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Quadratic Reciprocity and the Group Orders of Particle States

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ABSTRACT

The construction of inverse states in a finite field $\mathbb{F}_{P_\alpha}$ enables the organization of the mass scale by associating particle states with residue class designations. With the assumption of perfect flatness ($\Omega_{total} = 1.0$), this approach leads to the derivation of a cosmic seesaw congruence which unifies the concepts of space and mass. The law of quadratic reciprocity profoundly constrains the subgroup structure of the multiplicative group of units $\mathbb{F}_{P_\alpha}^*$ defined by the field. Four specific outcomes of this organization are (1) a reduction in the computational complexity of the mass state distribution by a factor of $\sim 10^{30}$, (2) the extension of the genetic divisor concept to the classification of subgroup orders, (3) the derivation of a simple numerical test for any prospective mass number based on the order of the integer, and (4) the identification of direct biological analogies to taxonomy and regulatory networks characteristic of cellular metabolism, tumor suppression, immunology, and evolution. It is generally concluded that the organizing principle legislated by the alliance of quadratic reciprocity with the cosmic seesaw creates a universal <u>optimized</u> structure that functions in the regulation of a broad range of complex phenomena.

I. Introduction

Arithmetic conditions relating particle masses can be defined on the basis of (A) the supersymmetric conservation of congruence and (B) the observed characteristics of particle reactions and stabilities [1]. Stated in the form of common divisors of the particle mass parameters, these relations can be interpreted as expressions of genetic elements that represent particle characteristics [2]. In order to illustrate this concept, it has been shown that the pion triplet ($\pi^{\pm}$, π^{o}) can be associated with the existence of a greatest common divisor $d_{o\pm}$ in a way that can account for both the highly similar physical properties of these particles and the observed $\pi^{\pm}/\pi^{o}$ mass splitting. These results support the conclusion that a corresponding statement holds generally for all particle multiplets. Classification of the respective physical states is achieved by association of the common divisors (genes) with residue class designations [3] in a finite field $\mathbb{F}_{P_\alpha}$.

The existence of the finite field $\mathbb{F}_{P_\alpha}$ and the corresponding group $\mathbb{F}^*_{P_\alpha}$ leads immediately to the definition of a new physical entity, the inverse state [3]. With this theoretical apparatus, it can be demonstrated that the concept of supersymmetry can be directly expressed in terms of hierarchical relationships between odd and even order subgroups of $\mathbb{F}^*_{P_\alpha}$, an outcome that automatically reflects itself in the phenomenon of fermion/boson pairing of individual particles. Accordingly, supersymmetric pairing originates as a group rather than a particle property. The status of the Higgs subgroup is singular; since it is found to have order 4, it is isolated from the hierarchical pattern and communicates globally to the mass scale with the seesaw congruence [3] through both (α) the specification of the generators of the physical masses and (β) the fusion of the concepts of mass and space. Overall, these results indicate the existence of a new universal organizing principle that simultaneously reveals itself through (1) the determination of the intrinsic physical properties of particle states and (2) the regulation of their respective interactions by specification of the unified strong-electroweak coupling constant α^*. The existence of the multiplicative group of units $\mathbb{F}^*_{P_\alpha}$ of the field $\mathbb{F}_{P_\alpha}$ enables the corresponding mass parameters and their constituent genes to possess a rich subgroup structure, the properties of which are the main subject of the present study. On the basis of the analysis given below, it is concluded that this organizational structure is <u>optimized</u> and that it plays a fundamental regulatory role in a wide range of complex phenomena.

II. $\mathbb{F}^*_{P_\alpha}$ Subgroup Structure and Quadratic Reciprocity

A. Specification of Subgroup Orders

The law of quadratic reciprocity [4], that was conjectured by Euler and independently discovered and proved by Gauss in 1796, founded the era of modern number theory and consequently stands as one of the most prominent theorems in the entirety of mathematics. Of corresponding primary importance to the organization of the physical mass scale [2] is the group structure of the particle states and its expression by the group $\mathbb{F}^*_{P_\alpha}$. We now demonstrate the existence of a connection between these two fundamental entities.

The relationship between quadratic reciprocity and the structure of $\mathbb{F}^*_{P_\alpha}$ is derived from the work of Zolotarev [5]. In his proof of quadratic reciprocity [5], Zolotarev made a direct link to permutations [6] of the elements of $\mathbb{F}^*_{P_\alpha}$ by using the following theorem which we now state in a form adapted from that given by Lemmermeyer [4].

THEOREM (Zolotarev)

For P_α an odd prime let the element $a \in \mathbb{F}^*_{P_\alpha}$ define the permutation σ_a of $\mathbb{F}^*_{P_\alpha}$ by mapping $t(\text{mod } P_\alpha)$ to $ta(\text{mod } P_\alpha)$.
Let f denote the order of a in $\mathbb{F}^*_{P_\alpha} \rightarrow a^f \equiv 1(\text{mod } P_\alpha)$.
Then, the permutation σ_a is the product of g cycles of length f where

$$fg = P_\alpha - 1. \tag{1}$$

Hence,

$$fg \equiv -1(\text{mod } P_\alpha). \tag{2}$$

Two results are known from previous studies [3,7] on the structure of the mass scale. They are (A) the relation

$$B_{\nu_e} B_{\nu_\mu} = (2q)(r/2) = [g_\alpha]_{P_\alpha} [g_\beta]^{-1}_{P_\alpha} \equiv -1(\text{mod } P_\alpha), \tag{3}$$

in which B_{ν_e} and B_{ν_μ} are the prospective mass numbers for the electron neutrino ν_e and the muon neutrino ν_μ, integers respectively given by the residue class representatives $(2q) = [g_\alpha]_{P_\alpha}$ and $(r/2) = [g_\beta]^{-1}_{P_\alpha}$, and (B)

