

SANDIA REPORT

SAND2001-1532
Unlimited Release
Printed May 2001

Superresolution and Synthetic Aperture Radar

Fred M. Dickey, Louis A. Romero, and Armin W. Doerry

Prepared by
Sandia National Laboratories
Albuquerque, New Mexico 87185 and Livermore, California 94550

Sandia is a multiprogram laboratory operated by Sandia Corporation,
a Lockheed Martin Company, for the United States Department of
Energy under Contract DE-AC04-94AL85000.

Approved for public release; further dissemination unlimited.



Sandia National Laboratories

Issued by Sandia National Laboratories, operated for the United States Department of Energy by Sandia Corporation.

NOTICE: This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government, nor any agency thereof, nor any of their employees, nor any of their contractors, subcontractors, or their employees, make any warranty, express or implied, or assume any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represent that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government, any agency thereof, or any of their contractors or subcontractors. The views and opinions expressed herein do not necessarily state or reflect those of the United States Government, any agency thereof, or any of their contractors.

Printed in the United States of America. This report has been reproduced directly from the best available copy.

Available to DOE and DOE contractors from
U.S. Department of Energy
Office of Scientific and Technical Information
P.O. Box 62
Oak Ridge, TN 37831

Telephone: (865)576-8401
Facsimile: (865)576-5728
E-Mail: reports@adonis.osti.gov
Online ordering: <http://www.doe.gov/bridge>

Available to the public from
U.S. Department of Commerce
National Technical Information Service
5285 Port Royal Rd
Springfield, VA 22161

Telephone: (800)553-6847
Facsimile: (703)605-6900
E-Mail: orders@ntis.fedworld.gov
Online order: <http://www.ntis.gov/ordering.htm>



SAND2001-1532
Unlimited Distribution
Printed May 2001

Superresolution and Synthetic Aperture Radar

Fred M. Dickey, Louis A. Romero, Armin W. Doerry
Sandia National Laboratories
PO Box 5800
Albuquerque, NM 87185

ABSTRACT

Superresolution concepts offer the potential of resolution beyond the classical limit. This great promise has not generally been realized. In this study we investigate the potential application of superresolution concepts to synthetic aperture radar. The analytical basis for superresolution theory is discussed. The application of the concept to synthetic aperture radar is investigated as an operator inversion problem. Generally, the operator inversion problem is *ill posed*. A criterion for judging superresolution processing of an image is presented.

ACKNOWLEDGEMENTS

The authors would like to thank John M. DeLaurentis for extensive discussions on the underlying functional analysis and critique of the manuscript. We would like to thank Stephen E. Yao for help with the figures and Bibliography.

This work was funded by the US DOE Office of Nonproliferation & National Security, Office of Research and Development, under the Advanced Radar System (ARS) project. This effort is directed by Randy Bell of DOE NN-20.

FOREWORD

The ability to resolve Synthetic Aperture Radar (SAR) images to finer resolutions than the system bandwidths allow is a tantalizing prospect. Seemingly superresolution offers “something for nothing”, or at least “something better than the system was designed for” if only we process ‘enough’ or ‘right’. Claims in this arena certainly warrant further investigation.

This report documents the research that was commissioned with the following questions:

“What exactly is superresolution?” and “What is not really superresolution?”

“Is superresolution possible?” and if so “to what degree?”

“What constrains superresolution?”

“How should we objectively test whether an image is superresolved?”

The answers to these questions ultimately lead to yet another question that is outside the scope of this report, namely “Should we incorporate some form of superresolution within our SAR designs?”

This page left intentionally blank.

CONTENTS

1 Introduction.....	9
2 Definition and Assumptions.....	11
3 The Ideal Model and Superresolution.....	14
4 Perturbations on the Ideal Model.....	24
5 Noise.....	26
6 A Test Criterion.....	35
7 Discussion.....	38
Summary.....	44
References.....	45
Bibliography.....	48

This page left intentionally blank.

1 Introduction

The purpose of this study is to evaluate the potential application of superresolution techniques to synthetic aperture radar (SAR). Superresolution is an attempt to extrapolate the Fourier transform of an image beyond the bandpass of the image acquisition system, SAR in this case. The concept of superresolution has been around in one form or another since 1952.¹ Superresolution is based on the idea that the Fourier transform of a finite object (finite field-of-view) is analytic and the knowledge of the image transform is limited by the bandpass of the imaging system. The part of the spectrum outside the bandpass of the system is obtained using the method of analytic continuation. The problem is that an analytic function plus noise is no longer analytic. Also, the process of analytic continuation involves derivatives, which are sensitive to noise. Interest in the subject swelled about 1970 and continued for several years. Despite considerable work in the area, the authors do not know of any significant application of the concept. In their book, Andrews and Hunt² list superresolution in the index as, “superresolution (myth of).” In an early paper, Di Francia³ addresses the degrees of freedom of an image. He concludes that the space-bandwidth product of the imaging system essentially limits the degrees of freedom of an image, effectively eliminating the prospects of superresolution. Bertero and De Mol give a very good summary of the superresolution problem in their chapter in *Progress In Optics*.⁴

There is considerable literature on the subject of superresolution. We have compiled a bibliography on the subject that is fairly complete, especially if references in the listed papers are considered. The papers are listed in chronological order and are grouped into four categories: General, Synthetic Aperture Radar, Ancillary and Books.

The “general” category is just those papers that generally address superresolution or provide related mathematical background material. “Synthetic Aperture Radar” papers are papers that specifically address superresolution processing of SAR imagery. Papers that may touch on the subject of superresolution in general or its applications to SAR imagery but are not considered that direct are put in the “Ancillary” category. Finally, books are put in a separate category.

The next section discusses assumptions that define the problem. The ideal model for superresolution is analyzed in Section 3. Perturbations on the ideal model are treated in Section 4. The impact of noise on the problem is discussed in Section 5. Finally, Section 6 discusses a test criterion for evaluating superresolution processing schemes.

2 Definition and Assumptions

To address the application of superresolution concepts to SAR it is important to define what is meant by superresolution. We define superresolution as:

Definition: Superresolution is the recovery of spectral information outside the bandpass of the system. It is assumed that the bandpass of the system is determined by a hard limit, a limit beyond which no spectral information is available. Either the frequency response is identically zero outside of the band limit or noise masking imposes an effective band limit.

This is the approach taken in the *Progress in Optics* chapter by Bertero and De Mol.⁴

A schematic of the spectral response of an ideal bandpass system is illustrated in Fig. 1. In the figure the system frequency response is zero outside the band $|f| \leq f_0$. A more general representation of a bandpass system is shown in Fig. 2. In this case the frequency is again zero outside the band defined by $|f| \leq f_0$. A third type system response is shown in Fig. 3. In this case the usable system bandwidth is limited by noise. The system design should always limit the bandwidth so that the part of the spectrum that is dominated by noise is excluded.

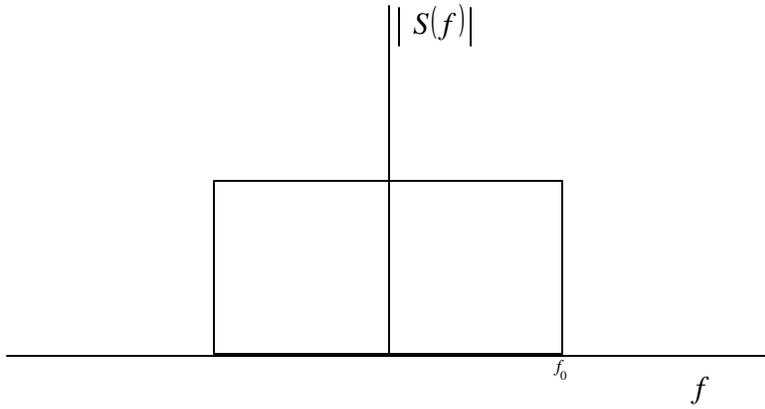


Fig. 1 Ideal bandpass system response.

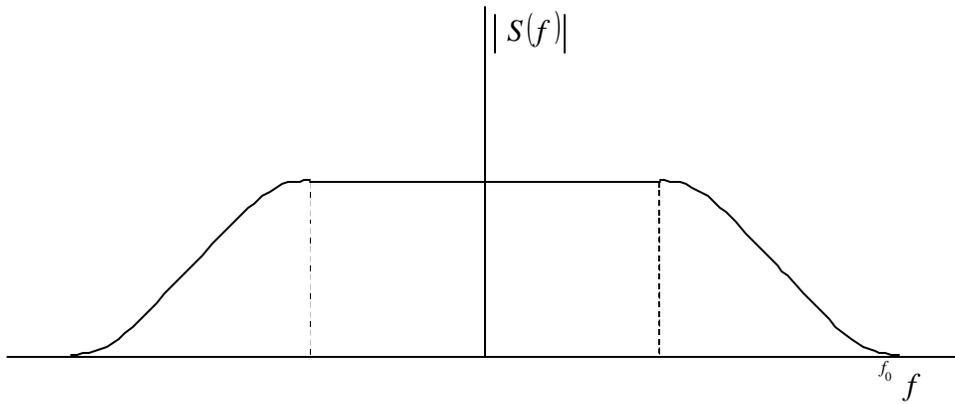


Fig. 2 Schematic of a general bandpass system response.

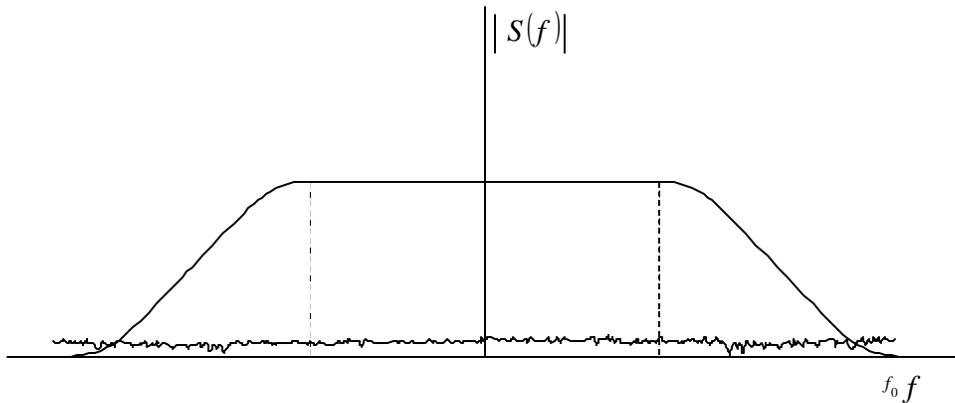


Fig. 3 Schematic of a noise limited system response.

