

# **SANDIA REPORT**

SAND2000-2217-Revised  
Unlimited Release  
Printed April 2001

Supersedes SAND2000-2217  
Dated September 2000

## **Using Vector Spherical Harmonics to Compute Antenna Mutual Impedance from Measured or Computed Fields**

Billy C. Brock

Prepared by  
Sandia National Laboratories  
Albuquerque, New Mexico 87185 and Livermore, California 94550

Sandia is a multiprogram laboratory operated by Sandia Corporation,  
a Lockheed Martin Company, for the United States Department of  
Energy under Contract DE-AC04-94AL85000.

Approved for public release; further dissemination unlimited.



**Sandia National Laboratories**

Issued by Sandia National Laboratories, operated for the United States Department of Energy by Sandia Corporation.

**NOTICE:** This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government, nor any agency thereof, nor any of their employees, nor any of their contractors, subcontractors, or their employees, make any warranty, express or implied, or assume any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represent that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government, any agency thereof, or any of their contractors or subcontractors. The views and opinions expressed herein do not necessarily state or reflect those of the United States Government, any agency thereof, or any of their contractors.

Printed in the United States of America. This report has been reproduced directly from the best available copy.

Available to DOE and DOE contractors from  
U.S. Department of Energy  
Office of Scientific and Technical Information  
P.O. Box 62  
Oak Ridge, TN 37831

Telephone: (865)576-8401  
Facsimile: (865)576-5728  
E-Mail: [reports@adonis.osti.gov](mailto:reports@adonis.osti.gov)  
Online ordering: <http://www.doe.gov/bridge>

Available to the public from  
U.S. Department of Commerce  
National Technical Information Service  
5285 Port Royal Rd  
Springfield, VA 22161

Telephone: (800)553-6847  
Facsimile: (703)605-6900  
E-Mail: [orders@ntis.fedworld.gov](mailto:orders@ntis.fedworld.gov)  
Online order: <http://www.ntis.gov/ordering.htm>



SAND2000-2217-Revised  
Unlimited Release  
Printed April 2001

Supercedes SAND2000-2217  
Dated September 2000

# Using Vector Spherical Harmonics to Compute Antenna Mutual Impedance from Measured or Computed Fields

Billy C. Brock  
Radar/Antenna Department  
Sandia National Laboratories  
P. O. Box 5800  
Albuquerque, NM 87123-0533

## Abstract

The mutual coupling that exists between the antenna elements in an antenna array can be described with a mutual impedance. The knowledge of this mutual impedance is critical to the successful design of the array. Computing the mutual impedance involves integrating vector products of fields over a surface, but the integrands can oscillate wildly over the integration surface, and are often difficult to integrate accurately. The method proposed and described here relies on the expansion of the fields in terms of vector spherical harmonics. The integrations over the closed surface are performed in closed form, leaving the mutual impedance expressed as a sum of products of expansion coefficients.

### **About this Revision**

This revision came about because of the investigation of a typographical error in one of the references for Appendix III. This typographical error had propagated into the original version of this report, in equation (III-9), which has been corrected in this revision. During the course of the investigation, it was discovered that the method of computing the Wigner  $3j$  symbols, originally given in Appendix III, was inadequate for large degree and order. A detailed description of a robust recursive algorithm is now included. Consequently, Appendix III is much expanded. Also, in the section on mutual impedance, a factor of  $r'^2$  was inadvertently excluded from some of the expressions. This has been corrected in this revision. Further scrutiny of the formulation of the mutual impedance has revealed significant simplification that has now been incorporated, including additional material in Appendix IV. Additionally, some of the text in the main body of the report has been moved to Appendix IV, and other parts have been revised in an effort to improve the clarity. Thus, the discovery of a typographical error in a reference, while somewhat minor in itself, provided an opportunity to further examine the mutual impedance calculation, leading to a substantial revision of this document.

## Contents

Introduction.....	5
Lorentz reciprocity theorem .....	6
Fields associated with different media .....	11
Reaction and mutual impedance.....	12
Implementation of the computation of the mutual impedance .....	14
Expansion of the antenna's field in vector spherical harmonics .....	14
Translation of the fields.....	17
Mutual impedance .....	20
Mutual impedance from radiation pattern measurement .....	23
Appendix I — Vector Spherical Harmonics.....	27
General vector harmonics.....	27
Vector harmonics in the spherical coordinate system .....	28
Special combinations of the vector spherical harmonics.....	33
Explicit forms for the associated Legendre function and its derivative .....	35
Explicit forms for the scalar spherical harmonic and its derivative .....	37
Explicit expressions for the $\bar{\mathbf{X}}_{n,m}(\theta, \phi)$ vector spherical harmonic:.....	38
Explicit expressions for the $\hat{\mathbf{r}} \times \bar{\mathbf{X}}_{n,m}(\theta, \phi)$ vector spherical harmonic:.....	39
Explicit expressions for the $\bar{\mathbf{M}}_{n,m}^{(i)}(\bar{\mathbf{r}})$ vector spherical harmonics:.....	40
Explicit expressions for the $\bar{\mathbf{N}}_{n,m}^{(i)}(\bar{\mathbf{r}})$ vector spherical harmonics:.....	41
Appendix II — Commutation Relations.....	43
Appendix III Addition Theorem for Vector Spherical Harmonics.....	45
Application to translation of electromagnetic field .....	47
Computing the Wigner 3j symbol and Clebsch-Gordon coefficient .....	48
Special values of the Wigner 3j symbol .....	49
Computing arbitrary values .....	50
Stable recursive algorithm .....	51
Case 1.....	54
Case 2.....	55
Case 3.....	55
Example families of Wigner 3j symbols .....	56
Appendix IV — Certain Integrals Containing Associated Legendre Functions .....	63
Establishing Orthogonality .....	65
Application of the differential equation.....	65
Relationship between the various integrals .....	66
Application of recursion relation .....	67
Evaluation of $\mathcal{I}_0(n, n', m) = \int_{-1}^1 P_n^m(x) P_{n'}^m(x) dx$ .....	67
Evaluation of $\mathcal{I}_3(n, n', m) = \int_{-1}^1 \frac{1}{1-x^2} P_n^m(x) P_{n'}^m(x) dx$ .....	71
Evaluation of $\mathcal{I}_1(n, n', m) = \int_{-1}^1 (1-x^2) \frac{d}{dx} P_n^m(x) \frac{d}{dx} P_{n'}^m(x) dx$ .....	73

Evaluation of $\mathcal{A}_2(n, n', m) = \int_{-1}^1 P_n^m(x) \frac{d}{dx} P_{n'}^m(x) dx$ .....	73
Evaluation of $I_1(n, n', m) = \int_{-1}^1 (1-x^2) \frac{d}{dx} P_n^m(x) \frac{d}{dx} P_{n'}^{-m}(x) dx$ .....	75
Evaluation of $I_2(n, n', m) = m \int_{-1}^1 P_n^m(x) \frac{d}{dx} P_{n'}^{-m}(x) dx$ .....	75
Evaluation of $I_3(n, n', m) = -m^2 \int_{-1}^1 \frac{1}{1-x^2} P_n^m(x) P_{n'}^{-m}(x) dx$ .....	76
Evaluation of $\mathcal{A}_4(n, n', m, m') = \int_{-1}^1 \left( P_n^m(x) \frac{d}{dx} P_{n'}^{m'}(x) + P_{n'}^{m'}(x) \frac{d}{dx} P_n^m(x) \right) dx$ .....	76
Evaluation of $m \left( \int_{-1}^1 P_n^m(x) \frac{d}{dx} P_{n'}^{-m}(x) dx + \int_{-1}^1 P_{n'}^{-m}(x) \frac{d}{dx} P_n^m(x) dx \right)$ .....	77
Evaluation of $\mathcal{A}_{\bar{\mathbf{x}} \times \bar{\mathbf{x}}}(n, m, n', m') = \int_0^{2\pi} d\phi' \int_0^\pi \bar{\mathbf{X}}_{n', m'}(\theta', \phi') \cdot \bar{\mathbf{X}}_{n, m}(\theta', \phi') \sin \theta' d\theta'$ .....	78
Evaluation of $\mathcal{A}_{\hat{\mathbf{r}} \times \bar{\mathbf{x}}}(n, m, n', m') = \int_0^{2\pi} d\phi' \int_0^\pi \hat{\mathbf{r}}' \times \bar{\mathbf{X}}_{n', m'}(\theta', \phi') \cdot \bar{\mathbf{X}}_{n, m}(\theta', \phi') \sin \theta' d\theta'$ .....	78
Appendix V — Relevant Network Parameters .....	81
The relationship between the $S$ parameters and the admittance matrix .....	81
The relationship between the $S$ parameters and the impedance matrix .....	82
Appendix VI — Alternate Evaluation of Impedance Integrals .....	85
References .....	87

## Introduction

A new method of computing the mutual impedance between two antennas is proposed and described in detail. The method utilizes an expansion of the vector fields in terms of the spherical vector harmonics.

