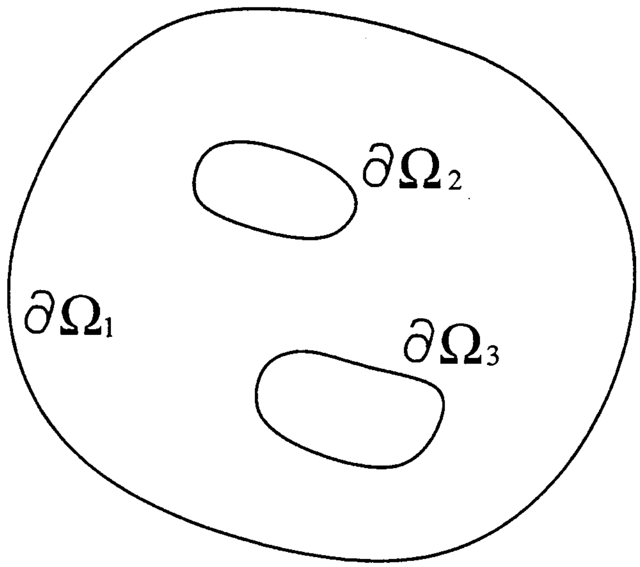

Figure 2. Domain with multiple components of the boundary.

Due to the trace theorem (see [1]), a function in $H^1(\Omega)^2$ has boundary data $\boldsymbol{g}$ in $H^{1/2}(\Omega)^2$. Thus, the problem with non-homogeneous boundary conditions can be stated as follows:

Problem 4. *Given* $\boldsymbol{f} \in H^{-1}(\Omega)^2$ *and* $\boldsymbol{g} \in H^{1/2}(\partial\Omega)^2$ *satisfying equation 31, find* $\boldsymbol{u} \in H^1(\Omega)^2$ *and* $\eta \in L_0^2(\Omega)$ *such that*

$$a(\boldsymbol{u}; \boldsymbol{u}, \tilde{\boldsymbol{u}}) - b(\tilde{\boldsymbol{u}}, \eta) = \langle \boldsymbol{f}, \tilde{\boldsymbol{u}} \rangle_{\mathrm{B}} \qquad \forall \tilde{\boldsymbol{u}} \in H_0^1(\Omega)^2 \, , \tag{32a}$$

$$b(\boldsymbol{u}, \tilde{\eta}) = 0 \qquad \forall \tilde{\eta} \in L_0^2(\Omega) \, , \tag{32b}$$

$$\boldsymbol{u} = \boldsymbol{g} \qquad \text{on } \partial\Omega \, . \tag{32c}$$

To establish the existence and uniqueness of solutions of Problem 4, we first need to demonstrate that we can find a function that satisfies the boundary condition but contributes little to the interior equations. To show this we need the following three lemmas. The first two of these are rather technical and will be stated without proof; they can be found in [6]. Below, the notation $d(\boldsymbol{x};\partial\Omega)$ denotes the minimum distance from a point $\boldsymbol{x}$ to the boundary $\partial\Omega$.

Lemma 1. *For all $\epsilon > 0$, there exists a function $\theta_\epsilon \in C^2(\bar{\Omega})$ such that*

$$\left.\begin{array}{ll} \theta_\epsilon = 1 & \textit{in a neighborhood of } \partial\Omega\,, \\ \theta_\epsilon(\boldsymbol{x}) = 0 & \textit{if } d(\boldsymbol{x};\partial\Omega) \geq 2\delta(\epsilon) \quad \big(\delta(\epsilon) = \exp(-1/\epsilon)\big)\,, \\ \left|\dfrac{\partial\theta_\epsilon}{\partial x_i}\right| \leq \dfrac{\epsilon}{d(\boldsymbol{x};\partial\Omega)} & \textit{if } d(\boldsymbol{x};\partial\Omega) \leq 2\delta(\epsilon)\,, \quad i = 1,2\,. \end{array}\right\} \tag{33}$$

Lemma 2. *There exists a constant $C = C(\Omega) > 0$ such that*

$$\left\|\frac{\phi}{d(.;\partial\Omega)}\right\|_0 \leq C|\phi|_1 \quad \forall\phi \in H_0^1(\Omega)\,. \tag{34}$$

The first of these asserts that there is a function θ_ϵ that is equal to 1 on a thin strip along the boundary, vanishes in most of the rest of the domain Ω, and has suitable bounds on its derivatives. The second gives a useful inequality bounding the L^2 norm of the ratio of a function ϕ to its distance to the boundary by the H^1 norm of ϕ.

Next we prove a lemma stating that for any suitable function $\boldsymbol{g}$ we can find a function $\boldsymbol{u}$ that satisfies the weighted incompressibility condition (equation 11b) and is equal to $\boldsymbol{g}$ along the boundary.