For our purposes there is effectively no difference between the system responses illustrated in Fig. 1, Fig. 2, and Fig. 3. In each case the superresolution problem is to recover the spectral information outside the bandpass of the system. Any attempt to flatten the response in Fig. 2 or Fig. 3 is a deblurring, inverse filtering, or Wiener filtering problem.⁴

In reviewing the SAR papers in the bibliography, we found that they did not generally address the imaging system model or noise, and if they did it was mainly a passing comment. Reference 5 does treat the system model and noise, and although it addresses SAR, it was put in the “General” category because of the broad applicability of the analysis. For this reason we make the following assumptions (or axioms).

Assumption – 1: Any significant treatment of superresolution applications in SAR must address the system model.

Assumption – 2: Any significant treatment of superresolution applications in SAR must address system noise.

Whether significant superresolution improvements can be obtained depends both on the system model and noise, and the two are generally related. It is of interest to make the system model as general as possible so that the results of the analysis will have broad applicability. The analysis can always be refined or extended for a specific system.

3 The Ideal Model and Superresolution

The general imaging problem is described by the equation,

$$s = L\mathbf{s} , \tag{1}$$

where \mathbf{s} is the function representing the object (i.e. the scene), L is a linear operator describing the imaging system, and s is the observation (data) collected by the imaging system. It should be noted that s in Eq. (1) may be an approximation to (image of) the object. This is the case for standard imaging in optics using a lens^{1,4} and it is also the case for SAR imaging⁵ if the standard processing is incorporated in the operator L . That is, the operator L can be a product of operators, or a complex operator, representing the transmitted signal, antenna pattern, receiver function, and SAR data processing. In general, the ideal imaging problem is the inversion of Eq. 1 written as

$$\mathbf{s} = L^{-1}s \tag{2}$$

where L^{-1} is the inverse operator. The inverse may not exist, and if it does it may not be bounded or continuous. When the inverse does not exist, one can use the pseudo-inverse, which is an inverse in least-mean-square sense.

The solution of the problem described by Eq. (1) or equivalently Eq. (2) belongs in the theory of operators on a Hilbert space (functional analysis). Generally, we will need only rudiments of the theory to address the problem of superresolution. A terse summary of most of the related Hilbert space facts is contained in the paper by Joyce and Root.⁵ A good introduction to functional analysis is given by Kreyszig.⁶ The inverse operator for compact self-adjoint (Hermitian) operators can be expressed in terms of the eigenvalues and eigenfunctions of the operator.^{7,8} It turns out that integral operators of the type that we will be interested in are generally compact. They are self-adjoint if they are

invariant (a convolution operator) with a real kernel. However, there are self-adjoint operators for which the kernel is neither real nor a convolution type. In the case that the operator of interest is not self-adjoint, but compact, we can form self-adjoint operators that have the properties needed to address the inversion.

For compact self-adjoint operator L on a Hilbert space H , the Hilbert-Schmidt theorem^{7,8} states that there is an orthonormal system of eigenfunctions, $\mathbf{j}_1, \mathbf{j}_2, \dots$, and associated eigenvalues I_1, I_2, \dots , such that every element $\mathbf{s} \in H$ has a unique representation

$$\mathbf{s} = \sum a_n \mathbf{j}_n, \quad (3)$$

and

$$\lim_{n \rightarrow 0} I_n = 0. \quad (4)$$

We state the obvious by noting that the a_n are independent. That is, for an arbitrary scene no inference can be made on a_n from any set of a_m for $a_m \neq a_n$.

Further, we can represent the inverse operator L^{-1} (Eq. (2)) as

$$L^{-1} \mathbf{q} = \sum \frac{1}{I_n} (\mathbf{j}_n, \mathbf{q}) \mathbf{j}_n, \quad (5)$$

where (\mathbf{j}, \mathbf{q}) denotes the inner product between \mathbf{j} and \mathbf{q} .

A problem is *well posed*⁵ if (1) the problem has a solution, (2) the solution is unique, and (3) the solution is a continuous function of the data. A problem is *ill posed* if it is not *well posed*. Equation (5) shows that condition (3) is not satisfied for compact self-adjoint operators. This is due to the division by the eigenvalues, which approach zero as n approaches infinity. The inverse is unbounded, or equivalently, discontinuous. This means that arbitrarily small changes in \mathbf{s} can result in arbitrarily large changes in

\mathbf{s} . Thus, linear inverse problems for compact self-adjoint operators are necessarily ill posed.

It is impractical to base an assessment of the potential of superresolution in SAR imaging on too detailed of a model of the SAR system. However, there are some very general assumptions about the model that may be reasonable that have significant impact on the analysis.

One such assumption is that the SAR imaging operator L can be represented as an integral operator with kernel $K(x, y)$ of the form

$$s(x) = L_K \mathbf{s} = \int_I K(x, y) \mathbf{s}(y) dy, \quad (6)$$

where I is a closed interval. We shall carry out the analysis in one-dimensional form; however, the formulations and results are readily extended to two (or higher) dimensions.

The operator represented by Eq. (6) is compact (completely continuous) if

$$\int_I \int_I |K(x, y)|^2 dx dy = M < \infty \quad (7)$$

The operator adjoint to L_K is an integral operator with kernel $K^*(y, x)$. It is self-adjoint if

$$K(x, y) = K^*(y, x). \quad (8)$$

This is the case if $K(x, y)$, is real and symmetric. Further if $K(x, y)$ has the form $K(x - y)$ the operator is a convolution type, and it is self-adjoint if it is real. The SAR data processing is sometimes modeled as a convolution (invariant) process. In general, the processing is quasi-invariant.

3.1 The Ideal SAR Model

We will use the model put forward by Joyce and Root.⁵ It is a linear invariant model that can be adapted to a quasi-invariant system model. In SAR imagery and tomography the collected measurement (data) can be arranged so as to constitute a band-limited Fourier transform of the space-truncated complex reflectivity of the object,^{9,10}

$$\tilde{s}(\mathbf{h}) = \text{rect}\left(\frac{\mathbf{h}}{2W}\right) \int_{-X/2}^{X/2} e^{-i2\mathbf{p}\mathbf{x}} \mathbf{s}(x) dx. \quad (9)$$

We can (inverse) Fourier transform Eq. (9) to obtain,

$$s(x) = \int_{-X/2}^{X/2} \frac{\sin 2\mathbf{p}W(x-y)}{\mathbf{p}(x-y)} \mathbf{s}(y) dy. \quad (10)$$

Superresolution can be applied to either Eq. (9) or Eq. (10). Both equations describe a process of collecting over an interval and then band-limiting the result. Although these two equations are equivalent, in that they are Fourier transform pairs, Eq. (10) represents the classical imaging problem. By classical imaging problem it is meant that $s(x)$ given by Eq. (10) is an image of $\mathbf{s}(x)$ with the resolution given by the implicit sinc function in the integrand. That is the resolution is inversely proportional to W (bandwidth). Only in the limit of large W do we have a true inversion.

The operator in Eq. (10) is compact, self-adjoint, and positive on the interval X . For any $X > 0$ and any $W > 0$, there is a countably infinite set $\mathbf{y}_0(x), \mathbf{y}_1(x), \mathbf{y}_2(x), \dots$ and a set of real positive numbers

$$I_0 > I_1 > I_2 > \dots \quad (11)$$

with the following properties:¹³

1) The $\mathbf{y}_n(x)$ are band-limited, orthonormal on the real line and complete in the space of band-limited functions (bandwidth $2W$):

$$\int_{-\infty}^{\infty} \mathbf{y}_i(x) \mathbf{y}_j(x) dx = \begin{cases} 0, & i \neq j \\ 1 & i = j \end{cases}. \quad (12)$$

2) In the interval $-X/2 \leq x \leq X/2$, the $\mathbf{y}_n(x)$ are orthogonal and complete on the interval X :

$$\int_{-X/2}^{X/2} \mathbf{y}_i(x) \mathbf{y}_j(x) dx = \begin{cases} 0, & i \neq j \\ \mathbf{I}_i & i = j \end{cases}. \quad (13)$$

3) For all values of x , real or complex,

$$\mathbf{I}_n \mathbf{y}_n(x) = \int_{-X/2}^{X/2} \frac{\sin 2\mathbf{p}W(x-y)}{\mathbf{p}(x-y)} \mathbf{y}_n(y) dy. \quad (14)$$

This notation conceals the fact that both the \mathbf{y} 's and the \mathbf{I} 's are functions of the product WX . That is, $\mathbf{I}_n = \mathbf{I}_n(c)$ and $\mathbf{y}_n(x) = \mathbf{y}_n(c, x)$, where

$$c = \mathbf{p}WX. \quad (15)$$

There are many other properties of the \mathbf{y} 's; we will only use them as needed. The \mathbf{y} 's are prolate spheroidal wave functions. They are extremely valuable in the treatment of space-limited and band-limited functions and to address the ability to achieve essentially space and band-limited functions. An example is the application of prolate spheroidal wave functions to the problem of optimum edge detection.¹¹ Harger¹² applies the prolate spheroidal wave functions to SAR antenna design and signal analysis. The properties of the prolate spheroidal wave functions have been extensively treated in the literature.^{13,14,15,16,17,18,19,20}

The analysis of the inversion (superresolution) problem can be carried out using the properties of the prolate spheroidal wave functions without explicit use of the fact that the defining operator is compact, self-adjoint, etc. Since the prolate spheroidal wave functions are complete in the interval I , property 2), we can expand the input as

$$\mathbf{s}(x) = \sum_0^{\infty} a_n \mathbf{y}_n(x). \quad (16)$$

Applying Eq. (14) to Eq. (10) and Eq. (16) gives the output as

$$s(x) = \sum_0^{\infty} \mathbf{I}_n a_n \mathbf{y}_n(x), \quad (17)$$

which is just the prolate spheroidal wave function expansion for the output. This says that to invert Eq. (10) we need only to expand the output in terms of the prolate spheroidal wave functions and divide each term by the corresponding eigenvalue, \mathbf{I}_n .