The mutual impedance is a circuit-theory quantity associated with a network. It relates the current,  $i_k$ , flowing into one port of the network to the open-circuit voltage,  $v_j$ , at another port. The definition is

$$z_{jk} = \left. \frac{v_j}{i_k} \right|_{i_m=0; m=1,2,\dots, m \neq k},$$

where  $j$ ,  $k$ , and  $m$  are indices which designate the various ports of the network. Alternatively, a mutual admittance can be defined,

$$y_{jk} = \left. \frac{i_j}{v_k} \right|_{v_m=0; m=1,2,\dots, m \neq k},$$

which relates the voltage,  $v_k$ , applied across one port to the short-circuit current,  $i_j$ , at another port. Mutual impedance (or admittance) exists between the antenna elements in an antenna array, and knowledge of this mutual impedance (or admittance) is critical to the successful design of the array. Because of the mutual coupling, the reflection coefficient looking into an element is different when it is embedded in an array with all the elements excited, compared to that for the isolated element. Thus, in order to tune the elements properly for minimum reflection in the active array, the mutual impedance (or admittance) is needed.

Obviously, one could build an array of antenna elements, and then measure the coupling between the elements. From this measurement, usually in the form of a scattering matrix, the mutual impedance (or admittance) is easily determined (see Appendix V). However, this is not very practical in many situations, when the number of elements is large.

As described below, the mutual impedance can be computed if the electric and magnetic fields for the elements are known. Often, especially when the array is composed of small, identical elements, it is practical to measure or compute the fields for the element. Ideally, these fields should be measured or computed in the array environment, with all other elements terminated in matched impedances, but not excited. However, this is not always practical, and, sometimes, useful information can be obtained with the element fields measured or computed in an environment where it is the only element present.

For most applications, the mutual impedance is needed for many different relative positions of the two antenna elements. When numerical methods, such as method of moments, finite-difference time-domain methods, and finite-element methods are used, it can be very time consuming to compute the fields at each new relative position of the antennas. Computer memory requirements also limit the practicality of these methods for large arrays. Thus, a field representation that allows easy translation to new positions

would be valuable for improving the efficiency of the computation as the relative position is iterated.

Computing the mutual impedance involves integrating vector products of fields over a surface. The integrands can oscillate wildly over the integration surface, and are often difficult to integrate accurately. The method described here relies on the expansion of the fields in terms of vector spherical harmonics. The integrations over the closed surface are performed in closed form, leaving the mutual impedance expressed as a sum of products of expansion coefficients.

The mutual impedance is described in terms of a physical observable called the reaction, introduced by Rumsey. The reaction theorem is related to the reciprocity theorem derived by Lorentz, and an understanding of the reciprocity theorem is helpful for understanding the reaction concept. In the following, the Lorentz reciprocity theorem is derived and examined in some detail. In order to understand better the generality and applicability of the theorem, it is derived in a very general form. The medium in which the elements are embedded is assumed linear and time-invariant, but not homogeneous or even isotropic. The validity of the theorem does place constraints on the medium, and these constraints will be stated.

After the discussion of the reciprocity theorem, an expression for the mutual impedance in terms of the reaction quantity is written. The reaction is a term contained in the mathematical statement of the reciprocity theorem, and thus the discussion of the reciprocity theorem is very relevant to understanding the mutual impedance.

Once the expression for mutual impedance is written, the fields are expanded in vector spherical harmonics, and the mutual impedance is ultimately written in terms of the expansion coefficients. This process is somewhat tedious and is described in detail. Fortunately, the final expression is simple, and it is straightforward to program a computer to perform the computation. The first advantage, of course, is the avoidance of the need to integrate a wildly oscillating integrand that is slow to converge. The second advantage is that mutual impedance can be computed for many sets of element positions, using a *single* measurement or computation of the fields around an element. However, a disadvantage of this method is the loss of numerical significance that occurs for small element spacing. This results, in part, from the need for a large number of harmonics to describe the translated field. Further study is needed to adequately assess the method.

## **Lorentz reciprocity theorem**

The reciprocity theorem derived by Lorentz [1, 2, 3] leads to a reaction concept [3, 4,5] that is useful for understanding and computing mutual coupling between two antennas, or more generally, between two sets of source currents. The reciprocity theorem is discussed below, but a particularly entertaining discussion of the reciprocity theorem is contained in Weeks [6]. The reaction quantity, which corresponds to terms contained in the statement of the reciprocity theorem, was introduced by Rumsey [5]. The reaction quantity is a physical observable associated with the reaction between the fields of two

sources. In an electrostatic system, Rumsey's reaction corresponds to the force exerted by one source of charge on another. He shows that, for monochromatic electromagnetic fields, the reaction is the difference between the instantaneous and average rates (over one period) at which one source performs work against the other.

Following [1], the reciprocity theorem will be developed in a general form. It is important to realize that the theorem is obtained by simply applying certain mathematical operations to fields associated with two independent sets of sources. The fields are required to satisfy Maxwell's equations, but they are not required to be related to each other, or even to exist at the same time. However, they are required to be associated with the same region of space. In addition, one would expect the validity of the theorem to require that the media associated with each set of fields be the same. While this is true in the isotropic case, it will be shown below that a more general relationship between the two media must hold, and for certain anisotropic media, the two media will not be the same.

The fields will be assumed time-harmonic (monochromatic) with dependence  $e^{j\omega t}$ . An electric current,  $\bar{\mathbf{J}}$ , is the usual true physical source for the fields. A fictitious magnetic current,  $\bar{\mathbf{M}}$ , will also be included, because of its convenience in handling the equivalent sources often associated with the tangential electric-field of apertures in conducting surfaces. The medium of interest will be assumed linear and time-invariant. However, it will not be assumed homogeneous or even isotropic. Thus, the medium will be characterized by dyadic electric permittivity and magnetic permeability, which are not necessarily symmetric,

$$\bar{\bar{\boldsymbol{\epsilon}}} \neq \bar{\bar{\boldsymbol{\epsilon}}}^\dagger, \quad (1.1)$$

$$\bar{\bar{\boldsymbol{\mu}}} \neq \bar{\bar{\boldsymbol{\mu}}}^\dagger, \quad (1.2)$$

where  $^\dagger$  indicates the transposed dyadic. The electric displacement field and magnetic flux density are

$$\bar{\mathbf{D}} = \bar{\bar{\boldsymbol{\epsilon}}} \cdot \bar{\mathbf{E}}, \quad (1.3)$$

and

$$\bar{\mathbf{B}} = \bar{\bar{\boldsymbol{\mu}}} \cdot \bar{\mathbf{H}}. \quad (1.4)$$

Suppose there exist two sets of independent sources,  $(\bar{\mathbf{J}}_1, \bar{\mathbf{M}}_1)$ , and  $(\bar{\mathbf{J}}_2, \bar{\mathbf{M}}_2)$ . The first set of sources is associated with the medium  $(\bar{\bar{\boldsymbol{\epsilon}}}, \bar{\bar{\boldsymbol{\mu}}})$ , and the second source is associated with the "transposed" medium  $(\bar{\bar{\boldsymbol{\epsilon}}}^\dagger, \bar{\bar{\boldsymbol{\mu}}}^\dagger)$ . At least when the medium is symmetric ( $\bar{\bar{\boldsymbol{\epsilon}}}^\dagger = \bar{\bar{\boldsymbol{\epsilon}}}$ ,  $\bar{\bar{\boldsymbol{\mu}}}^\dagger = \bar{\bar{\boldsymbol{\mu}}}$ ), it is natural to ask whether the two sets of sources are present at the same time. The theorem to be developed will be valid regardless of whether the sources are present at the same time. The phasor fields associated with each source satisfy

$$\nabla \times \bar{\mathbf{E}}_1 = -j\omega \bar{\bar{\boldsymbol{\mu}}} \cdot \bar{\mathbf{H}}_1 - \bar{\mathbf{M}}_1 \quad (1.5)$$

$$\nabla \times \bar{\mathbf{H}}_1 = j\omega \bar{\bar{\boldsymbol{\epsilon}}} \cdot \bar{\mathbf{E}}_1 + \bar{\mathbf{J}}_1 \quad (1.6)$$

and

$$\nabla \times \bar{\mathbf{E}}_2 = -j\omega\bar{\boldsymbol{\mu}}^\dagger \cdot \bar{\mathbf{H}}_2 - \bar{\mathbf{M}}_2 \quad (1.7)$$

$$\nabla \times \bar{\mathbf{H}}_2 = j\omega\bar{\boldsymbol{\epsilon}}^\dagger \cdot \bar{\mathbf{E}}_2 + \bar{\mathbf{J}}_2. \quad (1.8)$$

The reciprocity theorem is obtained by combining vector products of the fields and applying vector identities, with the fields subject to (1.5) through (1.8). Thus, the theorem begins as simply a mathematical relationship that is imposed because the fields are solutions of Maxwell's equations. We begin by forming the difference between the cross product between the electric field of the first source with the magnetic field of the second source and the cross product of the remaining electric and magnetic fields. The divergence of this difference is

$$\nabla \cdot (\bar{\mathbf{E}}_1 \times \bar{\mathbf{H}}_2 - \bar{\mathbf{E}}_2 \times \bar{\mathbf{H}}_1) = \bar{\mathbf{H}}_2 \cdot (\nabla \times \bar{\mathbf{E}}_1) - \bar{\mathbf{E}}_1 \cdot (\nabla \times \bar{\mathbf{H}}_2) - \bar{\mathbf{H}}_1 \cdot (\nabla \times \bar{\mathbf{E}}_2) + \bar{\mathbf{E}}_2 \cdot (\nabla \times \bar{\mathbf{H}}_1). \quad (1.9)$$