$$\alpha^* = \frac{m_{\nu_e}}{m_{\nu_\mu}} = \frac{B_{\nu_e}}{B_{\nu_\mu}} = \frac{2q}{r/2} = \frac{[g_\alpha]_{P_\alpha}}{[g_\beta]^{-1}_{P_\alpha}} = (34.26)^{-1}, \tag{4}$$

a statement which specifies the unified strong-electroweak coupling constant α^* in terms of the electron and muon neutrino masses m_{ν_e} and m_{ν_μ}. Accordingly, by combination of the theorem of Zolotarev given by Eq.(2) with the findings expressed by Eqs.(3) and (4), we immediately obtain the result

$$\underbrace{\alpha^* = \frac{m_{\nu_e}}{m_{\nu_\mu}}}_{\text{MASS RATIO}} = \frac{[g_\alpha]_{P_\alpha}}{[g_\beta]^{-1}_{P_\alpha}} = \frac{2q}{r/2} = \underbrace{\frac{f}{g} = \alpha^*}_{\text{SUBGROUP ORDERS}}. \tag{5}$$

Equation (5) states that α^* <u>simultaneously</u> specifies (a) the pair of generators designated by the Higgs seesaw [3] relation

$$[g_\alpha]_{P_\alpha}[g_\beta]^{-1}_{P_\alpha} \equiv [B_{Higgs}]^2_{P_\alpha}\ , \tag{6}$$

integers which correspond to the mass parameters of the electron and muon neutrinos, and (b) special subgroups of the group $\mathbb{F}^*_{P_\alpha}$. Hence, with (f, g) given alternatively by (2q, r/2) and (r/2, 2q), α^* selects the two special subgroups with orders

$$\delta = \{\delta_1, \delta_2\} = \{2q, r/2\} = \{[g_\alpha]_{P_\alpha}, [g_\beta]^{-1}_{P_\alpha}\}\ . \tag{7}$$

Of course, these two orders are primitive roots of the prime modulus P_α, a condition necessary for them to serve as generators [3].

The relationship expressed by Eq.(5) elevates the significance of the magnitude of the unified strong-electroweak coupling constant α^* to an exceptional status, since it simultaneously expresses fundamental mathematical and physical conclusions. Specifically, <u>quadratic reciprocity enables α^*, the cardinal nongravitational coupling constant, to identify special subgroups of $\mathbb{F}^*_{P_\alpha}$ and the orders of these subgroups are the identical generators (primitive roots) specified by the Higgs seesaw congruence given by Eq.(6)</u>. Further, since the orders of both subgroups are even, a previous study [2] has shown that these two groups automatically include the supersymmetric partners corresponding to each particle state in the group [2]. These findings immediately suggest the obvious conjecture that <u>all physical particles are members of these two designated subgroups</u>. In the following discussion, this hypothesis will be described as the δ-conjecture. We observe further that the δ-conjecture would be physically untenable for odd subgroup orders, since corresponding supersymmetric states would be fully excluded from representation [2].

The validity of this conjecture would be of enormous significance for the practical computation of the mass scale of particle states. Foremost, with $P_\alpha \cong 6\times10^{60}$ and $\delta \cong 10^{30}$, <u>a huge factor of ~ 10^{30} in reduction of the complexity of the problem is immediately achieved</u>. Of comparably high importance is the availability of a simple direct numerical test of any prospective mass number B_x through the evaluation of the order of the integer. Specifically,

$$(B_x)^\delta \equiv 1 \ (\mathrm{mod}\ P_\alpha) \tag{8}$$

must hold with

$$\delta \in \{2q, r/2\} = \{[g_\alpha]_{P_\alpha}, [g_\beta]^{-1}_{P_\alpha}\}. \tag{9}$$

We will accordingly assume the validity of the δ-conjecture in the following development. We know, however, that an exceptional set of particles must exist that cannot satisfy the test defined by Eqs. (8) and (9). Two cases are apparent. First, since the Higgs particle [3] has order 4 and 4 is not a divisor of either 2q or r/2, we have perforce

$$(B_{Higgs})^\delta \not\equiv 1 \ (\mathrm{mod}\ P_\alpha) \tag{10}$$

with δ given by Eq.(9). Second, since any generator (primitive root) must have maximal order (i.e. $P_\alpha - 1$) by definition, the mass numbers of the two neutrinos appearing in Eq.(3) also surely fail this test. It follows that the full complement of masses related by the cosmic (Higgs) seesaw congruence, the statement that fuses the concepts of mass and space [3], stands as a fundamental exception to the condition expressed by Eqs.(8) and (9). This interpretation identifies this small exclusive set of particles as the most elementary of all physical systems. Hence, the law of quadratic reciprocity confers dual physical and mathematical definitions on the concept of elementarity and certifies these systems as the organizers of the mass scale.

B. Structure of Subgroup Orders

The set of primitive root orders δ_1 and δ_2 specified by Eqs.(5) and (7) have explicit arithmetic structures [3,7] that are prospectively given by

$$\delta_1 = [g_\alpha]_{P_\alpha} = 2q = 2\cdot 3^2\cdot 7\cdot 13\cdot 31\cdot 37\cdot 47\cdot 53\cdot 59\cdot 71\cdot 73\cdot 83\cdot 107\cdot 109\cdot 113\cdot 137\cdot 139\cdot 149 \tag{11}$$

and

$$\delta_2 = [g_\beta]^{-1}_{P_\alpha} = r/2 = 2\cdot 5^2\cdot 11\cdot 17\cdot 19\cdot 23\cdot 29\cdot 41\cdot 43\cdot 61\cdot 67\cdot 79\cdot 89\cdot 97\cdot 101\cdot 103\cdot 127\cdot 131\cdot 151, \tag{12}$$

integers that obey the relations

$$[g_\alpha]_{P_\alpha} [g_\beta]^{-1}_{P_\alpha} = P_\alpha - 1 \tag{13}$$

with

$$[g_\beta]_{P_\alpha} = P_\alpha - g_\alpha , \tag{14}$$

as defined in earlier work [3] and represented in Eq.(3). We recall from a previous analysis [7] that the values for δ_1 and δ_2 given in Eqs.(11) and (12) are strongly constrained; they yield a value for the fine-structure constant $\alpha = q/r$ that is in full accord with the very accurately known ($\Delta\alpha/\alpha \sim 10^{-8}$) experimental figure [8]. Further, by inspection we see that