This is equivalent to Eq. (5) with $\mathbf{j}_n(x) = \mathbf{y}_n(x) / \sqrt{\mathbf{I}_n}$. The division by $\sqrt{\mathbf{I}_n}$ is required to make the functions \mathbf{y}_n orthonormal on the interval.

Even though $\mathbf{I}_n \rightarrow 0$ as $n \rightarrow \infty$, division by \mathbf{I}_n does not seem to be that much of a problem. Certainly, we can include a lot of terms in the inversion before we get very close to infinity. The real problem with superresolution is that for a fixed value of c is that the \mathbf{I}_n fall off to zero rapidly with increasing n once n has exceeded $(2/p)c = 2WX$.^{13,17,18,19} Further, Newsam and Barakat²¹ argue that the \mathbf{I}_n are distributed in a step-like fashion,

$$\begin{aligned} \mathbf{I}_n &\approx 1 && \text{if } n < 2WX \\ &\approx 0 && \text{if } n > 2WX \end{aligned} \quad (18)$$

with the change from unity to zero occurring in a strip of width $\approx \log n$ centered on $n = 2WX$. Furthermore, the eigenvalues decay exponentially with n , as $n \rightarrow \infty$. This is the heart of the superresolution problem. It is this rapid roll-off of the eigenvalues that limits the ability to recover information outside the bandpass of an essentially band-limited system. Clearly, terms in Eq. (17) would be lost in the noise for $n > 2WX$. It is very curious that, although Joyce and Root⁵ address the ill posed aspects of the inversion of Eq. (9) (equivalently Eq. (10)), they do not address this well known and problematic result. They do, in passing, reference Slepian and Pollak¹³ with respect to the general ill posed nature of the problem of inversion of compact self-adjoint operators.

One might consider an alternate scheme for divining the values for a_n for $n > 2WX$, by perhaps somehow transmogrifying or extrapolating in some manner from those values of a_n for $n < 2WX$. Extrapolation techniques of various sorts pervade the field of superresolution techniques. Clearly, however, for a general scene the independence of the a_n precludes this. Denying this undeniable truth and attempting to do so anyway then forces a correlation between the a_n that in fact destroys the ability of such an algorithm to superresolve a general scene. While perhaps allowing some images, or perhaps some targets within a specific image, to ‘look better’, such a technique would necessarily make other sorts of targets or images *less* accurately rendered in an image. One might consider this as actually *adding* distortion to a general image with results that are likely arguable and highly subjective. In any case, this is not superresolution as we have previously defined it. We reserve further comments on these sorts of algorithms to the discussion later in this report.

In some sense, one could stop at this point and consider the problem solved. However, further treatment of the matter would provide additional insight into the problem, and the added material may be of interest to the reader in its own right.

3.2 Degrees of Freedom of an Image

A very related problem is the number of degrees of freedom of an image. This subject has received considerable attention in the literature^{3,15,21,22,23} Di Francia³ uses (as do others) the imaging model of the previous section. Since the object (field-of-view) has a finite field of view and the imaging system has a finite bandwidth, he defines the *number of degrees of freedom of an image* as,

$$N = 2WX . \quad (19)$$

This definition is based on the sampling theorem; it is just the band-limited sampling rate times the extent of the image. The argument in the previous sections effectively limits the degrees of freedom of an image to the number given by Eq. (19).

Although a function cannot be simultaneously space-limited and band-limited, it can be so to a high degree of approximation. Such functions are considered “essentially” space and band-limited. Landau and Pollak treat this problem in great detail.¹⁵ We state the main theorem of their paper as:

Theorem 1: Let $g(x)$, of total energy 1, be band-limited to bandwidth $2W$, and let

$$\int_{-X/2}^{X/2} |g(x)|^2 dx = 1 - e^{-2} . \quad (20)$$

Then

$$\inf_{\{a_i\}} \int_{-\infty}^{\infty} \left| g(x) - \sum_0^{[2WX]+N} a_n \mathbf{j}_n \right|^2 dx < C e^{-2}, \quad [z] \text{ means the largest integer } \leq z, \quad (21)$$

is (a) true for all such g with $N=0$, $C=12$, if the \mathbf{j}_n are the prolate spheroidal wave functions; (b) false for some such g for any finite constants N and C if the \mathbf{j}_n are the sampling (sinc) functions.

If the expansion in part (a) of Theorem 1 is done in terms of prolate spheroidal wave functions we have the following theorem¹⁵

Theorem 2: Given $g(x)$ defined in Theorem 1, then

$$\int_{-\infty}^{\infty} \left| g(x) - \sum_0^{[2WX]} a_n \mathbf{y}_n \right|^2 dx \leq 12 \mathbf{e}_x^2, \quad [z] \text{ means the largest integer } \leq z, \quad (22)$$

where the a_n are the Fourier coefficients of its expansion in the \mathbf{y} 's (prolate spheroidal wave functions).

This theorem bounds the accuracy with which a function can be represented by a function with $2WX$ degrees of freedom. For example, if the energy in the function $g(x)$ falling outside the interval X is 10^{-p} , the mean square error in representing the function with $N = 2WX$ prolate spheroidal wave functions can be made less than 12×10^{-p} . Later we will discuss bounds on the use of sampling functions. It is not as bad as one might expect.

4 Perturbations on the Ideal Model

A natural question at this point is, what happens when the ideal model of Section 3.1 does not hold? There are several ways that this can be approached. The most general assumption, for our purposes, is a model in the form of Eq. (6). The associated operator is probably compact, but it may not be self-adjoint. If it is not self-adjoint, we can form the self-adjoint operators, $\hat{L}_K L_K$ and $L_K \hat{L}_K$ where \hat{L}_K is the adjoint of L_K . The eigenfunctions and eigenvalues of these self-adjoint operators would provide set of basis function for the analysis of the problem similar to the approach taken above. To do this, we would have to solve the eigenvalues and eigenfunctions. This approach is referred to as singular value decomposition (SVD).^{24,25} It is probably not necessary to take the SVD approach.

A sinc function kernel in Eq. (6) gives a spectral response represented in Fig. 1 and the analysis leading to Eq. (17) is applicable. If we perturb the kernel we would generally get a spectral response represented by Fig. 2. First order operator perturbation theory shows that small perturbations of the kernel can destroy the exponential decay in the eigenvalue spectrum described by Eq. (18). However, one can design large perturbations of the operator that do not destroy the exponential decay in the eigenvalue spectrum. One way to do this is to design perturbations that leave the form of the kernel unchanged, but change the value of c given by Eq. (15). In general, perturbations of the kernel in Eq. (6) will give a frequency response with a roll-off characterized in Fig. 2. Since there is always some noise there is always a cut-off frequency used in the system design/processing and we are back to the type system associated with Fig. 1.

Mathematically we can state the above considerations as follows. We assume that the data is of the form

$$s(x) = L_K \mathbf{S} = \int_I K(x, y) \mathbf{S}(y) dy. \quad (23)$$

Although Eq. (23) is the same form as Eq. (6), in this case it represents the collected data, not a classical image. The data, $s(x)$, has a Fourier transform that is generally represented by Fig. 3. Due to the presence of noise we assume that a band-limiting operator of the form

$$s'(x) = L_{BW} s = \int_I \frac{\sin 2pW(x-y)}{p(x-y)} s(y) dy, \quad (24)$$

is applied to the data $s(x)$. If an image processing operator, L_{ip} , is then applied to the band-limited data, $s'(x)$, we have a classical image of the form,

$$s'' = L_{ip} L_{BW} L_K \mathbf{S}. \quad (25)$$

Even if we assume that we can invert the operator L_{ip} , the inversion of the operator, L_{BW} , is limited to by the rapid decay of its eigenvalue spectrum as discussed in the previous discussion. Thus it appears that there is a practical limit beyond which spectral information about an image (data set) cannot be recovered. This limit is, of course, set by noise.

5 Noise

Clearly, noise is the major obstacle to achieving significant imaging system resolution improvement (superresolution gain) from bandwidth extrapolation beyond the system bandwidth. In an empirical study of band-limited image extrapolation based on a Gerchberg type algorithm, Smith and Marks²⁶ reported that the output began looking like the input at a signal-to-noise ratio (SNR) of 10^8 and the output became virtually equivalent to the input at a SNR of 10^{10} . They investigated the problem of extrapolating a truncated image function given that the image bandwidth is known. They used a sinc function as the image function. This should be equivalent to extrapolating the Fourier transform of a function given the extent of the function. Cox and Sheppard²⁷ take the interesting approach of treating the superresolution problem in terms of information theory. Their analysis is based on the invariance of information capacity. They state that it is not the spatial bandwidth but the information capacity of an imaging system that is constant. Further, they conclude that analytic continuation is essentially an attempt to increase the spatial bandwidth by reducing the SNR in the final image. Their analysis for the large space-bandwidth product case is plotted in Fig. 4. In the figure, SNR_I and SNR_O denote input and output signal-to-noise ratio respectively. In their paper they use bandwidth to define resolution, that is, the resolution improvement is a ratio of the output to the input bandwidth.