Now, substitute (1.5) through (1.8) for the curl of the fields

$$\begin{aligned} \nabla \cdot (\bar{\mathbf{E}}_1 \times \bar{\mathbf{H}}_2 - \bar{\mathbf{E}}_2 \times \bar{\mathbf{H}}_1) &= -j\omega(\bar{\mathbf{H}}_2 \cdot \bar{\boldsymbol{\mu}} \cdot \bar{\mathbf{H}}_1 - \bar{\mathbf{H}}_1 \cdot \bar{\boldsymbol{\mu}}^\dagger \cdot \bar{\mathbf{H}}_2) \\ &\quad + j\omega(\bar{\mathbf{E}}_2 \cdot \bar{\boldsymbol{\epsilon}} \cdot \bar{\mathbf{E}}_1 - \bar{\mathbf{E}}_1 \cdot \bar{\boldsymbol{\epsilon}}^\dagger \cdot \bar{\mathbf{E}}_2) \\ &\quad + \bar{\mathbf{E}}_2 \cdot \bar{\mathbf{J}}_1 - \bar{\mathbf{E}}_1 \cdot \bar{\mathbf{J}}_2 + \bar{\mathbf{H}}_1 \cdot \bar{\mathbf{M}}_2 - \bar{\mathbf{H}}_2 \cdot \bar{\mathbf{M}}_1 \end{aligned} \quad (1.10)$$

Since the transpose of a scalar is that same scalar, we see

$$\bar{\mathbf{a}} \cdot \bar{\bar{\mathbf{X}}} \cdot \bar{\mathbf{b}} = (\bar{\mathbf{a}} \cdot \bar{\bar{\mathbf{X}}} \cdot \bar{\mathbf{b}})^\dagger = \bar{\mathbf{b}} \cdot \bar{\bar{\mathbf{X}}}^\dagger \cdot \bar{\mathbf{a}}$$

for all vectors  $\bar{\mathbf{a}}, \bar{\mathbf{b}}$  and all dyadics  $\bar{\bar{\mathbf{X}}}$ . Thus, (1.10) becomes

$$\nabla \cdot (\bar{\mathbf{E}}_1 \times \bar{\mathbf{H}}_2 - \bar{\mathbf{E}}_2 \times \bar{\mathbf{H}}_1) = \bar{\mathbf{E}}_2 \cdot \bar{\mathbf{J}}_1 - \bar{\mathbf{E}}_1 \cdot \bar{\mathbf{J}}_2 + \bar{\mathbf{H}}_1 \cdot \bar{\mathbf{M}}_2 - \bar{\mathbf{H}}_2 \cdot \bar{\mathbf{M}}_1, \quad (1.11)$$

which is the differential form of the Lorentz reciprocity theorem. Integrating (1.11) over the volume containing the sources,

$$\begin{aligned} \iiint_V \nabla \cdot (\bar{\mathbf{E}}_1 \times \bar{\mathbf{H}}_2 - \bar{\mathbf{E}}_2 \times \bar{\mathbf{H}}_1) dV &= \oint\oint_\Sigma (\bar{\mathbf{E}}_1 \times \bar{\mathbf{H}}_2 - \bar{\mathbf{E}}_2 \times \bar{\mathbf{H}}_1) \cdot d\bar{\mathbf{s}} \\ &= \iiint_V (\bar{\mathbf{E}}_2 \cdot \bar{\mathbf{J}}_1 - \bar{\mathbf{E}}_1 \cdot \bar{\mathbf{J}}_2 + \bar{\mathbf{H}}_1 \cdot \bar{\mathbf{M}}_2 - \bar{\mathbf{H}}_2 \cdot \bar{\mathbf{M}}_1) dV, \end{aligned} \quad (1.12)$$

where the closed surface  $\Sigma$  encloses the volume  $V$ , and the surface normal points out of the volume. The integral form of the Lorentz reciprocity theorem is given by (1.12).

For most situations of interest, the electric permittivity and the magnetic permeability are scalars or symmetric dyadics, and the two sets of sources are radiating in the same medium. However, even when the constitutive parameters are non-symmetric dyadics, (1.11) and (1.12) still hold, provided  $\bar{\mathbf{E}}_2, \bar{\mathbf{H}}_2$  meet a very special condition:  $\bar{\mathbf{E}}_2, \bar{\mathbf{H}}_2$  must correspond to the fields when the second set of sources are embedded in a medium whose constitutive properties are the transpose of the constitutive properties of the medium in which the first set of sources are embedded, that is

$$\bar{\bar{\boldsymbol{\epsilon}}}_2 = \bar{\bar{\boldsymbol{\epsilon}}}_1^\dagger$$

and

$$\bar{\bar{\boldsymbol{\mu}}}_2 = \bar{\bar{\boldsymbol{\mu}}}_1^\dagger.$$

When the source currents exist in a finite volume and radiate into unbounded space, the fields are subject to the radiation condition [7]. The radiation condition says that the electric and magnetic fields become transverse to each other and propagate outward, so that

$$\lim_{r \rightarrow \infty} r (\nabla \times \bar{\mathbf{E}} + jk_0 \hat{\mathbf{r}} \times \bar{\mathbf{E}}) = 0. \quad (1.13)$$

Applying the radiation condition as  $r \rightarrow \infty$ , we have

$$\lim_{r \rightarrow \infty} r \bar{\mathbf{H}}_1 = \frac{k_0}{\omega} \bar{\bar{\boldsymbol{\mu}}}^{-1} \cdot \lim_{r \rightarrow \infty} r (\hat{\mathbf{r}} \times \bar{\mathbf{E}}_1), \quad (1.14)$$

and

$$\lim_{r \rightarrow \infty} r \bar{\mathbf{H}}_2 = \frac{k_0}{\omega} \bar{\bar{\boldsymbol{\mu}}}^{\dagger-1} \cdot \lim_{r \rightarrow \infty} r (\hat{\mathbf{r}} \times \bar{\mathbf{E}}_2). \quad (1.15)$$

The differential surface-area vector is

$$d\bar{\mathbf{s}} = \hat{\mathbf{r}} r^2 \sin \theta d\theta d\phi. \quad (1.16)$$

Thus, if the surface  $\Sigma$  is taken to be the surface of the sphere at  $r \rightarrow \infty$ , the surface integral in (1.12), becomes

$$\begin{aligned} \lim_{r \rightarrow \infty} \oiint_{\Sigma} (\bar{\mathbf{E}}_1 \times \bar{\mathbf{H}}_2 - \bar{\mathbf{E}}_2 \times \bar{\mathbf{H}}_1) \cdot d\bar{\mathbf{s}} = \\ \frac{k_0}{\omega} \lim_{r \rightarrow \infty} \oiint_{\Sigma} \left( \bar{\mathbf{E}}_1 \times \bar{\bar{\boldsymbol{\mu}}}^{\dagger-1} \cdot (\hat{\mathbf{r}} \times \bar{\mathbf{E}}_2) - \bar{\mathbf{E}}_2 \times \bar{\bar{\boldsymbol{\mu}}}^{-1} \cdot (\hat{\mathbf{r}} \times \bar{\mathbf{E}}_1) \right) \cdot \hat{\mathbf{r}} r^2 \sin \theta d\theta d\phi. \end{aligned} \quad (1.17)$$

A completely general form for  $\bar{\bar{\boldsymbol{\mu}}}^{-1}$  is

$$\bar{\bar{\boldsymbol{\mu}}}^{-1} = \bar{\mathbf{m}}_x \hat{\mathbf{x}} + \bar{\mathbf{m}}_y \hat{\mathbf{y}} + \bar{\mathbf{m}}_z \hat{\mathbf{z}}, \quad (1.18)$$

and, since the inverse of the transpose of a dyadic is the same as the transpose of the inverse,

$$(\bar{\bar{\boldsymbol{\mu}}}^\dagger)^{-1} = (\bar{\bar{\boldsymbol{\mu}}}^{-1})^\dagger = \hat{\mathbf{x}} \bar{\mathbf{m}}_x + \hat{\mathbf{y}} \bar{\mathbf{m}}_y + \hat{\mathbf{z}} \bar{\mathbf{m}}_z. \quad (1.19)$$

Substitution of (1.18) and (1.19) into (1.17) yields

$$\begin{aligned} \lim_{r \rightarrow \infty} \oiint_{\Sigma} (\bar{\mathbf{E}}_1 \times \bar{\mathbf{H}}_2 - \bar{\mathbf{E}}_2 \times \bar{\mathbf{H}}_1) \cdot d\bar{\mathbf{s}} \\ = \frac{k_0}{\omega} \lim_{r \rightarrow \infty} \oiint_{\Sigma} \left( \bar{\mathbf{E}}_1 \times (\hat{\mathbf{x}} \bar{\mathbf{m}}_x + \hat{\mathbf{y}} \bar{\mathbf{m}}_y + \hat{\mathbf{z}} \bar{\mathbf{m}}_z) \cdot (\hat{\mathbf{r}} \times \bar{\mathbf{E}}_2) \right. \\ \left. - \bar{\mathbf{E}}_2 \times (\bar{\mathbf{m}}_x \hat{\mathbf{x}} + \bar{\mathbf{m}}_y \hat{\mathbf{y}} + \bar{\mathbf{m}}_z \hat{\mathbf{z}}) \cdot (\hat{\mathbf{r}} \times \bar{\mathbf{E}}_1) \right) \cdot \hat{\mathbf{r}} r^2 \sin \theta d\theta d\phi. \end{aligned} \quad (1.20)$$