$$\gcd(\delta_1, \delta_2) = 2 \tag{15}$$

and that the general forms of δ_1 and δ_2 can be written as

$$\delta_1 = 2 \cdot 3^2 \cdot p_1 \cdot p_2 \ldots p_{16} \tag{16}$$

$$\delta_2 = 2 \cdot 5^2 \cdot q_1 \cdot q_2 \ldots q_{17}, \tag{17}$$

in which the two sets of prime factors $\{p_i\}$ and $\{q_j\}$ are such that

$$\{p_i\} \cap \{q_j\} = \phi , \tag{18}$$

since Eq.(15) holds. Accordingly, we then have

$$P_\alpha - 1 = 2^2 \cdot 3^2 \cdot 5^2 \prod_{i=1}^{16} p_i \prod_{j=1}^{17} q_j \qquad (19)$$

in conformance with earlier work [3,7]. We observe further that Eq.(15) expresses the minimum value possible, since acceptance of the δ-conjecture requires that δ_1 and δ_2 be even.

1. Group Structure for P = 61

A previous study [2] has shown how the group structure can be generally arranged in a hierarchical pattern of subgroup inclusion relationships and presented in the format of a digraph. In order to develop a specific picture of the subgroup organization associated with the large prime P_α , we now examine the pattern of the subgroup inclusions for the prime P = 61 ≡ 1(mod 4), a modulus which has the factor structure $P - 1 = 60 = 2^2 \cdot 3 \cdot 5$ and the corresponding divisor set given by

$$\{d\} = \{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\}. \qquad (20)$$

The divisors 2, 6, 10, and 30 in {d} are primitive roots [9] of P = 61. In accord with the description given for the prime P = 37 ≡ 1(mod 4) presented in the earlier work [2], the digraph for the ten subgroup orders {2, 3, 4, 5, 6, 10, 12, 15, 20, 30} shown in Fig. (1) is obtained. We see immediately that the four highly connected inner subgroups, with corresponding orders of 2, 6, 10, and 30, are all even orders and primitive roots. Further, these four primitive roots can be organized into the set {(2, 30), (6, 10)} of pairs (g_1, g_2) that each satisfy the congruence

$$g_1 g_2 \equiv -1 (\text{mod } P), \qquad (21)$$

the seesaw condition [3] given by Eq.(3) and a relation that arose in an earlier study [3] of primitive root inverse mappings for prime moduli of the form P ≡ 1(mod 4). Also apparent is the asymmetry present among these four groups; specifically, 6 is not a divisor of 10, hence, this cluster of four subgroups possesses a two-fold, not a four-fold axis of symmetry. Importantly, the isolation in the digraph of the subgroup of order 4, the subgroup corresponding to the supersymmetric Higgs pair [3], is manifest; it possesses a minimal connection to this central cluster of primitive root order subgroups (nodes). The pattern shown in Fig. (1) also suggests that all primitive roots that are divisors of P − 1, for moduli P ≡ 1(mod 4), are even integers. This hypothesis can be proved [10].

LEMMA

Let prime P ≡ 1 (mod 4) and suppose that x is an odd divisor of P − 1. Then x is a quadratic residue mod P, hence, not a primitive root. It follows that any primitive root of P that is contained in the set of divisors of P − 1, if it exists, must be even.

PROOF: (x/p) = (p/x) because p = 1(mod 4) and x is odd [9,11]. But (p/x) = (1/x) = 1 because x divides P − 1. Q.E.D.

This result supports the δ-conjecture, reduces the search for primitve roots of the modulus that are divisors of P − 1, and identifies a special role for residue classes corresponding to fermions.

2. Group Structure for P = 157

The pattern illustrated in Fig. (1) would be substantively altered if 2 were not a primitive root of the modulus. The character of this change is readily visualized by consideration of a modulus P ≡ 1(mod 4) which has a group inclusion pattern isomorphic to that illustrated for P = 61 in Fig.(1)