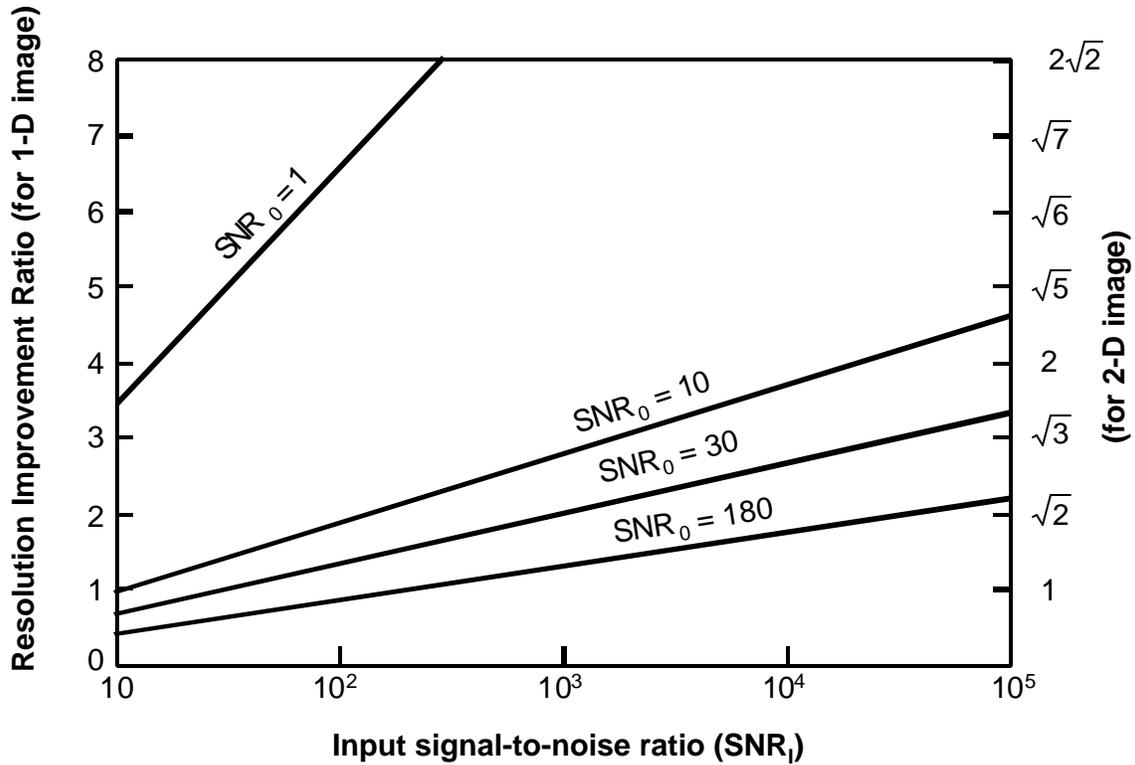


Fig. 4 Resolution improvement (ratio) as a function of input SNR for analytic continuation of a large space-bandwidth product image.

Bertero and De Mol,⁴ using the approach in Section 3.1 conclude 1) that superresolution is feasible only when the object is not too large compared to the resolution of the imaging system, and 2) the amount of achievable superresolution depends on the space-bandwidth-product (c) and the SNR. They analyze the noise effect in the small space-bandwidth-product case. They also point out that attempts have been made to add additional constraints such as image positivity to improve the extrapolation. We are not sure what constraints, if any, would be appropriate in the SAR case. Since the image/data SNR is a dominant factor in extending the bandwidth of the SAR image, it is appropriate to say something about the sources of noise. It is outside the scope of this work to treat the sources of noise (or errors) in radar systems in any detail. We might

mention that there is clutter noise, receiver noise, noise due to system and waveform nonlinearities (multiplicative noise), and noise due to sampling. We consider errors associated with sampling appropriate to this study.

The analysis we have presented is for continuous variables, but any numerical processing would certainly involve sampled data. We have been working under the tacit assumption that the most favorable solutions are those using continuous variable analysis. The discrete case would be an approximation, although a good one. In fact, Fiddy and Hall²⁸ argue that superresolution applied to sampled data is not unique if the number of samples is finite. It is well known (Shannon sampling theorem) that a band-limited function can be represented exactly if using a countable infinity of uniformly spaced samples. In practice an infinity of samples is never used. However, from Theorem 1, a function can be effectively band-limited and space-limited. The question is then, what is the bound on the error resulting from using a finite number of samples to represent a band-limited function? Landau and Pollak¹⁵ address this in an interesting theorem. We give an abridged version of that theorem.

Theorem 3: Let $g(x)$, of total energy 1, be band-limited to bandwidth $2W$, and let

$$\int_{-x/2}^{x/2} |g(x)|^2 dx = 1 - \mathbf{e}_x^2. \quad (26)$$

Then, if $WX - [WX] \leq 1/2$, where $[z]$ means the largest integer $\leq z$, we have

$$\int_{-\infty}^{\infty} \left| g(x) - \sum_{|k \leq WX+1|} g\left(\frac{k}{2W}\right) \frac{\sin \mathbf{p}(2WX+1-k)}{\mathbf{p}(2WX+1-k)} \right|^2 \leq \mathbf{p}\mathbf{e}_x + \mathbf{e}_x^2. \quad (27)$$

This theorem should be compared with Theorems 1 and 2. The major result is that if “unconcentrated part,” \mathbf{e}_x^2 , of the energy is small, the errors in representing a function using the prolate spheroidal wave functions is proportional to \mathbf{e}_x^2 while the errors using sampling functions is proportional to \mathbf{e}_x . For example if the “unconcentrated part,” of the energy is 10^{-6} the error in the sampling function representation may be as large as $p \times 10^{-3}$ while the errors in the prolate spheroidal wave function representation is less than 12×10^{-6} . The sampling errors are part of the noise to be considered with respect to the application of superresolution techniques. It should be noted, that for hard-limited chirp type waveforms, this Theorem 3 can be turned around to estimate the error in sampling of the Fourier transform.

If the signal to noise ratio is large enough, we can use super-resolution techniques to resolve well past the limits of the classical resolution limit. However, for large space-bandwidth product signals the signal to noise ratios must be enormously large to achieve even a moderate amount of super-resolution. We will now estimate how large the signal to noise ratios must in fact be. The following argument, although somewhat heuristic with respect to the noise characterization, gives reasonable results and it is instructive.

For the noise case corresponding to Eq. (1), we are trying to solve the equation

$$L_c \mathbf{s} = \mathbf{s} + \mathbf{h}, \quad (28)$$

where \mathbf{s} is the noiseless data, and $\mathbf{h}(x)$ is the noise. We will assume that $\mathbf{h}(x)$ is white noise. This means that

$$\langle \mathbf{h}(x) \rangle = 0, \quad (29)$$

and

$$\langle \mathbf{h}(x)\mathbf{h}(z) \rangle = \mathbf{e}^2 \mathbf{d}(x-z). \quad (30)$$

Here \mathbf{e} represents the level of the noise. The following lemma will be used to characterize the noise.

Lemma 1: Let \mathbf{j}_k be the normalized eigenfunction associated with the eigenvalue \mathbf{l}_k of the operator L_c . Let \mathbf{h}_k be the k th component of the eigenvalue expansion of the noise. The expected value of \mathbf{h}_k^2 is given by

$$\langle \mathbf{h}_k^2 \rangle = \mathbf{e}^2. \quad (31)$$

Proof: We can write

$$\mathbf{h}_k = \int_X \mathbf{j}_k(x) \mathbf{h}(x) dx. \quad (32)$$

It follows that

$$\mathbf{h}_k^2 = \int_X \int_X \mathbf{j}_k(x) \mathbf{h}(x) \mathbf{j}_k(z) \mathbf{h}(z) dx dz. \quad (33)$$

Taking the expected value of this, and using Eq. (30), we find that

$$\langle \mathbf{h}_k^2 \rangle = \mathbf{e}^2 \int_X \int_X \mathbf{j}_k(x) \mathbf{j}_k(z) \mathbf{d}(x-z) dx dz = \mathbf{e}^2 \int_X \mathbf{j}_k^2(x) dx = \mathbf{e}^2. \quad (34)$$

QED

Radar processing is much more realistically modeled by bandlimited white noise.

Let $\mathbf{h}^W(x)$ represent the noise that results from bandlimiting the noise $\mathbf{h}(x)$ to a bandwidth $2W$. The autocorrelation function of \mathbf{h}^W satisfies

$$\langle \mathbf{h}^W(x) \mathbf{h}^W(y) \rangle = \mathbf{e}^2 \frac{\sin(2pW(x-y))}{pW(x-y)}. \quad (35)$$

The following lemma concerns the expected value of the eigenfunction expansion of the coefficients of \mathbf{h}^W .

Lemma 2: Let \mathbf{j}_k be the normalized eigenfunction associated with the eigenvalue \mathbf{I}_k of the operator L_c . Let \mathbf{h}_k^W be the k th component of the eigenvalue expansion of the noise. The expected value of $(\mathbf{h}_k^W)^2$ is given by

$$\langle (\mathbf{h}_k^W)^2 \rangle = \mathbf{I}_k \mathbf{e}^2. \quad (36)$$

Proof: We can write

$$\mathbf{h}_k^W = \int_X \mathbf{j}_k(x) \mathbf{h}(x) dx. \quad (37)$$

It follows that

$$(\mathbf{h}_k^W)^2 = \int_X \int_X \mathbf{j}_k(x) \mathbf{h}(x) \mathbf{j}_k(z) \mathbf{h}(z) dx dz. \quad (38)$$

Taking the expected value of this, and using Eq. (35), we find that

$$\langle (\mathbf{h}_k^W)^2 \rangle = \mathbf{e}^2 \int_X \int_X \mathbf{j}_k(x) \mathbf{j}_k(z) \frac{\sin(2\mathbf{p}(x-z))}{\mathbf{p}(x-z)} dx dz = \mathbf{I}_k \mathbf{e}^2 \int_X \mathbf{j}_k^2(x) dx = \mathbf{I}_k \mathbf{e}^2. \quad (39)$$

QED

We believe that the radar data is correctly represented as

$$s_o = \text{rect}\left(\frac{x}{X}\right) [h * \{L\mathbf{s} + \mathbf{h}\}]. \quad (40)$$

where h is the impulse response of a bandlimiting filter. This equation says that since the imaging operator L bandlimits the data we should filter the signal and noise to the bandwidth $2W$ and retain the data corresponding to the field-of-view. This is reasonable because the data extent and field-of-view are essentially the same for large space-bandwidth-product systems. This allows us to write Eq. (40) as

$$s_o = \text{rect}\left(\frac{x}{X}\right) [L\mathbf{s} + h * \mathbf{h}]. \quad (41)$$

The rect function in Eq. (41) allows us to expand the right hand side over the interval X .