Applying a cyclic permutation of the triple-vector products on the right of (1.20)

$$\lim_{r \rightarrow \infty} \oiint_{\Sigma} (\bar{\mathbf{E}}_1 \times \bar{\mathbf{H}}_2 - \bar{\mathbf{E}}_2 \times \bar{\mathbf{H}}_1) \cdot d\bar{\mathbf{s}} = k_0 \frac{1}{\omega} \lim_{r \rightarrow \infty} \oiint_{\Sigma} \left( \begin{aligned} &\bar{\mathbf{E}}_1 \times [\hat{\mathbf{x}}\hat{\mathbf{r}} \cdot (\bar{\mathbf{E}}_2 \times \bar{\mathbf{m}}_x) + \hat{\mathbf{y}}\hat{\mathbf{r}} \cdot (\bar{\mathbf{E}}_2 \times \bar{\mathbf{m}}_y) + \hat{\mathbf{z}}\hat{\mathbf{r}} \cdot (\bar{\mathbf{E}}_2 \times \bar{\mathbf{m}}_z)] \\ & - \bar{\mathbf{E}}_2 \times [\bar{\mathbf{m}}_x \hat{\mathbf{r}} \cdot (\bar{\mathbf{E}}_1 \times \hat{\mathbf{x}}) + \bar{\mathbf{m}}_y \hat{\mathbf{r}} \cdot (\bar{\mathbf{E}}_1 \times \hat{\mathbf{y}}) + \bar{\mathbf{m}}_z \hat{\mathbf{r}} \cdot (\bar{\mathbf{E}}_1 \times \hat{\mathbf{z}})] \end{aligned} \right) \cdot \hat{\mathbf{r}} r^2 \sin \theta d\theta d\phi, \quad (1.21)$$

and, grouping the scalar factors in each term, we see that pairs of terms cancel, and the surface integral is zero,

$$\lim_{r \rightarrow \infty} \oiint_{\Sigma} (\bar{\mathbf{E}}_1 \times \bar{\mathbf{H}}_2 - \bar{\mathbf{E}}_2 \times \bar{\mathbf{H}}_1) \cdot d\bar{\mathbf{s}} = k_0 \frac{1}{\omega} \lim_{r \rightarrow \infty} \oiint_{\Sigma} \left( \begin{aligned} &\hat{\mathbf{r}} \cdot (\bar{\mathbf{E}}_1 \times \hat{\mathbf{x}}) (\bar{\mathbf{E}}_2 \times \bar{\mathbf{m}}_x) \cdot \hat{\mathbf{r}} \\ & + \hat{\mathbf{r}} \cdot (\bar{\mathbf{E}}_1 \times \hat{\mathbf{y}}) (\bar{\mathbf{E}}_2 \times \bar{\mathbf{m}}_y) \cdot \hat{\mathbf{r}} \\ & + \hat{\mathbf{r}} \cdot (\bar{\mathbf{E}}_1 \times \hat{\mathbf{z}}) (\bar{\mathbf{E}}_2 \times \bar{\mathbf{m}}_z) \cdot \hat{\mathbf{r}} \\ & - \hat{\mathbf{r}} \cdot (\bar{\mathbf{E}}_2 \times \bar{\mathbf{m}}_x) (\bar{\mathbf{E}}_1 \times \hat{\mathbf{x}}) \cdot \hat{\mathbf{r}} \\ & - \hat{\mathbf{r}} \cdot (\bar{\mathbf{E}}_2 \times \bar{\mathbf{m}}_y) (\bar{\mathbf{E}}_1 \times \hat{\mathbf{y}}) \cdot \hat{\mathbf{r}} \\ & - \hat{\mathbf{r}} \cdot (\bar{\mathbf{E}}_2 \times \bar{\mathbf{m}}_z) (\bar{\mathbf{E}}_1 \times \hat{\mathbf{z}}) \cdot \hat{\mathbf{r}} \end{aligned} \right) r^2 \sin \theta d\theta d\phi = 0. \quad (1.22)$$

To reiterate,

$$\lim_{r \rightarrow \infty} \oiint_{\Sigma} (\bar{\mathbf{E}}_1 \times \bar{\mathbf{H}}_2 - \bar{\mathbf{E}}_2 \times \bar{\mathbf{H}}_1) \cdot d\bar{\mathbf{s}} = 0, \quad (1.23)$$

as long as the fields  $\bar{\mathbf{E}}_2, \bar{\mathbf{H}}_2$  are associated with a set of sources embedded in a medium whose constitutive properties are the transpose of the constitutive properties of the medium in which the sources for  $\bar{\mathbf{E}}_1, \bar{\mathbf{H}}_1$  are embedded, that is

$$\bar{\bar{\mathbf{e}}}_2 = \bar{\bar{\mathbf{e}}}_1^\dagger,$$

and

$$\bar{\bar{\boldsymbol{\mu}}}_2 = \bar{\bar{\boldsymbol{\mu}}}_1^\dagger,$$

at each point in space. This covers the cases where magnetic permeability and electric permittivity are any combination of scalars, symmetric dyadics, or even non-symmetric dyadics. No spatial derivatives of the constitutive parameters were used, and the reciprocity theorem (1.12) and the result (1.23) are valid for inhomogeneous as well as homogeneous media.

Inserting (1.23) into (1.12) we see that

$$\iiint_V (\bar{\mathbf{E}}_2 \cdot \bar{\mathbf{J}}_1 - \bar{\mathbf{E}}_1 \cdot \bar{\mathbf{J}}_2 + \bar{\mathbf{H}}_1 \cdot \bar{\mathbf{M}}_2 - \bar{\mathbf{H}}_2 \cdot \bar{\mathbf{M}}_1) dV = 0, \quad (1.24)$$

when currents contained in a finite volume radiate into unbounded space.

Suppose each set of sources is localized and the sets are contained in non-overlapping, finite, closed volumes  $V_1$  and  $V_2$ . The volume integral can be broken into two pieces

$$\begin{aligned} \iiint_V (\bar{\mathbf{E}}_2 \cdot \bar{\mathbf{J}}_1 - \bar{\mathbf{E}}_1 \cdot \bar{\mathbf{J}}_2 + \bar{\mathbf{H}}_1 \cdot \bar{\mathbf{M}}_2 - \bar{\mathbf{H}}_2 \cdot \bar{\mathbf{M}}_1) dV &= \iiint_{V_1} (\bar{\mathbf{E}}_2 \cdot \bar{\mathbf{J}}_1 - \bar{\mathbf{H}}_2 \cdot \bar{\mathbf{M}}_1) dV \\ &- \iiint_{V_2} (\bar{\mathbf{E}}_1 \cdot \bar{\mathbf{J}}_2 - \bar{\mathbf{H}}_1 \cdot \bar{\mathbf{M}}_2) dV, \end{aligned} \quad (1.25)$$

where  $V$  is a closed volume containing all of the sources,  $V_1$  is the closed volume containing only sources  $\bar{\mathbf{J}}_1, \bar{\mathbf{M}}_1$ , and  $V_2$  is the closed volume containing only sources  $\bar{\mathbf{J}}_2, \bar{\mathbf{M}}_2$ . Because of (1.24), (1.25) becomes

$$\iiint_{V_1} (\bar{\mathbf{E}}_2 \cdot \bar{\mathbf{J}}_1 - \bar{\mathbf{H}}_2 \cdot \bar{\mathbf{M}}_1) dV = \iiint_{V_2} (\bar{\mathbf{E}}_1 \cdot \bar{\mathbf{J}}_2 - \bar{\mathbf{H}}_1 \cdot \bar{\mathbf{M}}_2) dV. \quad (1.26)$$

Suppose we choose to integrate (1.12) over the source-free volume, the volume  $V$  less the two closed volumes containing the sources,  $V_1$  and  $V_2$ . The surface integral in (1.12) will now contain three separate parts,

$$\oiint_{\Sigma} (\bar{\mathbf{E}}_1 \times \bar{\mathbf{H}}_2 - \bar{\mathbf{E}}_2 \times \bar{\mathbf{H}}_1) \cdot d\bar{\mathbf{s}} + \oiint_{\Sigma_1} (\bar{\mathbf{E}}_1 \times \bar{\mathbf{H}}_2 - \bar{\mathbf{E}}_2 \times \bar{\mathbf{H}}_1) \cdot d\bar{\mathbf{s}} + \oiint_{\Sigma_2} (\bar{\mathbf{E}}_1 \times \bar{\mathbf{H}}_2 - \bar{\mathbf{E}}_2 \times \bar{\mathbf{H}}_1) \cdot d\bar{\mathbf{s}} \quad (1.27)$$

where  $\Sigma$  is the surface of volume  $V$ , and  $\Sigma_1, \Sigma_2$  are the surfaces of volumes  $V_1$  and  $V_2$ , respectively. In each of the integrals, the direction of  $d\bar{\mathbf{s}}$  is outward from the enclosed source-free volume. This means that in the integrals over  $\Sigma_1, \Sigma_2$ ,  $d\bar{\mathbf{s}}$  is pointing into the volume containing the sources. Now, the volume integral on the right side of (1.12) will be zero since no sources are contained within the volume. Also, as previously shown, the integral over the outer surface  $\Sigma$  will be zero as we allow  $r \rightarrow \infty$ . In this case, the Lorentz reciprocity theorem reduces to

$$\oiint_{\Sigma_1} (\bar{\mathbf{E}}_1 \times \bar{\mathbf{H}}_2 - \bar{\mathbf{E}}_2 \times \bar{\mathbf{H}}_1) \cdot d\bar{\mathbf{s}} + \oiint_{\Sigma_2} (\bar{\mathbf{E}}_1 \times \bar{\mathbf{H}}_2 - \bar{\mathbf{E}}_2 \times \bar{\mathbf{H}}_1) \cdot d\bar{\mathbf{s}} = 0. \quad (1.28)$$