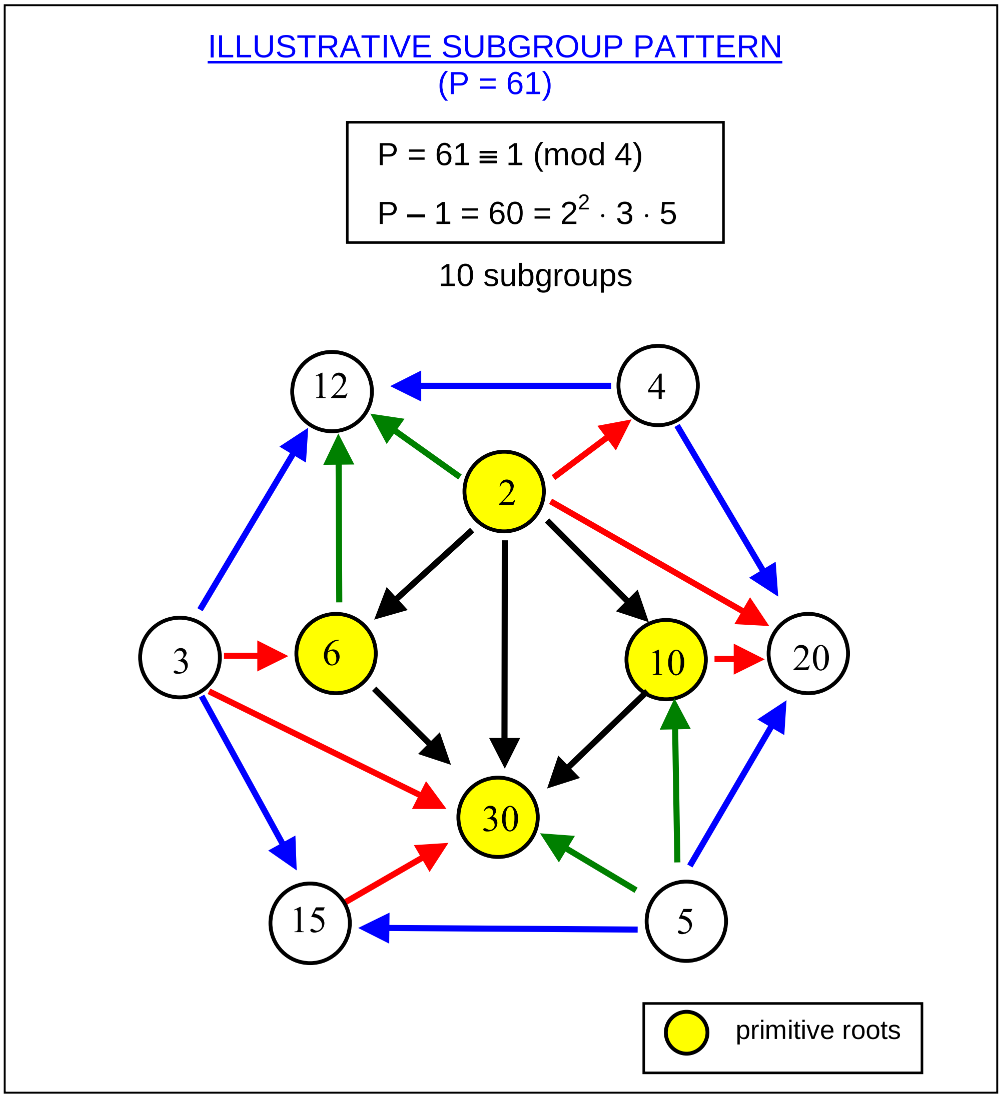


Fig.(1): Hierarchical pattern of subgroup inclusion relationships for the ten subgroups {2,3,4,5,6,10,12,15,20,30} associated with the prime P = 61 ≡ 1(mod 4). The vertices identify the subgroup orders and the arrows indicate subgroup inclusion. We observe that the four highly connected interior nodes, corresponding to the subgroup orders 2, 6, 10, and 30, are all primitive roots of the modulus. The group of order 4 corresponds to the supersymmetric Higgs pairs.

and for which the integer 2 is <u>not</u> a primitive root. The prime P = 157 ≡ 1(mod 4), with $P - 1 = 156 = 2^2 \cdot 3 \cdot 13$ and divisor set

$$\{d\} = \{1, 2, 3, 4, 6, 12, 13, 26, 39, 52, 78, 156\}, \qquad (22)$$

satisfies these conditions. Fig.(2) displays the digraph corresponding to the subgroup organization of the group $\mathbb{F}^*_{157}$. In this case, the only divisors that are primitive roots are $g_1 = 6$ and $g_2 = 26$, integers which perforce also satisfy the congruence

$$g_1 g_2 \equiv -1 (\text{mod } 157), \qquad (23)$$

the seesaw condition stated in Eq.(21).

With the exception of the group of order 4, which corresponds to the Higgs system [3] and stands in singular remove from the centrally located primitive root nodes (in this case, group orders $g_1 = 6$ and $g_2 = 26$), there generally exist two clusters of subgroups respectively connected directly to the generating subgroups having orders g_1 and g_2. Specifically, with the exclusion of the trivial subgroup of order unity and itself, each subgroup respectively includes

$$n_i = d(g_i) - 2, \quad i = 1, 2 \tag{24}$$

subgroups of lesser order. In the simple example for P = 157 given in Fig.(2), $n_1 = 2$ and $n_2 = 2$. The quantity $d(g_i)$ in Eq.(24) denotes the customary arithmetical function [11,12] which gives the number of divisors of the integers g_i.