Lemma 3. *For each $\boldsymbol{g} \in H^{1/2}(\partial\Omega)^2$ satisfying Equation 31, there exists a function $\boldsymbol{u} \in H^1(\Omega)^2$ such that*

$$\nabla\cdot(\mathrm{B}\boldsymbol{u}) = 0\,, \quad \boldsymbol{u}|_{\partial\Omega} = \boldsymbol{g}\,.$$

Proof. Let $\boldsymbol{w}$ be any function of $H^1(\Omega)^2$ that satisfies $\boldsymbol{w} = \boldsymbol{g}$ on $\partial\Omega$. By Green's formula we have

$$\int_\Omega \nabla\cdot(\mathrm{B}\boldsymbol{w})d\Omega = \int_{\partial\Omega} \boldsymbol{g}\cdot\boldsymbol{n}\,\mathrm{B}ds = 0.$$

Therefore $\nabla\cdot(\mathrm{B}\boldsymbol{w}) \in L_0^2(\Omega)$. Because the operator B, presented in section 4, is isomorphic from $V^\perp$ (the $H^1(\Omega)^2$ complement of V) to $M' = L_0^2(\Omega)$, there is a $\boldsymbol{v}$ in $V^\perp$ such that

$$\nabla\cdot(\mathrm{B}\boldsymbol{v}) = \nabla\cdot(\mathrm{B}\boldsymbol{w})\,.$$

Along the boundary $\boldsymbol{v}$ is 0, as it is an element of $V^{\perp}$. Setting $\boldsymbol{u} = \boldsymbol{w} - \boldsymbol{v}$ gives the desired result. □

Now we can prove the following lemma, similar to one found in [7], showing that we can find a function $\boldsymbol{u}$ which satisfies the boundary condition 30 and the weighted incompressibility condition, but also satisfies inequalities regarding the nonlinear terms.

Lemma 4. *Given a function $\boldsymbol{g} \in H^{1/2}(\partial\Omega)^2$ satisfying condition 31, for any $\epsilon > 0$ there exists a function $\boldsymbol{u}_0 = \boldsymbol{u}_0(\epsilon) \in H^1(\Omega)^2$ such that*

$$\nabla \cdot (\mathrm{B}\boldsymbol{u}_0) = 0\,, \quad \boldsymbol{u}_0|_{\partial\Omega} = \boldsymbol{g}\,, \tag{35}$$

$$|a_i(\boldsymbol{v}; \boldsymbol{u}_0, \boldsymbol{v})| \le \epsilon |\boldsymbol{v}|_1^2 \quad \forall \boldsymbol{v} \in V\,, i = 1, 2\,, \tag{36}$$

$$|a_4(\boldsymbol{u}_0; \boldsymbol{v}, \boldsymbol{v})| \le \epsilon |\boldsymbol{v}|_1^2 \quad \forall \boldsymbol{v} \in V\,. \tag{37}$$

Proof: By Lemma 3, there exists a function $\boldsymbol{w}_0 \in H^1(\Omega)^2$ such that

$$\nabla \cdot (\mathrm{B}\boldsymbol{w}_0) = 0\,, \quad \boldsymbol{w}_0|_{\partial\Omega} = \boldsymbol{g}\,.$$

We can express $\boldsymbol{w}_0$ in terms of a stream function $\psi_0 \in H^2(\Omega)$ by

$$\boldsymbol{w}_0 = \frac{1}{\mathrm{B}} \nabla \wedge \psi_0 = \frac{1}{\mathrm{B}} (-\partial_y \psi_0, \partial_x \psi_0)\,.$$

For all $\mu > 0$, introduce the function

$$\boldsymbol{u}_{0\mu} = \frac{1}{\mathrm{B}} \nabla \wedge (\theta_\mu \psi_0)\,,$$

where θ_μ is defined as in Lemma 1. Note that $\boldsymbol{u}_{0\mu}$ is in $H^1(\Omega)^2$ and that conditions 35 are satisfied. Now for $\boldsymbol{x}$ such that $d(\boldsymbol{x}; \partial\Omega) \le 2\delta(\mu)$, Lemma 1 gives

$$|\partial_{x_j}(\theta_\mu \psi_0)| \le |\psi_0(\partial_{x_j}\theta_\mu)| + |\theta_\mu(\partial_{x_j}\psi_0)| \le \frac{\mu}{d(\boldsymbol{x}; \partial\Omega)} |\psi_0(\boldsymbol{x})| + |\partial_{x_j}\psi_0(\boldsymbol{x})|\, |\theta_\mu|\,,$$