## Fields associated with different media

Suppose the two sets of sources are contained within different media,  $\bar{\boldsymbol{\mu}}_1, \bar{\boldsymbol{\epsilon}}_1$  for source 1, and  $\bar{\boldsymbol{\mu}}_2, \bar{\boldsymbol{\epsilon}}_2$  for source 2. With this situation, (1.10) becomes

$$\begin{aligned} \nabla \cdot (\bar{\mathbf{E}}_1 \times \bar{\mathbf{H}}_2 - \bar{\mathbf{E}}_2 \times \bar{\mathbf{H}}_1) &= -j\omega (\bar{\mathbf{H}}_2 \cdot \bar{\boldsymbol{\mu}}_1 \cdot \bar{\mathbf{H}}_1 - \bar{\mathbf{H}}_1 \cdot \bar{\boldsymbol{\mu}}_2 \cdot \bar{\mathbf{H}}_2) \\ &+ j\omega (\bar{\mathbf{E}}_2 \cdot \bar{\boldsymbol{\epsilon}}_1 \cdot \bar{\mathbf{E}}_1 - \bar{\mathbf{E}}_1 \cdot \bar{\boldsymbol{\epsilon}}_2 \cdot \bar{\mathbf{E}}_2) \\ &+ \bar{\mathbf{E}}_2 \cdot \bar{\mathbf{J}}_1 - \bar{\mathbf{E}}_1 \cdot \bar{\mathbf{J}}_2 + \bar{\mathbf{H}}_1 \cdot \bar{\mathbf{M}}_2 - \bar{\mathbf{H}}_2 \cdot \bar{\mathbf{M}}_1 \end{aligned} \quad (1.29)$$

As shown previously, in order for (1.29) to reduce to the usual forms of Lorentz's reciprocity theorem, (1.11) and (1.12), the media must have the transpose relationship

$$\bar{\boldsymbol{\epsilon}}_2 = \bar{\boldsymbol{\epsilon}}_1^\dagger, \quad (1.30)$$

$$\bar{\boldsymbol{\mu}}_2 = \bar{\boldsymbol{\mu}}_1^\dagger. \quad (1.31)$$

However, when the fields are associated with media that do not satisfy the transpose relationship, the reciprocity theorem is not as simple, but still it can be stated that

$$\begin{aligned} \oiint_{\Sigma} (\bar{\mathbf{E}}_1 \times \bar{\mathbf{H}}_2 - \bar{\mathbf{E}}_2 \times \bar{\mathbf{H}}_1) \cdot d\bar{\mathbf{s}} &= \iiint_V (\bar{\mathbf{E}}_2 \cdot \bar{\mathbf{J}}_1 - \bar{\mathbf{E}}_1 \cdot \bar{\mathbf{J}}_2 + \bar{\mathbf{H}}_1 \cdot \bar{\mathbf{M}}_2 - \bar{\mathbf{H}}_2 \cdot \bar{\mathbf{M}}_1) dV \\ &+ j\omega \iiint_V (\bar{\mathbf{E}}_2 \cdot \bar{\bar{\boldsymbol{\epsilon}}}_1 \cdot \bar{\mathbf{E}}_1 - \bar{\mathbf{H}}_2 \cdot \bar{\bar{\boldsymbol{\mu}}}_1 \cdot \bar{\mathbf{H}}_1 - \bar{\mathbf{E}}_1 \cdot \bar{\bar{\boldsymbol{\epsilon}}}_2 \cdot \bar{\mathbf{E}}_2 + \bar{\mathbf{H}}_1 \cdot \bar{\bar{\boldsymbol{\mu}}}_2 \cdot \bar{\mathbf{H}}_2) dV. \end{aligned} \quad (1.32)$$

The additional volume integral compensates for the different materials, but now the integration must be extended to the entire volume where the media properties differ. When (1.30) and (1.31) hold, this last volume integral is zero because the integrand itself is zero.

When the media are isotropic (scalar permittivity and permeability), then

$$\begin{aligned} \oiint_{\Sigma} (\bar{\mathbf{E}}_1 \times \bar{\mathbf{H}}_2 - \bar{\mathbf{E}}_2 \times \bar{\mathbf{H}}_1) \cdot d\bar{\mathbf{s}} &= \iiint_V (\bar{\mathbf{E}}_2 \cdot \bar{\mathbf{J}}_1 - \bar{\mathbf{E}}_1 \cdot \bar{\mathbf{J}}_2 + \bar{\mathbf{H}}_1 \cdot \bar{\mathbf{M}}_2 - \bar{\mathbf{H}}_2 \cdot \bar{\mathbf{M}}_1) dV \\ &+ j\omega \iiint_V (\epsilon' \bar{\mathbf{E}}_1 \cdot \bar{\mathbf{E}}_2 - \mu' \bar{\mathbf{H}}_1 \cdot \bar{\mathbf{H}}_2) dV, \end{aligned} \quad (1.33)$$

where

$$\epsilon' = \epsilon_1 - \epsilon_2, \quad (1.34)$$

and

$$\mu' = \mu_1 - \mu_2. \quad (1.35)$$

Consider the situation as  $r \rightarrow \infty$ . Using (1.13) in the left-hand expression of (1.33), we see that

$$\lim_{r \rightarrow \infty} \oiint_{\Sigma} (\bar{\mathbf{E}}_1 \times \bar{\mathbf{H}}_2 - \bar{\mathbf{E}}_2 \times \bar{\mathbf{H}}_1) \cdot d\bar{\mathbf{s}} = \lim_{r \rightarrow \infty} \oiint_{\Sigma} \begin{pmatrix} \bar{\mathbf{E}}_1 \times \left( \frac{k_0}{\omega} \bar{\bar{\boldsymbol{\mu}}}_2^{-1} \cdot (\hat{\mathbf{r}} \times \bar{\mathbf{E}}_2) \right) \\ -\bar{\mathbf{E}}_2 \times \left( \frac{k_0}{\omega} \bar{\bar{\boldsymbol{\mu}}}_1^{-1} \cdot (\hat{\mathbf{r}} \times \bar{\mathbf{E}}_1) \right) \end{pmatrix} \cdot d\bar{\mathbf{s}} \quad (1.36)$$

In general, the surface integral does not go to zero as  $r \rightarrow \infty$  when the different media extend to  $r \rightarrow \infty$ . However, it is reasonable to assume that the region of differing media is finite, so as  $r \rightarrow \infty$ ,  $\bar{\bar{\boldsymbol{\mu}}}_1 \rightarrow \mu_0$  and  $\bar{\bar{\boldsymbol{\mu}}}_2 \rightarrow \mu_0$ . With this assumption, the surface integral does go to zero at  $r \rightarrow \infty$ . Thus, in the general case where the currents are contained in a finite volume of differing media and radiate into unbounded space

$$\begin{aligned} \iiint_V (\bar{\mathbf{E}}_2 \cdot \bar{\mathbf{J}}_1 - \bar{\mathbf{E}}_1 \cdot \bar{\mathbf{J}}_2 + \bar{\mathbf{H}}_1 \cdot \bar{\mathbf{M}}_2 - \bar{\mathbf{H}}_2 \cdot \bar{\mathbf{M}}_1) dV &= j\omega \iiint_V (\bar{\mathbf{E}}_1 \cdot \bar{\bar{\boldsymbol{\epsilon}}}_2 \cdot \bar{\mathbf{E}}_2 - \bar{\mathbf{H}}_1 \cdot \bar{\bar{\boldsymbol{\mu}}}_2 \cdot \bar{\mathbf{H}}_2) dV \\ &- j\omega \iiint_V (\bar{\mathbf{E}}_2 \cdot \bar{\bar{\boldsymbol{\epsilon}}}_1 \cdot \bar{\mathbf{E}}_1 - \bar{\mathbf{H}}_2 \cdot \bar{\bar{\boldsymbol{\mu}}}_1 \cdot \bar{\mathbf{H}}_1) dV. \end{aligned} \quad (1.37)$$