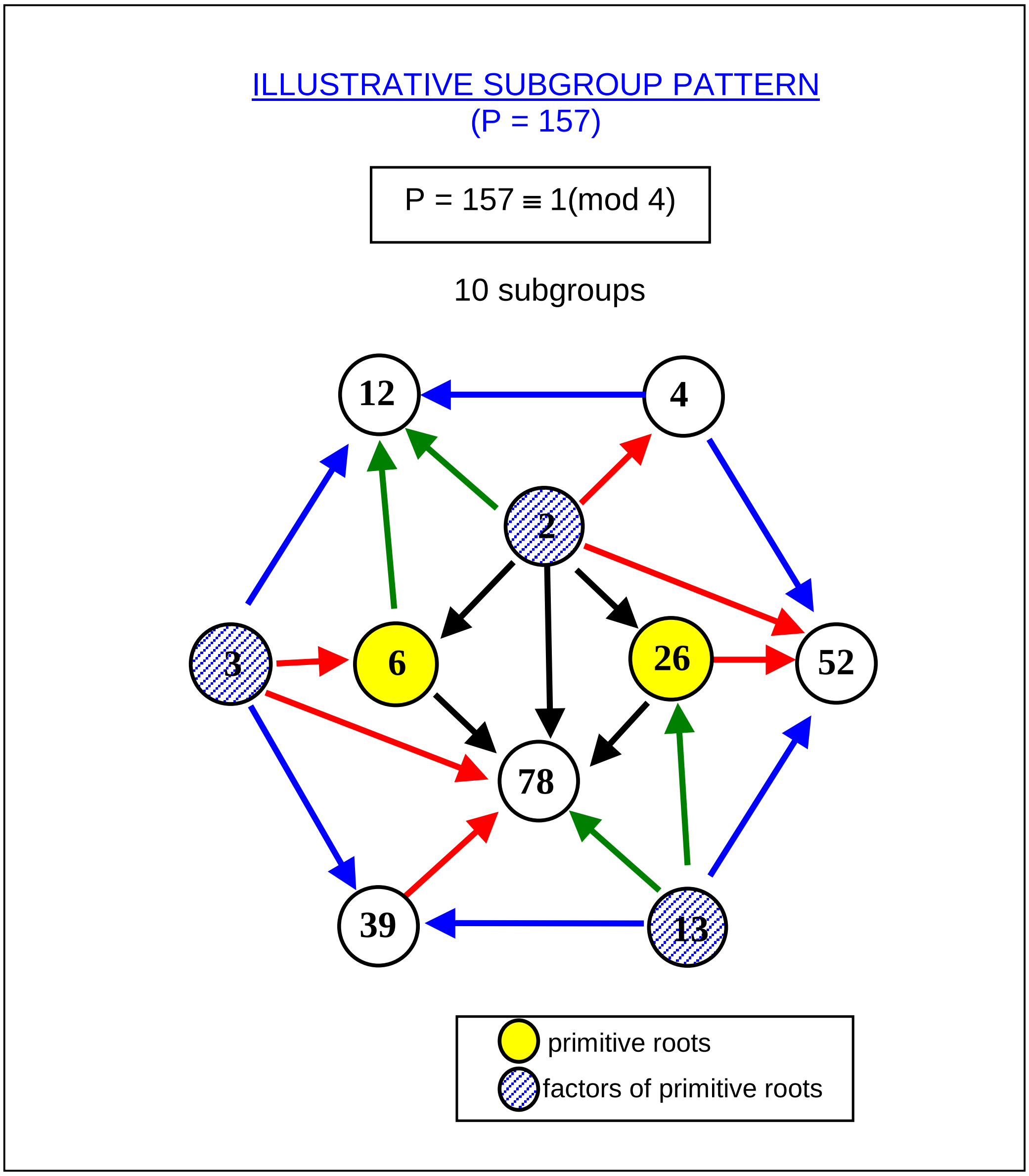


Fig. (2): Hierarchical pattern of subgroup inclusion relationships for the ten subgroups {2,3,4,6,12,13,26,39,52,78} associated with the prime P = 157 ≡ 1(mod 4). The vertices identify the subgroup orders and the arrows indicate subgroup inclusion. The pattern is isomorphic to the digraph shown in Fig. (1) for P = 61. The interior nodes corresponding to the subgroup orders 6 and 26 are primitive roots of the modulus. Primitive roots and corresponding factors of the primitive roots are color designated. The group of order 4 corresponds to the Higgs subgroup.

3. Group Structure for P_α

The results given above enable us now to represent key features of the subgroup organization for the large prime P_α. This modulus has two primitive root orders δ_1 and δ_2, given respectively by Eqs.(11) and (12), integers which are divisors of $P_\alpha - 1$ that satisfy the seesaw congruence [3]. We also know that the integer 2 is not a primitive root of P_α; the minimum primitive root [3] is 14.

A reduced schematic of the subgroup pattern for the modulus P_α is illustrated in Fig.(3). Subgroups associated with divisors of $P_\alpha - 1$ that are not divisors of δ_1 or δ_2 are deleted in this representation. This reduction, which corresponds physically to the acceptance of the δ-conjecture, greatly simplifies the subgroup structure. Quantitatively, since $n_1 + n_2 \cong 1.2 \times 10^6$ and $d(P_\alpha - 1) \cong 2.3 \times 10^{11}$, the number of physically relevant subgroups is diminished by a factor of $\sim 10^5$.