so that

$$\begin{aligned}
|\boldsymbol{u}_{0\mu}(\boldsymbol{x})|^2 &= \left| \frac{1}{\mathrm{B}} \partial_x(\theta_\mu \psi_0) \right|^2 + \left| \frac{1}{\mathrm{B}} \partial_y(\theta_\mu \psi_0) \right|^2 \\
&\le \frac{1}{\mathrm{B}_m^2} \left[\left(\frac{\mu}{d(\boldsymbol{x}; \partial\Omega)} |\psi_0(\boldsymbol{x})| + |\partial_x \psi_0(\boldsymbol{x})|\, |\theta_\mu| \right)^2 \right. \\
&\qquad \left. + \left(\frac{\mu}{d(\boldsymbol{x}; \partial\Omega)} |\psi_0(\boldsymbol{x})| + |\partial_y \psi_0(\boldsymbol{x})|\, |\theta_\mu| \right)^2 \right] \\
&\le \frac{2}{\mathrm{B}_m^2} \left[2\mu^2 \frac{|\psi_0|^2}{d^2(\boldsymbol{x}; \partial\Omega)} + |\theta_\mu|^2 |\nabla \psi_0|^2 \right].
\end{aligned}$$

Let $\boldsymbol{v} = (v_1, v_2)$ belong to V. We have

$$\|v_i u_{0\mu j}\|_0^2 \leq \int_\Omega |v_i(\boldsymbol{x})|^2 |u_{0\mu j}(\boldsymbol{x})|^2 \mathrm{d}\Omega$$
$$\leq \frac{2}{\mathrm{B}_m^2}\left[2\mu^2 \int_\Omega |\psi_0|^2 \frac{|v_i|^2}{d^2(\boldsymbol{x};\partial\Omega)} \mathrm{d}\Omega + \int_\Omega |\theta_\mu|^2 |v_i|^2 |\nabla\psi_0|^2 \mathrm{d}\Omega\right].$$

For the first term, Sobolev embedding gives that $H^2(\Omega) \hookrightarrow \mathcal{C}^0(\Omega)$, so $\max_{\boldsymbol{x}} |\psi_0(\boldsymbol{x})|^2 < \infty$. Moreover, since ψ_0 is determined by $\boldsymbol{g}$, we can express $\max_{\boldsymbol{x}} |\psi_0|^2$ as $C_1(\boldsymbol{g})$ and bound the first integral by $C_1(\boldsymbol{g}) \|v_i / d(\cdot;\partial\Omega)\|_0^2$. For the second term, we have that $|\theta_\mu| \leq 1$ if $d(x;\partial\Omega) \leq 2\delta(\mu)$ and $|\theta_\mu| = 0$ if $d(x;\partial\Omega) \geq 2\mu$, so the above is

$$\|v_i u_{0\mu j}\|_0^2 \leq \frac{2}{\mathrm{B}_m^2}\left[2C_1(\boldsymbol{g})\mu^2 \left\|\frac{v_i}{d(\cdot;\partial\Omega)}\right\|_0^2 + \int_{d(x;\partial\Omega)\leq 2\delta(\mu)} |v_i|^2 |\nabla\psi_0|^2 \mathrm{d}\Omega\right].$$

By Hölder's inequality we have

$$\int_{d(x;\partial\Omega)\leq 2\delta(\mu)} |v_i|^2 |\nabla\psi_0|^2 \mathrm{d}\Omega$$
$$\leq \left(\int_{d(x;\partial\Omega)\leq 2\delta(\mu)} |v_i|^4 \mathrm{d}\Omega\right)^{1/2} \left(\int_{d(x;\partial\Omega)\leq 2\delta(\mu)} |\nabla\psi_0|^4 \mathrm{d}\Omega\right)^{1/2}.$$

Since $H^1(\Omega) \hookrightarrow L^4(\Omega)$, we also have that $(\int |v_i|^4 \mathrm{d}\Omega)^{1/2} \leq C_2(\Omega)|v_i|_1^2$. Applying these results and Lemma 2 gives

$$\|v_i u_{0\mu j}\|_0^2 \leq \frac{2}{\mathrm{B}_m^2}\Bigg[2C_1(\boldsymbol{g})C_3(\Omega)\mu^2 |v_i|_1^2$$
$$+ C_2(\Omega)|v_i|_1^2 \left(\int_{d(x;\partial\Omega)\leq 2\delta(\mu)} |\nabla\psi_0|^4 \mathrm{d}\Omega\right)^{1/2}\Bigg].$$