## Reaction and mutual impedance

Now suppose the volume of integration is the closed volume containing only sources  $\bar{\mathbf{J}}_1, \bar{\mathbf{M}}_1$ , or sources  $\bar{\mathbf{J}}_2, \bar{\mathbf{M}}_2$ . In the first case, we have, from (1.12)

$$\oiint_{\Sigma_1} (\bar{\mathbf{E}}_1 \times \bar{\mathbf{H}}_2 - \bar{\mathbf{E}}_2 \times \bar{\mathbf{H}}_1) \cdot d\bar{\mathbf{s}} = \iiint_{V_1} (\bar{\mathbf{E}}_2 \cdot \bar{\mathbf{J}}_1 - \bar{\mathbf{H}}_2 \cdot \bar{\mathbf{M}}_1) dV, \quad (1.38)$$

while in the second case,

$$\oiint_{\Sigma_2} (\bar{\mathbf{E}}_1 \times \bar{\mathbf{H}}_2 - \bar{\mathbf{E}}_2 \times \bar{\mathbf{H}}_1) \cdot d\bar{\mathbf{s}} = -\iiint_{V_2} (\bar{\mathbf{E}}_1 \cdot \bar{\mathbf{J}}_2 - \bar{\mathbf{H}}_1 \cdot \bar{\mathbf{M}}_2) dV. \quad (1.39)$$

Rumsey [5] defines the right hand sides of (1.38) and (1.39) as the reaction,  $\langle 1, 2 \rangle$ , between source 1 and 2, and  $\langle 2, 1 \rangle$  between sources 2 and 1, respectively. In Rumsey's notation, the first designator in  $\langle :, : \rangle$  indicates the source located inside the volume of integration. Specifically

$$\langle 1, 2 \rangle = \iiint_{V_1} (\bar{\mathbf{E}}_2 \cdot \bar{\mathbf{J}}_1 - \bar{\mathbf{H}}_2 \cdot \bar{\mathbf{M}}_1) dV, \quad (1.40)$$

and

$$\langle 2, 1 \rangle = \iiint_{V_2} (\bar{\mathbf{E}}_1 \cdot \bar{\mathbf{J}}_2 - \bar{\mathbf{H}}_1 \cdot \bar{\mathbf{M}}_2) dV. \quad (1.41)$$

From (1.38) and (1.39), we also have

$$\langle 1, 2 \rangle = \oiint_{\Sigma_1} (\bar{\mathbf{E}}_1 \times \bar{\mathbf{H}}_2 - \bar{\mathbf{E}}_2 \times \bar{\mathbf{H}}_1) \cdot d\bar{\mathbf{s}}, \quad (1.42)$$

and,

$$\langle 2, 1 \rangle = \oiint_{\Sigma_2} (\bar{\mathbf{E}}_2 \times \bar{\mathbf{H}}_1 - \bar{\mathbf{E}}_1 \times \bar{\mathbf{H}}_2) \cdot d\bar{\mathbf{s}}. \quad (1.43)$$

Richmond [4] has shown that the reaction can also be written

$$v_{jk} i_j = -\langle j, k \rangle, \quad (1.44)$$

where  $v_{jk}$  is the voltage induced across the open-circuited terminals of source  $j$  in the presence of the fields,  $(\bar{\mathbf{E}}_k, \bar{\mathbf{H}}_k)$ , due to current  $i_k$  at the terminals of source  $k$ . The fields  $(\bar{\mathbf{E}}_j, \bar{\mathbf{H}}_j)$  are the result of applying terminal current  $i_j$  at source  $j$ .

In a multiport network, the currents at each port are related to the port voltages by an impedance matrix, as follows,

$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ z_{21} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ z_{n1} & \cdots & \cdots & z_{nn} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_n \end{bmatrix}. \quad (1.45)$$

Thus, from (1.44) and (1.45), the mutual impedance between port  $j$  and port  $k$  is

$$z_{jk} = \left. \frac{v_j}{i_k} \right|_{i_m=0, m \neq k} = \frac{v_{jk}}{i_k} = \frac{-\langle j, k \rangle}{i_j i_k} = -\frac{1}{i_j i_k} \oiint_{\Sigma_j} (\bar{\mathbf{E}}_j \times \bar{\mathbf{H}}_k - \bar{\mathbf{E}}_k \times \bar{\mathbf{H}}_j) \cdot d\bar{\mathbf{s}}. \quad (1.46)$$

When the fields associated with two antennas are known, then the mutual impedance between them can be found from (1.46). In this case, the currents on the antennas need not be known. Only the fields radiated when each antenna is excited with a known terminal current are necessary. If necessary, these fields can be obtained through measurement.

## Implementation of the computation of the mutual impedance

In order to compute the mutual impedance between two antennas, we will assume that the near fields associated with each antenna have already been obtained in some manner. Perhaps, the fields have been obtained through spherical-near-field measurement, method-of-moments computation, finite-element computation, or some other means that results in the complex frequency-domain phasor representation of the spatial dependence of the time-harmonic field associated with each antenna. Regardless of how the fields have been obtained, it will be convenient to write the fields as expansions in a set of orthogonal vector harmonics. The convenient set associated with spherical coordinates is the set of vector spherical harmonics [3, 8, 9, 10, 11]. The vector spherical harmonics and their application in expansions of electromagnetic fields are described in Appendix I.

Initially, it may seem that this approach unnecessarily complicates the formulation of the mutual impedance. The motivation lies in the fact that considerable effort is required to obtain the near electric (or magnetic) field for a particular antenna element. However, if this effort is expended once for the element of interest, then the procedure described here will allow the mutual impedance with another identical element to be obtained easily, for any number of different locations of the second element. The second element can be translated to any position relative to the first element, but we will not consider rotation. (The method can be extended to include rotation, however.) In addition, the integrand of (1.46) can oscillate wildly, causing difficulty in obtaining an accurate value for the mutual impedance by simply evaluating the integral. The use of vector spherical harmonics allows the integrations to be performed in closed form, and the expression for the mutual impedance is reduced to sums containing products of expansion coefficients and appropriate closed-form integrals.

In the expression for mutual impedance, (1.46), the fields associated with each element must be obtained in an environment that is consistent with the presence of the other antenna. For example, if the fields are obtained for an isolated element, the computed mutual impedance will be approximate, to the extent that the presence of the second element perturbs the fields away from the isolated-element fields.

### Expansion of the antenna's field in vector spherical harmonics

We will assume that the region around the antenna, in which we wish to expand the field, is characterized by scalar permittivity,  $\epsilon$ , and permeability,  $\mu$ . The electric field is written as an expansion in the normalized vector spherical harmonics

$$\bar{\mathbf{E}} = \sum_{n=1}^{\infty} \sum_{m=-n}^n \left[ b_{n,m}^{TE} \bar{\mathbf{M}}_{n,m}^{(2)}(\bar{\mathbf{r}}) + b_{n,m}^{TM} \bar{\mathbf{N}}_{n,m}^{(2)}(\bar{\mathbf{r}}) \right], \quad (2.1)$$

where the normalized vector spherical harmonics are

$$\bar{\mathbf{M}}_{n,m}^{(i)}(\bar{\mathbf{r}}) = jC_{n,m} \frac{e^{jm\phi}}{2\sqrt{\pi}} \left[ \frac{jm}{\sin\theta} z_n^{(i)}(kr) P_n^m(\cos\theta) \hat{\boldsymbol{\theta}} + \sin\theta z_n^{(i)}(kr) \frac{d}{dx} P_n^m(x) \Big|_{x=\cos\theta} \hat{\boldsymbol{\phi}} \right], \quad (2.2)$$

and

$$\bar{\mathbf{N}}_{n,m}^{(i)}(\bar{\mathbf{r}}) = jC_{n,m} \frac{e^{jm\phi}}{2\sqrt{\pi}} \left\{ \frac{z_n^{(i)}(kr)}{kr} n(n+1) P_n^m(\cos\theta) \hat{\mathbf{r}} + \frac{1}{kr} \frac{\partial}{\partial r} [r z_n^{(i)}(kr)] \left[ -\sin\theta \frac{d}{dx} P_n^m(x) \Big|_{x=\cos\theta} \hat{\boldsymbol{\theta}} + \frac{jm}{\sin\theta} P_n^m(\cos\theta) \hat{\boldsymbol{\phi}} \right] \right\}, \quad (2.3)$$

with

$$C_{n,m} \equiv \sqrt{\frac{(2n+1)(n-m)!}{n(n+1)(n+m)!}}. \quad (2.4)$$

The expansion in (2.1) is often referred to as a multipole expansion [8]. The  $n=1$  terms are the dipole terms, while  $n=2$  corresponds to the quadrupole terms, etc. In (2.2) and (2.3),  $z_n^{(i)}(kr)$  is one of the spherical Bessel's functions

$$z_n^{(i)}(kr) = \begin{cases} h_n^{(1)}(kr); & i=1 \\ h_n^{(2)}(kr); & i=2 \\ j_n(kr); & i=3 \\ y_n(kr); & i=4 \end{cases}, \quad (2.5)$$

and  $P_n^m(x)$  is the associated Legendre function of the first kind. The notation used here is consistent with the notation used by Jackson [8] and by Abramowitz and Stegun [9].

The vector spherical harmonics are described in detail in Appendix I. With the  $e^{j\omega t}$  time dependence, the  $\bar{\mathbf{M}}_{n,m}^{(2)}(\bar{\mathbf{r}})$  and  $\bar{\mathbf{N}}_{n,m}^{(2)}(\bar{\mathbf{r}})$  represent outwardly propagating waves.