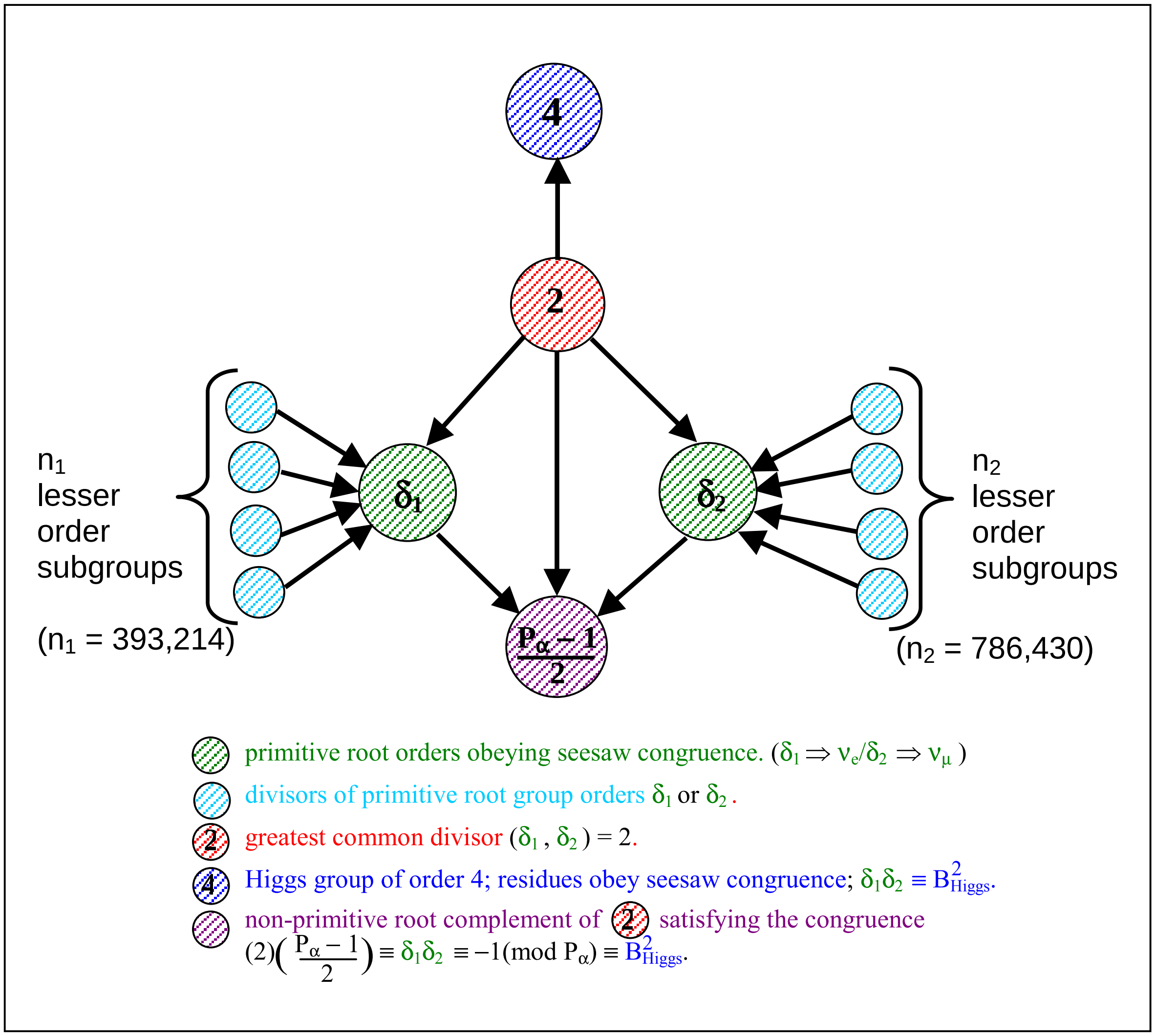


Fig (3): Reduced subgroup organizational pattern of the group $\mathbb{F}^*_{P_\alpha}$ for prime modulus $P_\alpha \equiv 1(\mathrm{mod}\ 4)$ indicating two subgroup clusters associated respectively with the primitive root order subgroups δ_1 and δ_2. $n_1 = d(\delta_1) - 2 = 3 \cdot 2^{17} - 2 = 393,214$ and $n_2 = d(\delta_2) - 2 = 3 \cdot 2^{18} - 2 = 786,430$. The Higgs subgroup stands in isolation from the generating subgroups. Additional subgroups corresponding to divisors of $P_\alpha - 1$ that are not contained in δ_1 and δ_2 are deleted from the representation in conformance with the δ-conjecture. Since $n_1 + n_2 \cong 1.2 \times 10^6$ and $d(P_\alpha - 1) \cong 2.3 \times 10^{11}$, the reduction in complexity by this elimination is very substantial.

C. Biological Analogy

The subgroup structure of the field group $\mathbb{F}^*_{P_\alpha}$ pictured in Fig.(3) inspires the construction of a biological analogy in which the abundant subgroups play the role of the evolved classes that stem from a central core genetic source. A reconfiguration of Fig. (3), which illustrates this analogy in the form of a group order taxonomy is shown in Fig.(4). The kingdom is represented by the ensemble composed of the subgroups of order 2, 4, and $(P_\alpha - 1)/2$. Directly following, two principal and distinct phyla are defined by the large subgroups with orders δ_1 and δ_2. These phyla then serve as the sources of numerous classes represented by the lesser order subgroups. We emphasize that this system of organization assumes the validity of the δ-conjecture based on the law of quadratic reciprocity stated above in **Section II.A.**

The comparison of the taxonomic pattern presented in Fig.(4) with the bacterial taxonomy descriptive of archaebacteria and eubacteria shown in Fig.(5) highlights the biological correspondence. From an ancestral procaryote (kingdom), two distantly related groups (archaebacteria and eubacteria) are evolved which subsequently experience further evolutionary branching into a range of specialized organisms [13].