Note that all terms in the right side are positive. Setting

$$\phi(\mu) = \left(\int_{d(x;\partial\Omega)\leq 2\delta(\mu)} |\nabla\psi_0(\boldsymbol{x})|^4 \mathrm{d}\Omega\right)^{1/4},$$

we get

$$\|v_i u_{0\mu j}\|_0 \leq C_4(\boldsymbol{g}, \Omega, \mathrm{B})(\mu + \phi(\mu))|v_i|_1. \tag{38}$$

Now we are able to show that conditions 36 and 37 hold. First, for $i = 1$, an integration by parts and Hölder's inequality give

$$|a_1(\boldsymbol{v}; \boldsymbol{u}_{0\mu}, \boldsymbol{v})| = \left|\int_\Omega (\boldsymbol{v}\cdot\nabla\boldsymbol{v})\cdot\boldsymbol{u}_{0\mu}\,\mathrm{B}\mathrm{d}\Omega\right| \leq \|\mathrm{B}\|_{L^\infty}^2 \|\nabla\boldsymbol{v}\|_0^2 \sum_{i,j=1}^{2} \|v_i u_{0\mu ij}\|_0^2$$
$$\leq C_5(\mu + \phi(\mu))|\boldsymbol{v}|_1^2.$$

Similarly, for $i = 2$ we have

$$
\begin{aligned}
|a_2(\boldsymbol{v}; \boldsymbol{u}_{0\mu}, \boldsymbol{v})| &= \left| \frac{\delta^2}{3} \int_\Omega (\boldsymbol{v}\cdot\nabla\mathrm{B})(\nabla\mathrm{B})\cdot(\boldsymbol{v}\cdot\nabla\boldsymbol{u}_{0\mu} - \boldsymbol{u}_{0\mu}\cdot\nabla\boldsymbol{v})\,\mathrm{B}\mathrm{d}\Omega \right| \\
&\leq C_6 \frac{\delta^2}{3} \|\nabla\mathrm{B}\|_{L^\infty}^2 \|\mathrm{B}\|_{L^\infty} \sum_{i,j=1}^{2} \int_\Omega |v_i(\boldsymbol{v}\cdot\nabla u_{0\mu j} - \boldsymbol{u}_{0\mu}\cdot\nabla v_j)|\,\mathrm{d}\Omega \\
&\leq C_6 \frac{\delta^2}{3} \|\nabla\mathrm{B}\|_{L^\infty}^2 \|\mathrm{B}\|_{L^\infty} \sum_{i,j=1}^{2} \int_\Omega \big(|u_{0\mu j}\boldsymbol{v}\cdot\nabla v_i| + |v_i\boldsymbol{u}_{0\mu}\cdot\nabla v_j|\big)\mathrm{d}\Omega \\
&\leq C_7(\mu + \phi(\mu))|\boldsymbol{v}|_1^2 ,
\end{aligned}
$$

using Hölder's inequality and the fact that B and it's derivatives are bounded.

Finally, for $i = 4$. we have

$$
\begin{aligned}
a_4(\boldsymbol{u}_{0\mu}; \boldsymbol{v}, \boldsymbol{v}) &= \frac{C_b}{\delta} \int_\Omega |\boldsymbol{u}_{0\mu}||\boldsymbol{v}|^2 \mathrm{d}\Omega \\
&\leq \frac{C_b}{\delta} \Big(\int_\Omega |\boldsymbol{u}_{0\mu}|^2 |\boldsymbol{v}|^2 \mathrm{d}\Omega \Big)^{1/2} \Big(\int_\Omega |\boldsymbol{v}|^2 \mathrm{d}\Omega \Big)^{1/2} \\
&\leq C_8(\mu + \phi(\mu))|\boldsymbol{v}|_1 \|\boldsymbol{v}\|_0 \leq C_9(\mu + \phi(\mu))|\boldsymbol{v}|_1^2 ,
\end{aligned}
$$

using equation 38, Hölder's inequality, and the Poincaré inequality.

Since $\lim_{\mu\to 0} \phi(\mu) = 0$, for any $\epsilon > 0$ we may choose μ small enough so that

$$\max(C_5, C_7, C_9)(\mu + \phi(\mu)) \leq \epsilon .$$

Setting $\boldsymbol{u}_0(\epsilon)$ to the corresponding $\boldsymbol{u}_{0\mu}$ now satisfies both conditions 35, 36, and 37. □

With the above results established, we can now prove the following theorem concerning existence of solutions of problem 4. The idea is to split the solution into two parts $\boldsymbol{u}_0$ and $\boldsymbol{w}$, where $\boldsymbol{u}_0$ has the properties of the function in lemma 4 and $\boldsymbol{w}$ satisfies an equation of the type in problem 2.