Using Eqs. (13) we obtain

$$s_o = \sum_k \mathbf{I}_k a_k \mathbf{y}_k + \frac{\mathbf{y}_k}{\sqrt{\mathbf{I}_k}} \int_X (h * \mathbf{h}) \frac{\mathbf{y}_k}{\sqrt{\mathbf{I}_k}} dx \quad (42)$$

where \mathbf{y}_k are the prolate spheroidal wave functions. Applying lemma 2 gives

$$\begin{aligned} s_o &= \sum_k \mathbf{I}_k a_k \mathbf{y}_k + \mathbf{e} \sqrt{\mathbf{I}_k} \frac{\mathbf{y}_k}{\sqrt{\mathbf{I}_k}} \\ &= \sum_k \mathbf{I}_k a_k \mathbf{y}_k + \mathbf{e} \mathbf{y}_k \end{aligned} \quad (43)$$

The first term on the right in Eq. (43) represents the signal (data) and the second term is the noise.

For the operator L_c (with c large) the eigenvalues \mathbf{I}_k are very close to unity as long as

$$k \leq N_c = \frac{2c}{p} = 2WX. \quad (44)$$

For $k > N_c$ the eigenvalues are close to zero. This allows us to write the signal energy as

$$E_s^2 = \int_X \left| \sum_k \mathbf{I}_k a_k \mathbf{y}_k \right|^2 dx \cong \sum_0^{N_c} |a_k|^2 \quad (45)$$

and the noise energy in the k th ($k > N_c$) term in the noise expansion as

$$E_h^2 = \mathbf{I}_k \mathbf{e}^2. \quad (46)$$

We should be able to invert our problem to include term of order k if the noise energy in the k th term is much smaller than the signal energy. This implies a SNR requirement given by

$$\sqrt{\mathbf{I}_k} \gg \frac{\mathbf{e}}{E_h}. \quad (47)$$

This shows that we can tolerate large levels of noise if we only invert the data using the first N_c eigenfunctions. When we do this we achieve the classical resolution. We achieve super-resolution by using more eigenfunctions in our inversion. It is commonly assumed that in the absence of noise if we used $2N_c$ eigenfunctions then we have effectively doubled our bandwidth. We believe that this is true, but the argument making this precise is somewhat subtle. We will assume that this is in fact the case. Certainly, we will have doubled the number of degrees of freedom.

We now ask the question: For a given space-bandwidth product (c), what SNR is needed in order to double our effective bandwidth. To answer this question we use the fact that for large values of c , we have

$$I_{2N_c} = e^{-p\sqrt{c}/\ln(2)}. \quad (48)$$

This result is easily obtained using equations in Reference 17. In order to reliably invert the modes associated with all of these eigenfunctions, the noise must satisfy Eq. (47). This shows that the signal to noise ratio must satisfy

$$SNR = e^{p\sqrt{c}/2\ln(2)}. \quad (49)$$

This result is derived for one-dimensional signals. For two-dimensional signals, the required signal to noise ratio is squared.

There are a few things that should be said about this result. First, if for example $2WX = 20$ (a small value in practice), then Eq. (49) requires a SNR greater than 3.3×10^5 to improve the resolution by a factor of 2. Although this analysis does not estimate the SNR of the superresolved image, it appears compatible with the data in Fig. 4, which is derived from an information theory point-of-view. In either approach, the results do not predict significant gain by superresolution processing. This result agrees with the SNR

requirement in Reference 4. They use the operator eigenfunction approach as we do. Newsam and Barakat²¹ treat the problem in a more detailed manner, relating the noise to a quantitative mean square error in the inversion.

Finally, it is generally believed that some moderate superresolution improvement is possible if the space-bandwidth product (c) is small. However, in that case the problem is probably one of spectral shaping (inverse filtering). The low space-bandwidth product case suggests a possible extension of the above to the problem of imaging relatively isolated bright targets. For such targets the SNR “might” be greater than the average for the image. This suggests that one might cut out that part of the image and attempt to achieve superresolution due to the reduced space-bandwidth product. Note that the space-bandwidth product approximated by dividing the reduced image size by the system resolution. We were not able to obtain an analytical solution to this problem within the scope of this work. The problem could be treated along the lines of this work. This would require the eigenvalues and eigenfunctions associated with the imaging operator representing this problem. They would probably be calculated numerically. There are two reasons for some doubt that there may be much gained in this case. One is that even in this case the space-bandwidth product would still be too large to expect much superresolution gain. The other is that the SNR for the isolated part of the image may not be as large as first assumed. This is due to the fact that the tails of the impulse response for the surrounding scene would corrupt (add noise to) the data for the restricted part of the image. Selecting a portion of a larger image for processing is not the same as imaging an isolated object, such as the example of imaging an object in space.

6 A Test Criterion

A basic test criterion for superresolution is one that determines the ability of a superresolution method to extrapolate, with fidelity, the data spectrum beyond that of the effective system bandwidth. Fidelity is a key part of the measure of performance; it would be meaningless to add, by whatever means, uncorrelated spectral content outside the system bandwidth. We have defined superresolution as the recovery of spectral information that falls outside the bandpass of the system. This suggests a mean square error measure of superresolution gain. Clearly, the merit function should be normalized so that amplitude scaling does not produce erroneous errors.

As discussed by Davila and Hunt,²⁹ there are three quantities that should be addressed in the development of a merit function; the object \mathbf{s} , the bandlimited image (data) s , and the superresolved image \hat{s} . They propose a measure of superresolution gain SRG of the form

$$SRG = \frac{\|\mathbf{s} - s\|^2}{\|\mathbf{s} - \hat{s}\|^2}, \quad (50)$$

where $\|g\|$ is the norm of g . As the superresolved image \hat{s} approaches \mathbf{s} , $SRG \rightarrow \infty$. Further, $SRG \rightarrow 1$ as $\hat{s} \rightarrow s$, which should be the case if $s = \mathbf{s}$. One might argue that a possibly infinite measure is not a well normalized. This is readily resolved by defining the superresolution gain as

$$SRG_N = 1 - \frac{1}{SRG} = \frac{\|\mathbf{s} - s\|^2 - \|\mathbf{s} - \hat{s}\|^2}{\|\mathbf{s} - s\|^2}. \quad (51)$$

This form has $SRG_N = 1$ for $\hat{s} = \mathbf{s}$ (perfect superresolution), and $SRG_N = 0$ if $\hat{s} = s$ (no improvement). Also, for really poor superresolution processing, $\|\mathbf{s} - s\|^2 < \|\mathbf{s} - \hat{s}\|^2$, Eq. (51) gives a negative SRG_N . Finally, Parseval's theorem allows us to replace the quantities in Eq. (50) or Eq. (51) with their Fourier transforms. Equation (51) can be written in terms of Fourier transformed functions as

$$SRG_N = \frac{\|\tilde{\mathbf{s}} - \tilde{s}\|^2 - \|\tilde{\mathbf{s}} - \tilde{\hat{s}}\|^2}{\|\tilde{\mathbf{s}} - \tilde{s}\|^2}. \quad (52)$$

We can use Eq. (51), Eq. (52) or equivalently Eq. (50) as a guide in the development of tests of superresolution schemes (algorithm). These Equations tell us that we need a reference object \mathbf{s} and a bandlimited image s of \mathbf{s} . We can apply the algorithm in question to s and calculate the superresolution gain by the above equations. One approach would be to use a model for \mathbf{s} , add noise and simulate the image s of \mathbf{s} . We can then apply the algorithm to s to obtain \hat{s} . Another approach is to use a high quality SAR image to represent \mathbf{s} and bandlimit this result to represent s . In this case we need to add noise to s before applying the superresolution algorithm. In this case noise should always be added at this step because once an image is formed the noise is an inseparable part of the image. Finally one might develop a test target (perhaps a corner reflector array), for which \mathbf{s} is well defined, and obtain a SAR image s . One can then apply the superresolution algorithm and calculate the merit function. Note that noise does not need to be added to s in this case. However, one could add noise (above that which is already present) to explore the effects of increasing noise beyond that inherent in the specific imaging process. That is, when a high resolution SAR image is used as the object function, noise should be added after the image is degraded (filtered) to produce

lower resolution image that is used as a test image for superresolution studies. Finally, it should be emphasized again that noise is the determining factor in the superresolution question.

7 Discussion

We would like to start this section with an example that illustrates what is not superresolution. The example consists of taking an image, $s(x)$, and raising it to the n th power to obtain an “enhanced image,” $s^n(x)$. Generally, we do not need to require n to be an integer, greater than 1, or even positive. However, for simplicity, we will consider the case of positive n greater than 1. It is easy to see that this operation, in this case, will increase the bandwidth of the image. However, within the context of our definition, this is not superresolution. Although the bandwidth has been increased, the increased bandwidth is not related to any information contained in the part of the spectrum that falls outside the bandwidth of the system. Further, it is easy to see that this image enhancement would not generally give $SRG_N = 1$ in Eq. (51). If the image were a sinc function corresponding to a point target, this process would appear to enhance the resolution and the peak-to-sidelobe ratio. However, if the image were a poorly resolved square wave, this processing would, in the limit of large n , produce narrow spikes of the same periodicity (which is not a superresolved image). It should be clear that this processing will, in most cases, add distortions. Finally, if the image is real and positive, no information is lost if the image is raised to a power. This is not generally the case if the image is complex or nonpositive.

This example is only one of many that may be used to “enhance” the appearance of an image. In fact such enhancements may be a valuable aid to the SAR image interpreter, or even as preprocessing steps to machine interpreters (e.g. automatic target recognition algorithms), but they should not be classed as resolution (superresolution) enhancements. Another defining test for superresolution would be to investigate what the

algorithm/method produces for an object that is considerably smaller than the classical resolution of the imaging system. To be specific, assume an isolated target that has radar reflectivity represented by a rect function of width r and the system resolution is mr , where $m \gg 1$. The question is, if the superresolution algorithm is applied without limit does it reproduce the target at resolution r or does it produce an impulse that has a width much less than r . If the algorithm produces a narrow impulse in this case it lacks fidelity and would not be considered superresolution as defined in this study. Algorithms that model an object as (that is, presume an object is composed of) a collection of impulses (point targets) seemingly do just this. Extruding non-impulse scene features through image impulse models might seem somewhat presumptuous. Clearly, such a result would not give a value of unity for Eq. (51).