The magnetic field intensity is obtained by substituting (2.1) into the curl equation for the electric field,

$$\bar{\mathbf{H}} = j \frac{k}{\omega\mu} \sum_{n=1}^{\infty} \sum_{m=-n}^n \left[ b_{n,m}^{TE} \bar{\mathbf{N}}_{n,m}^{(i)}(\bar{\mathbf{r}}) + b_{n,m}^{TM} \bar{\mathbf{M}}_{n,m}^{(i)}(\bar{\mathbf{r}}) \right]. \quad (2.6)$$

The coefficients  $b_{n,m}^{TE}$  describe the strength of the transverse-electric (TE) components of the radiated field, while coefficients  $b_{n,m}^{TM}$  describe the strength of the transverse-magnetic (TM) components. When the antenna can be enclosed in a sphere of radius  $a$ , the series usually can be truncated at degree  $n \approx ka$  [12], but in critical cases, such as this, it is advisable to use  $n \approx ka + n_1$ , where  $n_1 \approx 3\sqrt[3]{ka}$  [13, 14].

Using the asymptotic expansion of the spherical Hankel's function, the vector harmonics in the far-field region are approximated at large radius,  $r$ , by

$$\bar{\mathbf{M}}_{n,m}^{(2)}(\bar{\mathbf{r}}) \cong \frac{j^{n+1} e^{-jkr}}{kr} \bar{\mathbf{X}}_{n,m}(\theta, \phi), \quad (2.7)$$

and

$$\bar{\mathbf{N}}_{n,m}^{(2)}(\bar{\mathbf{r}}) \cong \frac{j^n e^{-jkr}}{kr} \hat{\mathbf{r}} \times \bar{\mathbf{X}}_{n,m}(\theta, \phi), \quad (2.8)$$

where the radially independent vector spherical harmonic,  $\bar{\mathbf{X}}_{n,m}$ , is given by

$$\begin{aligned} \bar{\mathbf{X}}_{n,m}(\theta, \phi) &= \frac{1}{z_n^{(i)}(kr)} \bar{\mathbf{M}}_{n,m}^{(i)} \\ &= j C_{n,m} \frac{e^{jm\phi}}{2\sqrt{\pi}} \left[ \frac{jm}{\sin\theta} P_n^m(\cos\theta) \hat{\boldsymbol{\theta}} + \sin\theta \frac{d}{dx} P_n^m(x) \Big|_{x=\cos\theta} \hat{\boldsymbol{\phi}} \right]. \end{aligned} \quad (2.9)$$

Thus, the far-field expressions for the outward-propagating fields are

$$\bar{\mathbf{E}} = \frac{e^{-jkr}}{kr} \sum_{n=1}^{\infty} \sum_{m=-n}^n j^n \left[ j b_{n,m}^{TE} \bar{\mathbf{X}}_{n,m}(\theta, \phi) + b_{n,m}^{TM} \hat{\mathbf{r}} \times \bar{\mathbf{X}}_{n,m}(\theta, \phi) \right], \quad (2.10)$$

and

$$\bar{\mathbf{H}} = \frac{e^{-jkr}}{kr} \frac{k}{\omega\mu} \sum_{n=1}^{\infty} \sum_{m=-n}^n j^n \left[ j b_{n,m}^{TE} \hat{\mathbf{r}} \times \bar{\mathbf{X}}_{n,m}(\theta, \phi) - b_{n,m}^{TM} \bar{\mathbf{X}}_{n,m}(\theta, \phi) \right] = \frac{k}{\omega\mu} \hat{\mathbf{r}} \times \bar{\mathbf{E}}. \quad (2.11)$$

Suppose we have obtained  $\bar{\mathbf{E}}(\bar{\mathbf{r}})$  at  $\bar{\mathbf{r}} = \bar{\mathbf{r}}_0$ . Then,

$$b_{n,m}^{TE} = \frac{1}{\mathcal{G}_{\bar{\mathbf{M}}\bar{\mathbf{M}}}(m, n; m, n)} \int_0^{2\pi} d\phi \int_0^\pi \sin\theta d\theta \bar{\mathbf{E}}(\bar{\mathbf{r}}_0) \cdot (\bar{\mathbf{M}}_{n,m}^{(i)}(\bar{\mathbf{r}}_0))^*, \quad (2.12)$$

and

$$b_{n,m}^{TM} = \frac{1}{\mathcal{G}_{\bar{\mathbf{N}}\bar{\mathbf{N}}}(m, n; m, n)} \int_0^{2\pi} d\phi \int_0^\pi \sin\theta d\theta \bar{\mathbf{E}}(\bar{\mathbf{r}}_0) \cdot (\bar{\mathbf{N}}_{n,m}^{(i)}(\bar{\mathbf{r}}_0))^*, \quad (2.13)$$

where (see Appendix I)

$$\mathcal{G}_{\bar{\mathbf{M}}\bar{\mathbf{M}}}(m, n; m', n'; r_0) = \left| z_n^{(i)}(kr_0) \right|^2 \delta_{nn'} \delta_{mm'},$$

and

$$\mathcal{G}_{\bar{\mathbf{N}}\bar{\mathbf{N}}}(m, n; m', n'; r_0) = \left[ \left| z_n^{(i)}(kr_0) \right|^2 + \frac{1}{k^2 r_0^2} \frac{\partial}{\partial r} \left[ r_0 z_n^{(i)}(kr_0) \frac{\partial}{\partial r} \left[ r_0 z_n^{(i)*}(kr_0) \right] \right] \right] \delta_{nn'} \delta_{mm'}, \quad (2.14)$$

and

$$\delta_{kj} = \begin{cases} 1, & k = j \\ 0, & k \neq j \end{cases}. \quad (2.15)$$

If the magnetic field is obtained instead of the electric field,

$$b_{n,m}^{TE} = -j \frac{\omega\mu}{k \mathcal{G}_{\bar{\mathbf{N}}\bar{\mathbf{N}}}(m, n; m, n; r_0)} \int_0^{2\pi} d\phi \int_0^\pi \sin\theta d\theta \bar{\mathbf{H}}(\bar{\mathbf{r}}_0) \cdot (\bar{\mathbf{N}}_{n,m}^{(i)}(\bar{\mathbf{r}}_0))^*, \quad (2.16)$$

and

$$b_{n,m}^{TM} = -j \frac{\omega\mu}{k \mathcal{G}_{\bar{\mathbf{M}}}(m, n; m, n; r_0)} \int_0^{2\pi} d\phi \int_0^\pi \sin\theta d\theta \bar{\mathbf{H}}(\bar{\mathbf{r}}_0) \cdot (\bar{\mathbf{M}}_{n,m}^{(i)}(\bar{\mathbf{r}}_0))^* . \quad (2.17)$$

Note that the units associated with the coefficients,  $b_{n,m}^{TE}$  and  $b_{n,m}^{TM}$ , are volts/meter.

## Translation of the fields

Although the expression for mutual impedance (1.46) is valid for any two arbitrary antennas, the array problem is concerned with the mutual impedance between identical elements. Thus, we will only address the problem of computing the mutual impedance between identical elements. The extension to non-identical elements is simple and straightforward. The translation of the fields is accomplished through the application of an appropriate addition theorem. The addition theorem for vector spherical harmonics, described in Appendix III, relates harmonics evaluated at  $\bar{\mathbf{r}}$  to harmonics evaluated at  $\bar{\mathbf{r}}'$ , where  $\bar{\mathbf{r}}'$  is measured from the origin of a second coordinate system, whose axes are parallel to those of the original system. The origin of the second coordinate system is located in the first coordinate system by  $\bar{\mathbf{r}}''$ . These three vectors are related by addition as

$$\bar{\mathbf{r}} = \bar{\mathbf{r}}'' + \bar{\mathbf{r}}' , \quad (2.18)$$

so the theorem is called an addition theorem.

Antenna 1 is located at the origin of coordinate system 1 (unprimed), and antenna 2 is located at the origin of coordinate system 2 (primed). The origin of coordinate system 2 is located from the origin of antenna 1 by position vector  $\bar{\mathbf{r}}''$ ,

$$\bar{\mathbf{r}}'' = r'' [\sin\theta'' \cos\phi'' \hat{\mathbf{x}}_1 + \sin\theta'' \sin\phi'' \hat{\mathbf{y}}_1 + \cos\theta'' \hat{\mathbf{z}}_1] . \quad (2.19)$$

The geometry is illustrated in Figure 1. To obtain  $z_{21}$ , the integration in (1.46) will be performed over a sphere that surrounds antenna 2. It is important to ensure that the sphere does not contain antenna 1, as well.

As illustrated in Figure 1,  $\bar{\mathbf{r}}$  locates the field point on the sphere of integration with respect to the origin of coordinate system 1 (unprimed), while  $\bar{\mathbf{r}}'$  locates the same field point from the origin of coordinate system 2 (primed). In fact, during the integration,  $|\bar{\mathbf{r}}'|$  will be constant at the value of the radius chosen for the integration, even though the length of  $\bar{\mathbf{r}}$  changes. Also, note that  $\bar{\mathbf{r}}''$ , which locates origin 2 from coordinate system 1, is a constant vector; it does not move during the integration.