GROUP ORDER TAXONOMY

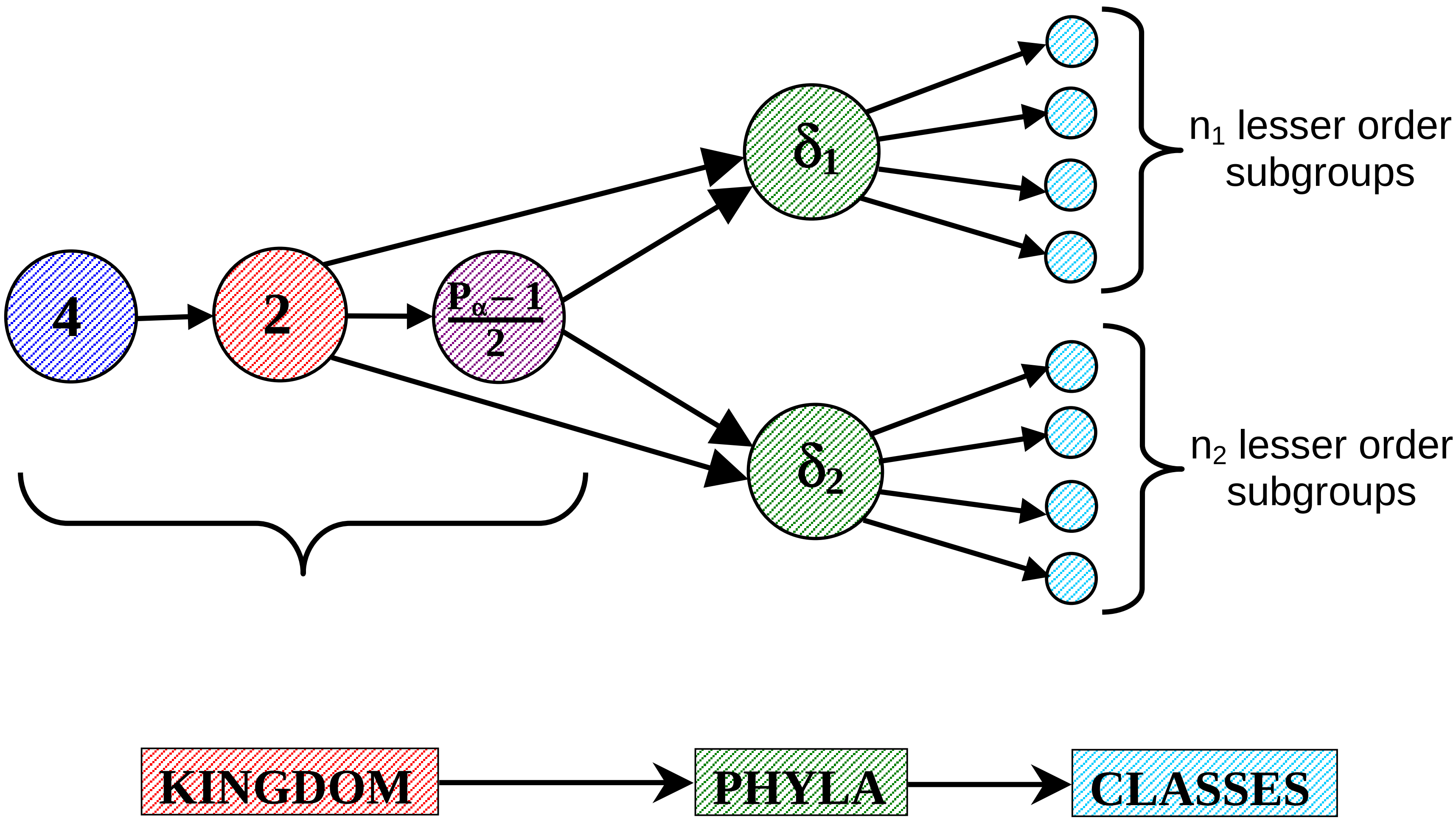
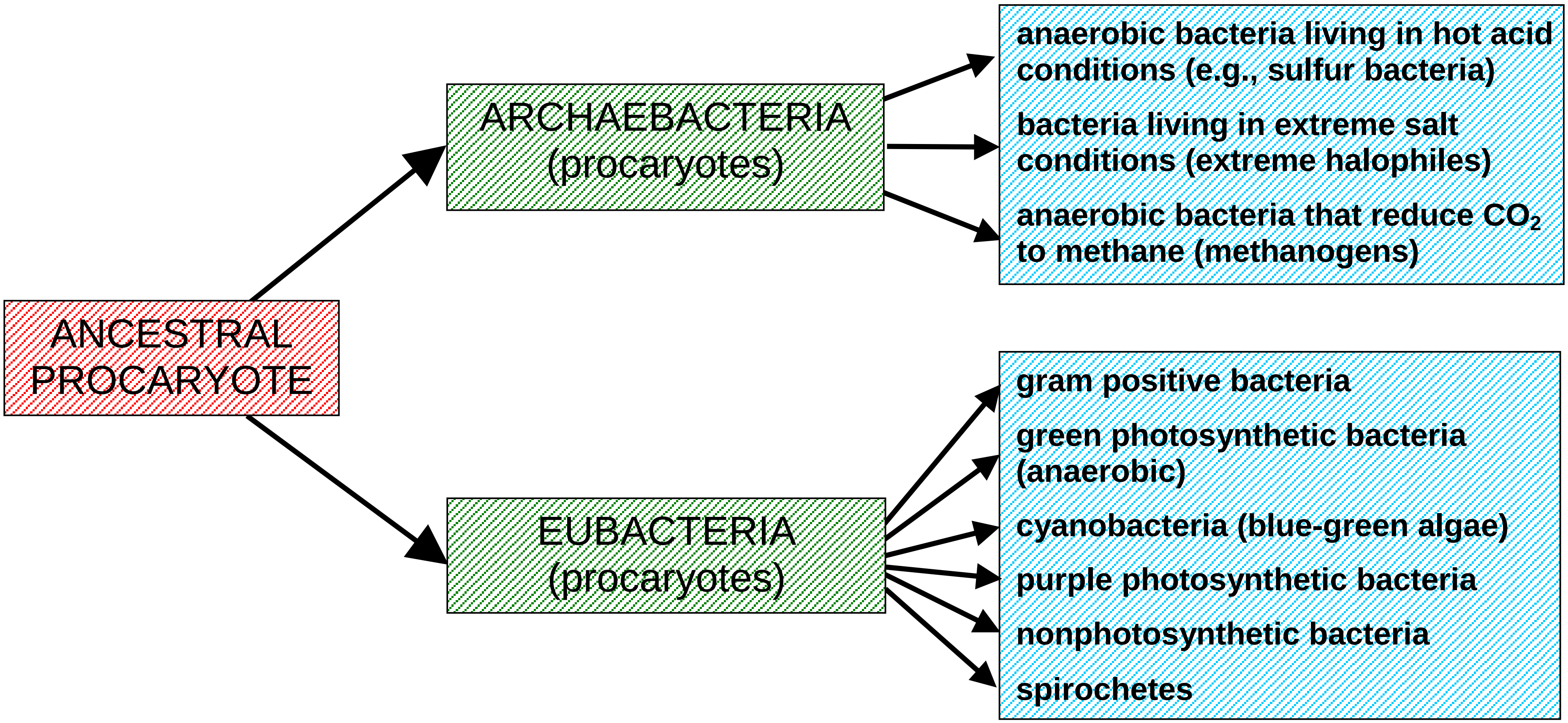


Fig. (4): Reconfiguration of Fig. (3) into a form corresponding to an evolutionary biological pattern illustrating the customary kingdom $\rightarrow$ phyla $\rightarrow$ classes taxonomic paradigm.

BACTERIAL TAXONOMY



B. Alberts, D. Bray, J. Lewis, M. Raff, K. Roberts, and J. D. Watson, *Molecular Biology of The Cell*, Second edition (Garland Publishing, Inc., New York/London, 1989) p. 11.

Fig. (5): Evolution of archaebacteria and eubacteria from an ancestral procaryote (kingdom) into an array of specialized organisms. The pattern mimics the group order taxonomy illustrated in Fig. (4).

D. Extension of Genetic Divisor Concept

A profound consequence of the subgroup ordering legislated by Zolotarev's proof [5] of quadratic reciprocity with the use of the theorem on permutations is the doubling of the genetic function [2] of the divisors and their classification with the residue classes of $\mathbb{F}_{P_\alpha}$. On the basis of the pattern shown in Figs. (3) and (4), the genetic divisor interpretation can be applied to both (A) the designation of subgroup orders and (B) the specification of the masses and intrinsic attributes of individual particle systems [2]. In relation to the former, with the exception of the common divisor 2 given in Eq.(15), Fig. (3) shows the existence of two fully distinct gene pools that are respectively defined by the divisors of the primitive roots δ_1 and δ_2. These correspond to the two phyla shown in Fig. (4). Hence, selected genetic constituents of $P_\alpha - 1$ also define the subgroup orders.