Theorem 5. *Given $\boldsymbol{f} \in H^{-1}(\Omega)^2$ and $\boldsymbol{g} \in H^{1/2}(\partial\Omega)^2$ satisfying condition 31, there exists at least one pair $(\boldsymbol{u}, \eta) \in H^1(\Omega)^2 \times L_0^2(\Omega)$ that solves problem 4.*

Proof: Choose $\boldsymbol{u}_0 \in H^1(\Omega)^2$ as in lemma 4 so that it satisfies conditions 35, 36, and 37, with $\epsilon < 4\nu$, and set $\boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{w}$. Because

$$\begin{aligned} a(\boldsymbol{u}_0 + \boldsymbol{w}; \boldsymbol{u}_0 + \boldsymbol{w}, \tilde{\boldsymbol{u}}) = a(\boldsymbol{w}; \boldsymbol{w}, \tilde{\boldsymbol{u}}) &+ a(\boldsymbol{u}_0; \boldsymbol{u}_0, \tilde{\boldsymbol{u}}) + a_1(\boldsymbol{u}_0; \boldsymbol{w}, \tilde{\boldsymbol{u}}) \\ &+ a_2(\boldsymbol{u}_0; \boldsymbol{w}, \tilde{\boldsymbol{u}}) + a_1(\boldsymbol{w}; \boldsymbol{u}_0, \tilde{\boldsymbol{u}}) + a_2(\boldsymbol{w}; \boldsymbol{u}_0, \tilde{\boldsymbol{u}}) \\ &+ a_4(\boldsymbol{u}_0 + \boldsymbol{w}; \boldsymbol{u}_0, \tilde{\boldsymbol{u}}) - a_4(\boldsymbol{u}_0; \boldsymbol{u}_0, \tilde{\boldsymbol{u}}) \\ &+ a_4(\boldsymbol{u}_0 + \boldsymbol{w}; \boldsymbol{w}, \tilde{\boldsymbol{u}}) - a_4(\boldsymbol{w}; \boldsymbol{w}, \tilde{\boldsymbol{u}}) \,, \end{aligned}$$

Problem 4 can be stated as follows:

Find a pair $(\boldsymbol{w}, \eta) \in H_0^2(\Omega)^2 \times L_0^2(\Omega)$ *such that*

$$\begin{aligned} \tilde{a}(\boldsymbol{w}; \boldsymbol{w}, \tilde{\boldsymbol{u}}) - b(\tilde{\boldsymbol{u}}, \eta) &= \langle \boldsymbol{f}, \tilde{\boldsymbol{u}} \rangle_{\mathrm{B}} - a(\boldsymbol{u}_0; \boldsymbol{u}_0, \tilde{\boldsymbol{u}}) \qquad \forall \tilde{\boldsymbol{u}} \in H_0^1(\Omega)^2 \,, \\ b(\boldsymbol{w}, \tilde{\eta}) &= 0 \qquad \forall \tilde{\eta} \in L_0^2(\Omega) \,, \end{aligned}$$

where

$$\begin{aligned} \tilde{a}(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v}) = a(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v}) &+ a_1(\boldsymbol{u}_0; \boldsymbol{u}, \boldsymbol{v}) + a_2(\boldsymbol{u}_0; \boldsymbol{u}, \boldsymbol{v}) + a_1(\boldsymbol{u}; \boldsymbol{u}_0, \boldsymbol{v}) \\ &+ a_2(\boldsymbol{u}; \boldsymbol{u}_0, \boldsymbol{v}) + a_4(\boldsymbol{u}_0 + \boldsymbol{u}; \boldsymbol{u}_0, \boldsymbol{v}) - a_4(\boldsymbol{u}_0; \boldsymbol{u}_0, \boldsymbol{v}) \\ &+ a_4(\boldsymbol{u}_0 + \boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v}) - a_4(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v}) \,. \end{aligned}$$

This problem now fits the framework of Problem 2 with $X = H_0^1(\Omega)$, $M = L_0^2(\Omega)$, and replace $a(\cdot;\cdot,\cdot)$ with $\tilde{a}(\cdot;\cdot,\cdot)$ and $\langle \boldsymbol{f}, \tilde{\boldsymbol{u}} \rangle_{\mathrm{B}}$ with $\langle \boldsymbol{f}, \tilde{\boldsymbol{u}} \rangle_{\mathrm{B}} - a(\boldsymbol{u}_0; \boldsymbol{u}_0, \tilde{\boldsymbol{u}})$. We then need only to check the conditions of Theorem 1. The second and third conditions are the same as before, so only condition 1 needs to be verified. For all $\boldsymbol{v}, \boldsymbol{w} \in V$ we have