There are several papers that present some approach to the application of superresolution techniques to SAR imaging. These papers are difficult to address in that they frequently have one or more of the following characteristics:

- a) The algorithm is nonlinear.
- b) The author does not provide an analysis of the algorithm.
- c) The data is too limited to draw broad conclusions.
- d) A quantitative measure of performance is not included.
- e) The paper is dealing more with image enhancement than superresolution, as defined in this study.

Although one cannot prove a negative, it is generally difficult to see how an algorithm, by virtue of being nonlinear, can increase the degrees of freedom of an image. In fact, the degrees-of-freedom of an image is a property of the imaging system, not the

image. If one assumes that the number of degrees-of-freedom of an image is effectively set by the system and that this number is a conserved quantity (as suggested in Reference 27), then it is difficult to see how copious amounts of processing would increase the number of degrees-of-freedom of an image. Having said this, we don't want to throw out any algorithms out of hand. If an algorithm appears to produce significantly improved images, it should be analyzed in detail. Unfortunately for most nonlinear algorithms, analysis is a major task.

As discussed in Section 3, the superresolution problem is ill posed. Techniques for mitigating the ill posed nature of the problem are referred to as regularization techniques. Regularization is the process of modifying the original problem so that it becomes less sensitive to small perturbations of the data and, at the same time, the solution is close to that of the original problem (See Reference 24, Chapter 1). These are conflicting goals. Generally, making the trade-off between these two goals is an empirical problem. In their highly analytical paper, Joyce and Root⁵ propose linear-precision gauges as an approach to regularization. They do not address the rapid fall-off of the eigenvalues in the original problem. They also do not give any examples of the application of their iterative algorithm.

In an interesting paper, Delves, Pryde and Luttrell³⁰ give an algorithm for superresolving isolated point targets in a uniform background. Their approach is an iterative algorithm that requires a statistical estimate of $|f|^2$ where f is the target they wish to superresolve. They also assumed a 1-% additive noise level. Simulation results are given. The space-bandwidth-product associated with this problem appears to be

small. Luttrell and Oliver³¹ give a more detailed exposition of the basic concepts in a related paper.

In a short paper, Guglielmi, Castanie and Piau³² investigate the Gerchberg-Papoulis generalized inversion to achieve superresolution. They use a stochastic inversion based on prior knowledge based on a priori probability distributions. The noise level does not appear to be explicitly given. The paper gives results of one-dimensional simulations for a radar target consisting of a rect function and two point targets. In such simulations, there is always a question of how well the SAR imaging process and noise are modeled.

Stankwitz and Kosek³³ propose using spatially variant apodization (SVA) to implement superresolution processing (Super-SVA). Assuming that the image data is bandlimited, they apply a nonlinear apodization to the image plane data that is designed to eliminate sidelobes from bright targets. This, generally, produces spectral content outside the system bandlimit, which is typically appended to the original spectrum for reprocessing. The resulting data is then effectively inverse filtered to give an appearance of enhanced resolution. The process can be applied iteratively. This approach does not appear to be superresolution in the sense of this study. It might be more appropriately considered image enhancement in the sense of the example at the start of this section, which is also a nonlinear process. (We do note that SVA enjoys some measure of popularity specifically as an image enhancement tool.)

Novak, Owirka and Weaver³⁴ report significant target recognition performance using what they call enhanced resolution SAR data. The image enhancing algorithm is

described in detail by Benitz.³⁵ This work also involves selective sidelobe reduction and a point target model.

Sidney, Bowling and Cuomo³⁶ discuss superresolution methods in the context of bandwidth extrapolation that involves sidelobe removal. They also discuss extended coherent processing that is based on a rotating point target model. They give interesting results for simple targets; they state, “The results demonstrate the validity of the principles discussed earlier and they illustrate the effects that can be observed with well-defined targets that are not very complex.” For a simple reentry vehicle type target they show a bandwidth extension by a factor of six. They also state that there are three important factors that will effect the success of bandwidth extraction, they are: 1) conformity actual target reflectivity with the limited-number-of-points model, 2) systematic errors that can distort signals relative to the model assumptions, 3) the signal-to-noise ratio.

A particularly interesting paper is by DeGraaf³⁷ wherein he compares a number of superresolution techniques. He states that these techniques generally “exploit a point scattering (sinusoidal signal history) model” to various degrees. Of special note is a figure containing an array of images processed by the various techniques of a common real (in the sense of non-synthetic) data set. Differences in the “superresolved” images are clearly obvious, and discussed.

There are several other papers^{38,39,40,41,42,43} that generally fit in class of techniques discussed above. It is not practical to discuss each of them in detail within the scope of this study. In fact, the above cursory discussion of some papers is not meant to obviate the need for the reader to pursue these papers and their references further. In addition we

have included a bibliography that includes related papers not specifically cited in this study.

Summary

The concept of superresolution has been around for many years. The original concept as well as the definition of superresolution in this study dealt with the possibility of recovering Fourier transformation about the target (radar cross section) that falls outside the effective bandpass of the imaging (SAR) system. Over the years several authors have investigated the potential of achieving significant superresolution of images. Unfortunately, the answer is that significant superresolution improvement on an image (data set) is not practical. The limit is set by the high sensitivity of superresolution to noise and equivalent system errors. The result is that, if an application requires a given resolution the resolution must be obtained by an equivalent real or synthetic aperture. It can not be obtained by post image processing.

There are image processing techniques that can generally be classed as image enhancement that may be beneficial to an application such as visual image interpretation or automatic target recognition. However, they are not truly superresolution techniques as defined here . “Better looking” does not equate to “more accurately resolved.”

References

-
- ¹ J. W. Goodman, *Introduction to Fourier Optics*, McGraw-Hill, New York (1968).
 - ² H. C. Andrews and B. R. Hunt, *Digital Image Restoration*, Prentice-Hall, New Jersey (1977).
 - ³ G. T. Di Francia, "Degrees of Freedom of an Image," *Journal of The Optical Society of America*, 59 (7), 799-804 (1969).
 - ⁴ M. Bertero, C. De Mol "Super-Resolution by Data Inversion," Chapter III *Progress In Optics*, Vol. XXXVI, E. Wolf (Ed.), North-Holland, Amsterdam, 130 – 178, (1997).
 - ⁵ L. S. Joyce and W. L. Root, "Precision Bounds in Superresolution Processing," *Journal of The Optical Society of America*, 1 (2), 149-168 (1984).
 - ⁶ E. Kreyszig, *Introductory Functional Analysis With Applications*, John Wiley and Sons, New York, (1978).
 - ⁷ F. Riesz And B. Sz.-Nagy, *Functional Analysis*, Dover Publications, Inc. New York, (1990).
 - ⁸ M. Reed and B. Simon, *Methods of Modern Mathematical Physics I: Function Analysis*, Academic Press Inc., New York, (1972).
 - ⁹ J. L. Walker, "Range-Doppler Imaging of Rotating Objects," *IEEE Trans. on Aerospace and Electronic Systems*, AES-16 (1), 23-52, (1980).
 - ¹⁰ D. C. Munson, Jr., J. D. O'Brien, and W. K. Jenkins, "A Tomographic Formulation of Spotlight-Mode Synthetic Aperture Radar," *Proceedings of the IEEE*, 71 (8), 917-925, (1983).
 - ¹¹ F. M. Dickey and K. S. Shanmugam, "Optimum edge detection filter," *Applied Optics*, 16 (1), 145-148 (1977).
 - ¹² R. O. Harger, *Synthetic Aperture Radar Systems: Theory and Design*, Academic Press, New York, (1970).
 - ¹³ D. Slepian and H. O. Pollak, "Prolate Spheroidal Wave Functions, Fourier Analysis and Uncertainty – I," *The Bell System Technical Journal*, 43-63 (1961).
 - ¹⁴ H. J. Landau and H. O. Pollak, "Prolate Spheroidal Wave Functions, Fourier Analysis and Uncertainty – II," *The Bell System Technical Journal*, 65- 84 (1961).
 - ¹⁵ H. J. Landau and H. O. Pollak, "Prolate Spheroidal Wave Functions, Fourier Analysis and Uncertainty – III: The Dimension of the Space of Essentially Time and Band-Limited Signals," *The Bell System Technical Journal*, 1295- 1336 (1962).
 - ¹⁶ D. Slepian, "Prolate Spheroidal Wave Functions, Fourier Analysis and Uncertainty – IV: Extensions to Many Dimensions; Generalized Prolate Spheroidal Functions," *The Bell System Technical Journal*, 3009-3057 (1964).
 - ¹⁷ D. Slepian and E. Sonnenblick, "Eigenvalues Associated with Prolate Spheroidal Wave Functions of Zero Order," *The Bell System Technical Journal*, 1745-1759 (1965).
 - ¹⁸ D. Slepian, "Some Asymptotic Expansions for Prolate Spheroidal Wave Functions," *J. Math. And Phys.*, (44), 99-140 (1965).
 - ¹⁹ H. J. Landau and H. Widom, "Eigenvalue Distribution of Time and Frequency Limiting," *Journal of Mathematical Analysis and Applications*, 77, 469-481, (1980).
 - ²⁰ R. Frieden, "Evaluation, Design and Extrapolation Methods for Optical Signals, Based on Use of The Prolate Functions," Chapter VIII, *Progress In Optics*, Vol. IX, E. Wolf (Ed.), North-Holland, Amsterdam, 311 – 406, (1971).