The singular isolated placement of the Higgs group of order 4 is clearly illustrated in Fig. (3). It communicates to the pair of principal subgroups δ_1 and δ_2 indirectly through the group with order 2, the magnitude of the greatest common divisor of δ_1 and δ_2 and the identifying divisor of fermions [1,14]. Since Eq.(6) gives the relation

$$\delta_1\delta_2 \equiv B^2_{\mathrm{Higgs}}(\mathrm{mod}\,P_\alpha)\,, \tag{25}$$

it follows that the seesaw congruence enables the Higgs system (subgroup) to specify (1) the masses of the particles (ν_e and ν_μ) that correspond to the generators of the mass scale [3], (2) the magnitude [3] of the unified strong-electroweak coupling constant α^*, and (3) the organizational pattern of the subgroup structure of all particle states. These consequences are the direct result of the law of quadratic reciprocity in alliance with the unified concept of space and mass expressed explicitly by the cosmic seesaw relation [3].

III. Conclusions

The law of quadratic reciprocity powerfully constrains the subgroup structure of the mass scale of physical particle states. The overall reduction in mathematical complexity is estimated to be a factor of $\sim 10^{30}$. This large simplification in the subgroup pattern is further augmented by the existence of a direct numerical test of any prospective mass number that is based on the order of the integer. This system of particle state organization enables the genetic divisor concept to play a double role; it specifies both (1) the properties of individual particles and (2) the orders and subgroup relationships of the corresponding group structure of the particle systems. The resulting architecture encourages the construction of a biological analogy in which the pattern aptly follows the customary kingdom $\rightarrow$ phylum $\rightarrow$ class $\rightarrow$ order $\rightarrow$ family $\rightarrow$ *et cetera* paradigm. A key feature is the existence of two principal highly connected subgroups (nodes) that define corresponding genetically distinct gene pools of subgroup orders. This characteristic, the presence of a relatively small number of highly linked nodes (hubs), is a basic property of scale-free networks. Accordingly, a second similarity to biological systems emerges, since there are well founded indications that structures of this general form are important in metabolic networks [15], the action of tumor-suppressor genes [16], the mechanism of immunoglobulin gene rearrangement [17], and evolution [18]. The wide significance of the pattern illustrated in Figs. (3) and (4) leads to the conclusion that the constraint on the group relationships expressed jointly by quadratic reciprocity and the seesaw congruence creates a universal optimized structure that plays a fundamental regulatory role in a large array of complex phenomena.

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References

1. Y. Dai, A. B. Borisov, J. W. Longworth, K. Boyer, and C. K. Rhodes, "Gamma-Ray Bursts and the Particle Mass Scale," *Proceedings of the International Conference on Electromagnetics in Advanced Applications*, edited by R. Graglia (Politecnico di Torino, 1999), p. 3.
2. Y. Dai, A. B. Borisov, K. Boyer, and Charles K. Rhodes, "Determination of Supersymmetric Particle Masses and Attributes with Genetic Divisors," manuscript in preparation.
3. Y. Dai, A. B. Borisov, K. Boyer, and C. K. Rhodes, "Computation with Inverse States in a Finite Field $\mathbb{F}_{p_\alpha}$: The Muon Neutrino Mass, the Unified Strong-Electroweak Coupling Constant, and the Higgs Mass," Sandia National Laboratories, Report SAND2000-2043, August 2000.
4. F. Lemmermeyer, *Reciprocity Laws* (Springer-Verlag, Berlin/New York, 2000).
5. G. Zolotarev, "Nouvelle démonstration de la loi de réciprocité de Legendre," *Nouv. Ann. Math* (2), **11,** 354 (1872).
6. J. D. Dixon and B. Mortimer, *Permutation Groups* (Springer-Verlag, Berlin/New York, 1996).
7. Y. Dai, A. B. Borisov, and C. K. Rhodes, "Nature, Number, and Universality," manuscript in preparation.
8. Peter J. Mohr and Barry N. Taylor, "CODATA Recommended Values of the Fundamental Physical Constants: 1998," *J. Phys. Chem. Ref. Data* **28,** 1713 (1999).
9. P. L. Tschebyscheff, *Theorie der Congruenzen* (Chelsea Publishing Co., New York, 1972).
10. Proof constructed by Jeremy T. Teitelbaum.
11. Kenneth Ireland and Michael Rosen, *A Classical Introduction to Modern Number Theory,* Second Edition (Springer-Verlag, Berlin/Heidelberg/New York, 1998).
12. G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers* (Clarendon Press, Oxford, 1960).
13. B. Alberts, D. Bray, J. Lewis, M. Raff, K. Roberts, and J. Watson, *Molecular Biology of the Cell*, Second edition (Garland Publishing, Inc., New York/London, 1989) p. 11.
14. Y. Dai, A. B. Borisov, J. W. Longworth, K. Boyer, and C. K. Rhodes, "Systematics of a Supersymmetric Power Residue Particle Mass Scale," manuscript in preparation.
15. H. Jeong, B. Tombor, R. Albert, Z. N. Oltrai, and A. -L. Barabási, "The Large-Scale Organization of Metabolic Networks," Nature **407,** 651 (2000).
16. B. Vogelstein, D. Lane, and A. J. Levine, "Surfing the p53 Network," Nature **408,** 307 (2000).
17. Abul K. Abbas, Andrew H. Lichtman, and Jordan S. Pober, *Cellular and Molecular Immunology,* (W. B. Saunders Company, Philadelphia, 1994) p. 79.
18. L. H. Hartwell, J. J. Hopfield, S. Leibler, and A. W. Murray, "From Molecular to Modular Cell Biology," Nature **402,** C47 (1999).

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