$$\begin{aligned} \tilde{a}(\boldsymbol{w}; \boldsymbol{v}, \boldsymbol{v}) = \nu |\boldsymbol{v}|_1^2 &+ a_4(\boldsymbol{w}; \boldsymbol{v}, \boldsymbol{v}) + a_1(\boldsymbol{v}; \boldsymbol{u}_0, \boldsymbol{v}) \\ &+ a_2(\boldsymbol{v}; \boldsymbol{u}_0, \boldsymbol{v}) + a_4(\boldsymbol{u}_0 + \boldsymbol{v}; \boldsymbol{u}_0, \boldsymbol{v}) - a_4(\boldsymbol{u}_0; \boldsymbol{u}_0, \boldsymbol{v}) \\ &+ a_4(\boldsymbol{u}_0 + \boldsymbol{w}; \boldsymbol{v}, \boldsymbol{v}) - a_4(\boldsymbol{w}; \boldsymbol{v}, \boldsymbol{v}) \,. \end{aligned}$$

By condition 36 of Lemma 4 we have that

$$|a_i(\boldsymbol{v}; \boldsymbol{u}_0, \boldsymbol{v})| \le \epsilon |\boldsymbol{v}|_1^2 \quad \forall \boldsymbol{v} \in V \quad i = 1, 2,$$

with $\epsilon < \nu/2$. Moreover, we have that

$$\begin{aligned} a_4(\boldsymbol{u}_0 + \boldsymbol{v}; \boldsymbol{u}_0, \boldsymbol{v}) - a_4(\boldsymbol{u}_0; \boldsymbol{u}_0, \boldsymbol{v}) &= \frac{C_b}{\delta} \int_\Omega (|\boldsymbol{u}_0 + \boldsymbol{v}| - |\boldsymbol{u}_0|) \boldsymbol{u}_0 \cdot \boldsymbol{v} \, \mathrm{d}\Omega \\ &\le \frac{C_b}{\delta} \int_\Omega \big| |\boldsymbol{u}_0 + \boldsymbol{v}| - |\boldsymbol{u}_0| \big| |\boldsymbol{u}_0| |\boldsymbol{v}| \mathrm{d}\Omega \\ &\le \frac{C_b}{\delta} \int_\Omega |\boldsymbol{v}| |\boldsymbol{u}_0| |\boldsymbol{v}| \mathrm{d}\Omega = a_4(\boldsymbol{u}_0; \boldsymbol{v}, \boldsymbol{v}) \le \epsilon |\boldsymbol{v}|_1^2 \,, \end{aligned}$$

and similarly that

$$\begin{aligned} a_4(\boldsymbol{u}_0+\boldsymbol{w};\boldsymbol{v},\boldsymbol{v}) - a_4(\boldsymbol{w};\boldsymbol{v},\boldsymbol{v}) &= \frac{C_b}{\delta}\int_\Omega \big(|\boldsymbol{u}_0+\boldsymbol{w}| - |\boldsymbol{w}|\big)\boldsymbol{v}\cdot\boldsymbol{v}\,\mathrm{d}\Omega \\ &\le \frac{C_b}{\delta}\int_\Omega \big||\boldsymbol{u}_0+\boldsymbol{w}| - |\boldsymbol{w}|\big||\boldsymbol{v}|^2\mathrm{d}\Omega \\ &\le \frac{C_b}{\delta}\int_\Omega |\boldsymbol{u}_0||\boldsymbol{v}|^2\mathrm{d}\Omega = a_4(\boldsymbol{u}_0;\boldsymbol{v},\boldsymbol{v}) \le \epsilon|\boldsymbol{v}|_1^2 . \end{aligned}$$

Therefore, since $a_4(\boldsymbol{w};\boldsymbol{v},\boldsymbol{v}) \ge 0$, we have

$$\tilde{a}(\boldsymbol{w};\boldsymbol{v},\boldsymbol{v}) \ge (\nu - 4\epsilon)|\boldsymbol{v}|_1^2 \quad \forall \boldsymbol{v},\boldsymbol{w}\in V .$$

Thus, the conditions of Theorem 1 are satisfied and we have that Problem 4 has at least one solution in $H^1(\Omega)^2 \times L_0^2(\Omega)$, which proves the theorem. $\square$