-
- ²¹ G. Newsam and R. Barakat, "Essential Dimension as a Well-Defined Number of Degrees of Freedom of Finite-Convolution Operators Appearing in Optics," *Journal of The Optical Society of America*, 2 (11), 2040-2045 (1985).
- ²² M. Bendinelli, A. Consortini, and L. Ronchi, "Degrees of Freedom, and Eigenfunctions, for The Noisy Image," *Journal of The Optical Society of America*, 64 (11), 1498-1502 (1974).
- ²³ C. W. Helstrom, "Resolvable Degrees of Freedom in Observation of a Coherent Object," *Journal of The Optical Society of America*, 67 (6), 833-838 (1977).
- ²⁴ H. Stark, *Image Recovery Theory and Application*, Academic Press Inc., New York, (1987).
- ²⁵ G. H. Golub and F. Van Loan, *Matrix Computations*, The John Hopkins University Press, Baltimore, (1989).
- ²⁶ David K. Smith and Robert J. Marks II, "Closed Form Bandlimited Image Extrapolation," *Applied Optics*, 20 (14), 2476-2483 (1981).
- ²⁷ I. J. Cox and C. J. Sheppard, "Information Capacity and Resolution in an Optical System," *Journal of The Optical Society of America*, 3 (8), 1152-1158 (1986).
- ²⁸ M. A. Fiddy and T. J. Hall, "Nonuniqueness of Superresolution Techniques Applied to Sampled Data," *Journal of The Optical Society of America*, 71 (11), 1406-1407 (1981).
- ²⁹ C. A. Davila and B. R. Hunt, "Training of a Neural Network for Image Superresolution Based on a Non-Linear Interpolative Vector Quantizer," *Applied Optics*, 39 (20), 3473-3485 (2000).
- ³⁰ L. M. Delves, G. C. Pryde, and S. P. Luttrell, "A Superresolution Algorithm for SAR Images," IOP Publishing Ltd., 681-703 (1988).
- ³¹ S. P. Luttrell and C. J. Oliver, "Prior Knowledge in Synthetic-Aperture Radar Processing," *J. Phys. D: Applied Phys.*, 19, 333-356 (1986).
- ³² V. Guglielmi, F. Castanie, and P. Piau, "Application of Superresolution methods to Synthetic Aperture Radar data," *Proc. IGARSS 3*, 2289-2291 (1995).
- ³³ H.C. Stankwitz and M.R. Kosek, "Super-Resolution for SAR/ISAR RCS Measurement Using Spatially Variant Apodization," *Proceedings of the Antenna Measurement Techniques Association (AMTA) 17th Annual Meeting and Symposium*, Williamsburg, VA, 13-17 November 1995.
- ³⁴ L.M. Novak, G. J. Owirka, and A. L. Weaver, "Automatic Target Recognition Using Enhanced Resolution SAR Data," *Proc. IEEE*, 157-175 (1999).
- ³⁵ Gerald R. Benitz, "High-Definition Vector Imaging," *Lincoln Laboratory Journal*, 10 (2), 147-170 (1997).
- ³⁶ S. L. Borison, S. B. Bowling, and K. M. Cutomo, "Superresolution Methods for Wideband Radar," *The Lincoln Laboratory Journal*, 5 (3), 441-461 (1992).
- ³⁷ S. R. DeGraaf, "SAR Imaging via Spectral Estimation Methods," *Conference Record of The Twenty-Eighth Asilomar Conference on Signals, Systems & Computers*, Pacific Grove, California, (1994).
- ³⁸ W. F. Gabriel, "Superresolution Techniques and ISAR Imaging," *Proc. IEEE*, 48-55 (1989).
- ³⁹ W. F. Gabriel, "Superresolution Techniques in The Range Domain," *Proc. IEEE*, 263-267 (1990).
- ⁴⁰ T. J. Abatzoglou, "Superresolution signal processing and its application," *Proc. SPIE* 2562, 88-98 (1995).

⁴¹ E. K. Walton, A. Moghaddar, and C. DeMattio, "Superresolution Radar Target Imaging," Proc. Symposium Antenna Measurement Techniques Association, 12/3-12/6 (1991).

⁴² Alenjandro E. Brito, Shiu Hung Chan, and Sergio D. Cabrera, "Application of a WNE Superresolution to MSTAR SAR Images," Proc. IASTED International Conference, (1998).

⁴³ Xu Xiaojian and Huang Peikang, "Superresolution Techniques with Applications to Microwave Imaging," International Conference Radar, 485-487 (1992).

Bibliography

General

D. Slepian and H. O. Pollak, "Prolate Spheroidal Wave Functions, Fourier Analysis and Uncertainty – I," The Bell System Technical Journal, 43-63 (1961).

H. J. Landau and H. O. Pollak, "Prolate Spheroidal Wave Functions, Fourier Analysis and Uncertainty – II," The Bell System Technical Journal, 65- 84 (1961).

H. J. Landau and H. O. Pollak, "Prolate Spheroidal Wave Functions, Fourier Analysis and Uncertainty – III: The Dimension of the Space of Essentially Time and Band-Limited Signals," The Bell System Technical Journal, 1295- 1336 (1962).

David Slepian, "Prolate Spheroidal Wave Functions, Fourier Analysis and Uncertainty – IV: Extensions to Many Dimensions; Generalized Prolate Spheroidal Functions," The Bell System Technical Journal, 3009-3057 (1964).

David Slepian and Estelle Sonnenblick, "Eigenvalues Associated with Prolate Spheroidal Wave Functions of Zero Order," The Bell System Technical Journal, 1745-1759 (1965).

David Slepian, "Some Asymptotic Expansions for Prolate Spheroidal Wave Functions," J. Math. And Phys., (44), 99-140 (1965).

G. Toraldo Di Francia, "Degrees of Freedom of an Image," Journal of The Optical Society of America, 59 (7), 799-804 (1969).

G. Toraldo Di Francia, "Optics as Scattering," Physics Today, 32-38 (1973).

M. Bendinelli, A. Consortini, and L. Ronchi, "Degrees of Freedom, and Eigenfunctions, for The Noisy Image," Journal of The Optical Society of America, 64 (11), 1498-1502 (1974).

F. Gori, "Integral Equations for Incoherent Imagery," Journal of The Optical Society of America, 64 (9), 1237-1243 (1974).

G. R. Boyer, "Pupil Filters for Moderate Super-Resolution," Applied Optics, 15 (12), 3089-3093 (1976).

Carl W. Helstrom, "Resolvable Degrees of Freedom in Observation of a Coherent Object," Journal of The Optical Society of America, 67 (6), 833-838 (1977).

Yakov I. Khurgin and Vitaly P. Yakovlev, "Progress in The Soviet Union on The Theory and Applications of Bandlimited Functions," Proc. of IEEE, 65 (7), 1005-1029, (1977).

C. K. Rushforth and R. L. Frost, "Comparison of Some Algorithms for Reconstructing Space-Limited Images," *Journal of The Optical Society of America*, 70(12), 1539-1544 (1980).

H. J. Landau and H. Widom, "Eigenvalue Distribution of Time and Frequency Limiting," *Journal of Mathematical Analysis and Applications*, 77, 469-481, (1980).

M. A. Fiddy and T. J. Hall, "Nonuniqueness of Superresolution Techniques Applied to Sampled Data," *Journal of The Optical Society of America*, 71 (11), 1406-1407 (1981).

David K. Smith and Robert J. Marks II, "Closed Form Bandlimited Image Extrapolation," *Applied Optics*, 20 (14), 2476-2483 (1981).

M. Bertero and E. R. Pike, "Resolution in Diffraction-Limited Imaging, A Singular Value Analysis: I. The Case of Coherent Illumination," *Optica Acta*, 29 (6), 727-746 (1982).

M. Bertero, P. Boccacci, and E. R. Pike, "Resolution in Diffraction-Limited Imaging, A Singular Value Analysis: II. The Case of Coherent Illumination," *Optica Acta*, 29 (12), 1599-1611 (1982).

Charles L. Byrne, Raymond M. Fitzgerald, Michael A. Fiddy, Trevor J. Hall, and Angela M. Darling, "Image Restoration and Resolution Enhancement," *Journal of The Optical Society of America*, 73 (11), 1481-1487 (1983).

Lawrence S. Joyce and William L. Root, "Precision Bounds in Superresolution Processing," *Journal of The Optical Society of America*, 1 (2), 149-168 (1984).

Garry Newsam and Richard Barakat, "Essential Dimension as a Well-Defined Number of Degrees of Freedom of Finite-Convolution Operators Appearing in Optics," *Journal of The Optical Society of America*, 2 (11), 2040-2045 (1985).

R. Barakat and G. Newsam, "Algorithms for reconstruction of partially known, band-limited Fourier-transform pairs from noisy data," *Journal of The Optical Society of America A*, 2 (11), 2027-2039 (1985).

I. J. Cox and C. J. Sheppard, "Information Capacity and Resolution in an Optical System," *Journal of The Optical Society of America*, 3 (8), 1152-1158 (1986).

P. J. Sementilli, B. R. Hunt, and M. S. Nadar, "Analysis of The Limit to Superresolution in Incoherent Imaging," *Journal of The Optical Society of America*, 10 (11), 2265-2276 (1993).

David O. Walsh and Pamela A. Nielsen-Delaney, "Direct Method for Superresolution," *Journal of The Optical Society of America*, 11 (2), 572-579 (1994).

Carlos A. Davila and B. R. Hunt, "Training of a Neural Network for Image Superresolution Based on a Non-Linear Interpolative Vector Quantizer," *Applied Optics*, 39 (20), 3473-3485 (2000).

Synthetic Aperture Radar

J. L. Walker, "Range-Doppler Imaging of Rotating Objects," *IEEE Trans. on Aerospace and Electronic Systems*, AES-16 (1), 23-52, (1980).

D. C. Munson, Jr., J. D. O'Brien, and W. K. Jenkins, "A Tomographic Formulation of Spotlight-Mode Synthetic Aperture Radar," *Proceedings of the IEEE*, 71 (8), 917-925, (1983).

S. P. Luttrell and C. J. Oliver, "Prior Knowledge in Synthetic-Aperture Radar Processing," *J. Phys. D: Applied Phys.*, 19, 333-356 (1986).