For uniqueness, we introduce some notation. First, for any $\boldsymbol{u}_0 \in H_0^1(\Omega)$, set

$$\rho(\boldsymbol{u}_0) = \sup_{\boldsymbol{v}\in V} \frac{a_1(\boldsymbol{v};\boldsymbol{u}_0,\boldsymbol{v}) + a_2(\boldsymbol{v};\boldsymbol{u}_0,\boldsymbol{v}) + \big(a_4(\boldsymbol{u}_0+\boldsymbol{v};\boldsymbol{u}_0,\boldsymbol{v}) - a_4(\boldsymbol{u}_0;\boldsymbol{u}_0,\boldsymbol{v})\big)}{|\boldsymbol{v}|_1^2}$$

and

$$\|\tilde{\boldsymbol{f}}\|_{V'} = \sup_{\boldsymbol{v}\in V}\frac{\langle\tilde{\boldsymbol{f}},\boldsymbol{v}\rangle_{\mathrm{B}}}{|\boldsymbol{v}|_1^2} \quad \text{where } \langle\tilde{\boldsymbol{f}},\boldsymbol{v}\rangle_{\mathrm{B}} = \langle\boldsymbol{f},\boldsymbol{v}\rangle_{\mathrm{B}} - a(\boldsymbol{u}_0;\boldsymbol{u}_0,\boldsymbol{v}) .$$

Then set

$$\nu_0 = \inf_{\boldsymbol{u}_0\in H^1(\Omega),\boldsymbol{u}_0|_{\partial\Omega}=\boldsymbol{g}} \left\{\rho(\boldsymbol{u}_0) + (\mathcal{N}\|\tilde{\boldsymbol{f}}\|_{V'})^{1/2}\right\} .$$

Now we can state the following theorem.

Theorem 6. *Assume the hypotheses of theorem 5. If $\nu > \nu_0$, then problem 4 has a unique solution in $H^1(\Omega)^2 \times L_0^2(\Omega)$.*

Proof: As before, we satisfy the hypotheses of theorem 2 to obtain a uniqueness result. First, choose a $\boldsymbol{u}_0$ as in lemma 4 such that $\rho(\boldsymbol{u}_0) < \nu$. We have that

$$\begin{aligned} \tilde{a}(\boldsymbol{w};\boldsymbol{v},\boldsymbol{v}) =\ & a(\boldsymbol{w};\boldsymbol{v},\boldsymbol{v}) + a_1(\boldsymbol{v};\boldsymbol{u}_0,\boldsymbol{v}) + a_2(\boldsymbol{v};\boldsymbol{u}_0,\boldsymbol{v}) + a_4(\boldsymbol{u}_0+\boldsymbol{v};\boldsymbol{u}_0,\boldsymbol{v}) \\ & - a_4(\boldsymbol{u}_0;\boldsymbol{u}_0,\boldsymbol{v}) + a_4(\boldsymbol{u}_0+\boldsymbol{w};\boldsymbol{v},\boldsymbol{v}) - a_4(\boldsymbol{w};\boldsymbol{v},\boldsymbol{v}) \\ =\ & a_0(\boldsymbol{v},\boldsymbol{v}) + a_4(\boldsymbol{u}_0+\boldsymbol{w};\boldsymbol{v},\boldsymbol{v}) + a_1(\boldsymbol{v};\boldsymbol{u}_0,\boldsymbol{v}) + a_2(\boldsymbol{v};\boldsymbol{u}_0,\boldsymbol{v}) \\ & + a_4(\boldsymbol{u}_0+\boldsymbol{v};\boldsymbol{u}_0,\boldsymbol{v}) - a_4(\boldsymbol{u}_0;\boldsymbol{u}_0,\boldsymbol{v}) \\ \ge\ & \nu|\boldsymbol{v}|_1^2 + a_4(\boldsymbol{u}_0+\boldsymbol{w};\boldsymbol{v},\boldsymbol{v}) - \rho(\boldsymbol{u}_0)|\boldsymbol{v}|_1^2 \\ \ge\ & \big(\nu - \rho(\boldsymbol{u}_0)\big)|\boldsymbol{v}|_1^2 . \end{aligned}$$

Thus, we choose $\alpha = \nu - \rho(\boldsymbol{u}_0)$ and condition 1 holds. For condition 2 we have

$$\begin{aligned}
|\tilde{a}(\boldsymbol{w}_1; \boldsymbol{u}, \boldsymbol{v}) - \tilde{a}(\boldsymbol{w}_2; \boldsymbol{u}, \boldsymbol{v})| \leq\ & |a_1(\boldsymbol{w}_1 - \boldsymbol{w}_2; \boldsymbol{u}, \boldsymbol{v})| + |a_2(\boldsymbol{w}_1 - \boldsymbol{w}_2; \boldsymbol{u}, \boldsymbol{v})| \\
& + |a_4(\boldsymbol{u}_0 + \boldsymbol{w}_1; \boldsymbol{u}, \boldsymbol{v}) - a_4(\boldsymbol{u}_0 + \boldsymbol{w}_2; \boldsymbol{u}, \boldsymbol{v})| \\
\leq\ & |a_1(\boldsymbol{w}_1 - \boldsymbol{w}_2; \boldsymbol{u}, \boldsymbol{v})| + |a_2(\boldsymbol{w}_1 - \boldsymbol{w}_2; \boldsymbol{u}, \boldsymbol{v})| \\
& + |a_4^*(\boldsymbol{w}_1 - \boldsymbol{w}_2; \boldsymbol{u}, \boldsymbol{v})| \\
\leq\ & \mathcal{N} |\boldsymbol{w}_1 - \boldsymbol{w}_2|_1 |\boldsymbol{u}|_1 |\boldsymbol{v}|_1 .
\end{aligned}$$