U. Nickel, Dipl-Math, and Dr rer nat, "Angular Superresolution With Phased Array Radar: A View of Algorithms and Operational Constraints," *Proc. IEE*, 134 (1), 53-59 (1987).

L. M. Delves, G. C. Pryde, and S. P. Luttrell, "A Superresolution Algorithm for SAR Images," *IOP Publishing Ltd.*, 681-703 (1988).

W. F. Gabriel, "Superresolution Techniques and ISAR Imaging," *Proc. IEEE*, 48-55 (1989).

U. Nickel, "Applications of Superresolution Methods for Airborne Radar," *Proc. AGARD conference on High Resolution Air and Spaceborne Radar Held in The Hague*, 7-1 – 7-6 (1989).

W. F. Gabriel, "Superresolution Techniques in The Range Domain," *Proc. IEEE*, 263-267 (1990).

G. C. Pryde, "A Transputer Array Implementation of a Fast Algorithm for The Superresolution of Synthetic Aperture Radar Images," in *Applications of Transputers*, *Proc. International Conference*, 271-275 (1990).

Bradley W. Deats, David J. Farina, and Jeffrey F. Bull, "Superresolution Signal Processing Aids RCS Testing," *Microwaves & RF*, 74-82 (1991).

E. K. Walton, A. Moghaddar, and C. DeMattio, "Superresolution Radar Target Imaging," *Proc. Symposium Antenna Measurement Techniques Association*, 12/3-12/6 (1991).

Sindney L. Borison, Stephen B. Bowling, and Kevin M. Cutomo, "Superresolution Methods for Wideband Radar," *The Lincoln Laboratory Journal*, 5 (3), 441-461 (1992).

Xu Xiaojian and Huang Peikang, "Superresolution Techniques with Applications to Microwave Imaging," International Conference Radar, 485-487 (1992).

Xiao Baojia, Zhou Lixing, Lu Shu, and Sun Zhenyu, "The Application of MESE to SAR imaging," Proc. ISAE, 432-435 (1993).

W. F. Gabriel, "Improved Range Superresolution Via Bandwidth Extrapolation," Proc. IEEE Nat. Radar Conference, 123-127 (1993).

Sergio D. Cabrera, "Assessment of Superresolution in Synthetic Aperture Radar Imaging," from Univ. of Texas at El Paso, Pages 1-21, Appendix 1-3, Figures 4.1-4.11, 5.1-5.16, 6.1-6.26 (1994).

Stuart R. DeGraaf, "SAR Imaging via Spectral Estimation Methods," Conference Record of The Twenty-Eighth Asilomar Conference on Signals, Systems & Computers, Pacific Grove, California, (1994).

Theagenis J. Abatzoglou, "Superresolution signal processing and its application," Proc. SPIE 2562, 88-98 (1995).

V. Guglielmi, F. Castanie, and P. Piau, "Application of Superresolution methods to Synthetic Aperture Radar data," Proc. IGARSS 3, 2289-2291 (1995).

G. Mesnager, F. Castanie, and C. Lambert-Nebout, "Spatial Resolution Improvement Using a SPECAN Approach in SAR Data Processing," Proc. IGARSS 3, 2295-2297, (1995).

H.C. Stankwitz and M.R. Kosek, "Super-Resolution for SAR/ISAR RCS Measurement Using Spatially Variant Apodization," Proceedings of the Antenna Measurement Techniques Association (AMTA) 17th Annual Meeting and Symposium, Williamsburg, VA, 13-17 November 1995.

Firooz Sadjadi, "Enhancing Angular Resolution in Non-Coherent Radar Imagery," Proc. IEEE and Publishing House of Electronics Industry, 830-833 (1996).

Gerald R. Benitz, "High-Definition Vector Imaging," Lincoln Laboratory Journal, 10 (2), 147-170 (1997).

J. J. Green and B. R. Hunt, "Superresolution in a Synthetic Aperture Imaging System," Proc. IEEE, 865-868 (1997).

V. Guglielmi, F. Castanie, and P. Piau, "Superresolution algorithms for SAR images," Proc. SPIE 3170, 195-202 (1997).

Alenjandro E. Brito, Shiu Hung Chan, and Sergio D. Cabrera, "Application of a WNE Superresolution to MSTAR SAR Images," Proc. IASTED International Conference, (1998).

Frank M. Candocia and Jose C. Principe, "A Method Using Multiple Models to Super-Resolve SAR Imagery," Proc. SPIE 3370, 197-207 (1998).

R. K. Mehra and B. Ravichandran, "Superresolution Techniques for Active Remote Sensing," The American Institute of Physics, 15-20 (1998).

L.M. Novak, G. J. Owirka, and A. L. Weaver, "Automatic Target Recognition Using Enhanced Resolution SAR Data," Proc. IEEE, 157-175 (1999).

Ancillary

William F. Gabriel, "Tracking Closely Spaced Multiple Sources Via Spectral-Estimation Techniques," Naval Research Laboratory, 1-17 (1982).

T. J. Abatzoglou and L. K. Lam, "Radar Super Range Resolution and Bragg Cell Interferometry," Proc. SPIE, (1348), 471-483 (1990).

M. Boifot, "Improved Resolution by Utilising an Artificial Transmission Medium," Proc. IEE, 137 (5), 371-376 (1990).

Jenho Tsao, "High Resolution Radar Imaging With Small Subarrays," Antennas & Propagation Intl. Symp., (2), 753-757 (1990).

Z. D. Zhu, Z. R. Ye, J. Yin, and Z. S. She, "Studies of Superresolution Range-Doppler Imaging," Intl. Conf. On Radar, 541-544 (1992).

Benjamin C. Flores, Joyce Chen, and Ying Yiang, "An Overview of High-Resolution Spectral Estimators for Range-Doppler Imaging," Antennas & Propagation Society Int. Symp., (3), 1902-1905 (1993).

Gregory P. Otto and Weng Cho Chew, " Microwave Inverse Scattering – Local Shape Function Imaging for Improved Resolution of Strong Scatterers," Proc. IEEE Transactions on Microwave Theory and Techniques, 42 (1), 137-141 (1994).

Alejandro E. Brito, Shiu H. Chan, and Sergio D. Cabrera, "SAR Image Formation Using 2-D Re-Weighted Minimum Norm Extrapolation," Proc. SPIE – The International Society for Optical Engineering, (3721), 79-91 (1999).

Hiroyuki Ichikawa, "Temporal Superresolution: An Application of Frequency Filtering by a Grating in The Resonance Domain," Journal of Modern Optics, 47 (13), 2361-2375 (2000).

Eran Sabo, Zeev Zalevsky, David Mendlovic, Naim Konforti, and Irena Kiryushev, "Superresolution Optical System With Two Fixed Generalized Damman Gratings," *Applied Optics*, 39 (29), 5318-5325 (2000).

Books

J. W. Goodman, *Introduction to Fourier Optics*, McGraw-Hill, New York (1968).

G. A. Korn and T. M. Korn, *Mathematical Handbook for Scientists and Engineers*, McGraw-Hill, New York (1968).

R. Frieden, "Evaluation, Design and Extrapolation Methods for Optical Signals, Based on Use of The Prolate Functions," Chapter VIII, *Progress In Optics, Vol. IX*, E. Wolf (Ed.), North-Holland, Amsterdam, , 311 – 406, (1971).

M. Reed and B. Simon, *Methods of Modern Mathematical Physics I: Function Analysis*, Academic Press Inc., New York, (1972).

H. C. Andrews and B. R. Hunt, *Digital Image Restoration*, Prentice-Hall, New Jersey (1977).

R. F. Curtain and A. J. Pritchard, *Functional Analysis in Modern Applied Mathematics*, Academic Press Inc., New York, (1977).

E. Kreyszig, *Introductory Functional Analysis With Applications*, John Wiley and Sons, New York, (1978).

Henry Stark, *Image Recovery Theory and Application*, Academic Press Inc., New York, (1987).

G. H. Golub and F. Van Loan, *Matrix Computations*, The John Hopkins University Press, Baltimore, (1989).

F. Riesz And B. Sz.-Nagy, *Functional Analysis*, Dover Publications, Inc. New York, (1990).

M. Bertero, C. De Mol "Super-Resolution by Data Inversion," Chapter III *Progress In Optics, Vol. XXXVI*, E. Wolf (Ed.), North-Holland, Amsterdam, 130 – 178, (1997).

M. Bertero and P. Boccacci, *Introduction to Inverse Problems in Imaging*, IOP Publishing, Bristol, (1998).

DISTRIBUTION

Unlimited Distribution

1	MS 0509	M. W. Callahan	2300
1	MS 0529	B. C. Walker	2308
1	MS 0529	A. W. Doerry	2345
1	MS 0529	D. F. Dubbert	2345
1	MS 0529	S. S. Kawka	2345
1	MS 0529	M. S. Murray	2345
1	MS 0529	G. R. Sloan	2345
1	MS 0519	B. L. Remund	2348
1	MS 0519	S. D. Bensonhaver	2348
1	MS 0519	T. P. Bielek	2348
1	MS 0519	B. L. Burns	2348
1	MS 0519	S. M. Devonshire	2348
1	MS 0519	J. A. Hollowell	2348
1	MS 0537	R. M. Axline	2344
1	MS 0537	D. L. Bickel	2344
1	MS 0537	J. T. Cordaro	2344
1	MS 0537	W. H. Hensley	2344
1	MS 0843	D. D. Boozer	15423
1	MS 1207	C. V. Jakowatz, Jr.	5912
1	MS 1207	N. E. Doren	5912
1	MS 1207	P. H. Eichel	5912
1	MS 1207	I. A. Erteza	5912
1	MS 0161	G. H. Libman	11500
1	MS 0328	J. A. Wilder Jr.	2612
1	MS 0328	F. M. Dickey	2612
1	MS 1110	D. E. Womble	9214
1	MS 1110	L. A. Romero	9214
1	MS 1100	J. M. Delaurentis	9214
1	MS 9018	Central Technical Files	8945-1
2	MS 0899	Technical Library	4916
1	MS 0612	Review & Approval Desk for DOE/OSTI	4912
1		Randy Bell	DOE NN-20