Consequently we again choose $L \equiv \mathcal{N}$, and the inequality 18 becomes

$$\frac{\|\tilde{\boldsymbol{f}}\|_{V'}}{\left(\nu - \rho(\boldsymbol{u}_0)\right)^2} \mathcal{N} < 1 ,$$

or

$$\nu > \rho(\boldsymbol{u}_0) + \sqrt{\mathcal{N} \|\tilde{\boldsymbol{f}}\|_{V'}} .$$

Since we can choose $\boldsymbol{u}_0$, we take the infimum over all admissible $\boldsymbol{u}_0$, and arrive at the condition $\nu > \nu_0$. If this condition holds, then the conditions of theorem 2 are satisfied, and problem 32 has a unique solution in $H^1(\Omega)^2 \times L_0^2(\Omega)$.

6 Conclusion

In this report we have analyzed the a modified version of the great lake equations of Camassa, Holm, and Levermore. In particular, we have considered the time-dependent version of these equations and have added terms due to the Coriolis force, bottom drag, wind shear, and viscosity. We have shown that solutions of these modified equations exist. Moreover, these solutions are unique if a relation with the viscosity, nonlinear terms, and forcing is satisfied. If the equations have non-homogeneous Dirichlet boundary conditions, the existence result is unchanged, and uniqueness is similar though a slighty different relation must be satisfied.

References

[1] R. A. Adams. *Sobolev Spaces*, volume 65 of *Pure and applied mathematics (Academic Press)*. Academic Press, San Francisco, 1975.

[2] I. Babuška. The finite element method with Lagrangian multipliers. *Numer. Math.*, 20:179–192, 1973.

[3] F. Brezzi. On the existence, uniqueness and approximation of saddle-point problems arising from Lagrange multipliers. *RAIRO Anal. Numér.*, 8(R2):129–151, 1974.

[4] R. Camassa, D. D. Holm, and C. D. Levermore. Long-time effects of bottom topography in shallow water. *Physica D*, 98:258–286, 1996.

[5] R. Camassa, D. D. Holm, and C. D. Levermore. Long-time shallow water equations with a varying bottom. *Journal of Fluid Mechanics*, 349:173–189, 1997.

[6] V. Girault and P.-A. Raviart. *Finite Element Methods for Naviar-Stokes Equations: Theory and Algorithms*, volume 5 of *Springer series in computational mathematics.* Springer-Verlag, New York, 1986.

[7] E. Hopf. On non-linear partial differential equations. In *Lecture Series of the Symp. on Partial Diff. Equations.* University of Kansas, Lawrence, KS, 1955.

[8] Z. Kowalik and T. S. Murty. *Numerical Modeling of Ocean Dynamics.* World Scientific, New Jersey, 1993.

[9] O. A. Ladyzhenskaya. *The Mathematical Theory of Viscous Incompressible Flow.* Gordon and Breach, New York, 1963.

[10] C. D. Levermore, M. Oliver, and E. S. Titi. Global well-posedness for models of shallow water in a basin with a varying bottom. *Indiana Univ. Math. J.*, 45:479–510, 1996.

[11] C. D. Levermore, M. Oliver, and E. S. Titi. Global well-posedness for the lake equations. *Physica D*, 98:492–509, 1996.

[12] C. D. Levermore and M. Sammartino. A shallow water model with eddy viscosity for basins with varying bottom topography. *Nonlinearity*, 14:1493–1515, 2001.

[13] J. Pedlosky. *Geophysical Fluid Dynamics.* Springer-Verlag, New York, 1986.

[14] A. Wouk. *A Course of Applied Functional Analysis.* John Wiley & Sons, Ltd., New York, 1979.

DISTRIBUTION:

1	C. David Levermore Mathematics Building University of Maryland College Park, MD 20742-4015	
1	MS 1110	David Womble, 9214
1	MS 1110	Pavel Bochev, 9214
1	MS 1110	Rich Lehoucq, 9214
1	MS 1110	John Shadid, 9233
6	MS 1110	David Ropp, 9214
1	MS 9018	Central Technical Files, 8945-1
2	MS 0899	Technical Library, 9616
1	MS 0612	Review & Approval Desk, 9612