The fields for antenna 1 will be expanded in vector spherical harmonics as follows:

$$\bar{\mathbf{E}}_1 = \sum_{n=1}^{\infty} \sum_{m=-n}^n [b_{n,m}^{TE} \bar{\mathbf{M}}_{n,m}^{(2)}(\bar{\mathbf{r}}) + b_{n,m}^{TM} \bar{\mathbf{N}}_{n,m}^{(2)}(\bar{\mathbf{r}})] , \quad (2.20)$$

and

$$\bar{\mathbf{H}}_1 = j \frac{k}{\omega\mu} \sum_{n=1}^{\infty} \sum_{m=-n}^n [b_{n,m}^{TE} \bar{\mathbf{N}}_{n,m}^{(2)}(\bar{\mathbf{r}}) + b_{n,m}^{TM} \bar{\mathbf{M}}_{n,m}^{(2)}(\bar{\mathbf{r}})] , \quad (2.21)$$

using vector spherical harmonics associated with the unprimed (antenna 1) coordinate system. However, in order to integrate over a sphere around antenna 2, it is convenient to express these fields in terms of vector spherical harmonics associated with the primed coordinate system.

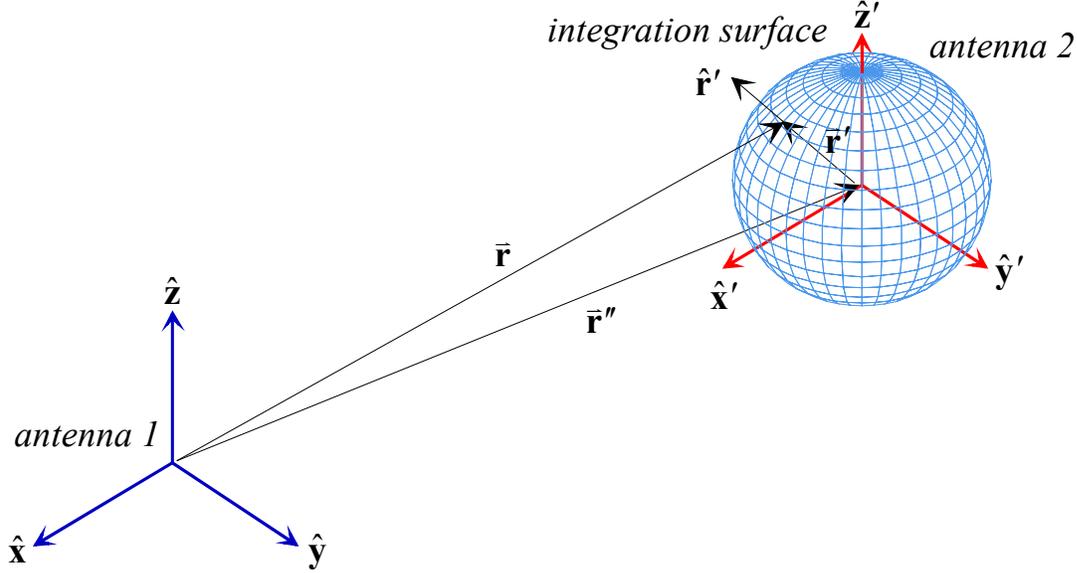


Figure 1 Geometry describing the relationship between the two antennas.

The vector-spherical-harmonic addition theorem says (see Appendix III) that for the additive relationship  $\bar{\mathbf{r}} = \bar{\mathbf{r}}' + \bar{\mathbf{r}}''$ ,

$$\bar{\mathbf{M}}_{n,m}^{(2)}(\bar{\mathbf{r}}) = \sum_{n',m'} A_{n',m';n,m} \bar{\mathbf{M}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') + B_{n',m';n,m} \bar{\mathbf{N}}_{n',m'}^{(i)}(\bar{\mathbf{r}}'), \quad (2.22)$$

and

$$\bar{\mathbf{N}}_{n',m'}^{(2)}(\bar{\mathbf{r}}) = \sum_{n',m'} A_{n',m';n,m} \bar{\mathbf{N}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') + B_{n',m';n,m} \bar{\mathbf{M}}_{n',m'}^{(i)}(\bar{\mathbf{r}}'), \quad (2.23)$$

where  $A_{n',m';n,m}$  and  $B_{n',m';n,m}$  are given in Appendix III. Both the vector spherical harmonics  $\bar{\mathbf{M}}_{n',m'}^{(i)}(\bar{\mathbf{r}}')$  and  $\bar{\mathbf{N}}_{n',m'}^{(i)}(\bar{\mathbf{r}}')$  contain spherical Bessel's functions  $z_n^{(i)}(kr')$ , the specific kind of which is determined by the relative size of  $r'$  and  $r''$  as follows

$$\text{in } \bar{\mathbf{M}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') \text{ and } \bar{\mathbf{N}}_{n',m'}^{(i)}(\bar{\mathbf{r}}'): z_n^{(i)}(kr') = \begin{cases} j_n(kr'), & r' < r'' \\ h_n^{(2)}(kr'), & r' > r'' \end{cases} \quad (2.24)$$

Similarly, both sets of coefficients  $A_{n',m';n,m}$  and  $B_{n',m';n,m}$  contain spherical Bessel's functions evaluated at  $r''$ , the specific kind of which is determined according to

$$\text{in } A_{n',m';n,m} \text{ and } B_{n',m';n,m} : z_{n'}^{(i)}(kr'') = \begin{cases} h_{n'}^{(2)}(kr''), & r' < r'' \\ j_{n'}(kr''), & r' > r'' \end{cases}. \quad (2.25)$$

It should be emphasized that, for the integration over the spherical surface around antenna 2 (Figure 1), these Bessel's functions are constant, since  $r'$  and  $r''$  are constant.

Substituting (2.22) and (2.23) into (2.20) and (2.21) gives the appropriate expressions for the fields associated with antenna 1

$$\bar{\mathbf{E}}_1 = \sum_{n=1}^{\infty} \sum_{m=-n}^n \left[ \begin{array}{l} b_{n,m}^{TE} \sum_{n',m'} \left( A_{n',m';n,m} \bar{\mathbf{M}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') + B_{n',m';n,m} \bar{\mathbf{N}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') \right) \\ + b_{n,m}^{TM} \sum_{n',m'} \left( A_{n',m';n,m} \bar{\mathbf{N}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') + B_{n',m';n,m} \bar{\mathbf{M}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') \right) \end{array} \right], \quad (2.26)$$

and

$$\bar{\mathbf{H}}_1 = j \frac{k}{\omega \mu} \sum_{n=1}^{\infty} \sum_{m=-n}^n \left[ \begin{array}{l} b_{n,m}^{TE} \sum_{n',m'} \left( A_{n',m';n,m} \bar{\mathbf{N}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') + B_{n',m';n,m} \bar{\mathbf{M}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') \right) \\ + b_{n,m}^{TM} \sum_{n',m'} \left( A_{n',m';n,m} \bar{\mathbf{M}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') + B_{n',m';n,m} \bar{\mathbf{N}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') \right) \end{array} \right], \quad (2.27)$$

or

$$\bar{\mathbf{E}}_1 = \sum_{n=1}^{\infty} \sum_{m=-n}^n \sum_{n',m'} \left( \begin{array}{l} \left( b_{n,m}^{TE} A_{n',m';n,m} + b_{n,m}^{TM} B_{n',m';n,m} \right) \bar{\mathbf{M}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') \\ + \left( b_{n,m}^{TE} B_{n',m';n,m} + b_{n,m}^{TM} A_{n',m';n,m} \right) \bar{\mathbf{N}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') \end{array} \right), \quad (2.28)$$

and

$$\bar{\mathbf{H}}_1 = j \frac{k}{\omega \mu} \sum_{n=1}^{\infty} \sum_{m=-n}^n \sum_{n',m'} \left( \begin{array}{l} \left( b_{n,m}^{TE} A_{n',m';n,m} + b_{n,m}^{TM} B_{n',m';n,m} \right) \bar{\mathbf{N}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') \\ + \left( b_{n,m}^{TE} B_{n',m';n,m} + b_{n,m}^{TM} A_{n',m';n,m} \right) \bar{\mathbf{M}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') \end{array} \right). \quad (2.29)$$

Exchanging the order of summation

$$\bar{\mathbf{E}}_1 = \sum_{n'=1}^{\infty} \sum_{m'=-n'}^{n'} \left( {}_1 b_{n',m'}^{TE} \bar{\mathbf{M}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') + {}_1 b_{n',m'}^{TM} \bar{\mathbf{N}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') \right), \quad (2.30)$$

and

$$\bar{\mathbf{H}}_1 = j \frac{k}{\omega \mu} \sum_{n'=1}^{\infty} \sum_{m'=-n'}^{n'} \left( {}_1 b_{n',m'}^{TM} \bar{\mathbf{N}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') + {}_1 b_{n',m'}^{TE} \bar{\mathbf{M}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') \right), \quad (2.31)$$

where

$${}_1 b_{n',m'}^{TE} = \sum_{n=1}^{\infty} \sum_{m=-n}^n \left( b_{n,m}^{TE} A_{n',m';n,m} + b_{n,m}^{TM} B_{n',m';n,m} \right), \quad (2.32)$$

and

$${}_1 b_{n',m'}^{TM} = \sum_{n=1}^{\infty} \sum_{m=-n}^n \left( b_{n,m}^{TE} B_{n',m';n,m} + b_{n,m}^{TM} A_{n',m';n,m} \right). \quad (2.33)$$

The pre-subscript is used to designate the coefficients as belonging to the field expansion for antenna 1.

The fields from antenna 2 are simply

$$\bar{\mathbf{E}}_2 = \sum_{n'=1}^{\infty} \sum_{m'=-n'}^{n'} \left[ {}_2b_{n',m'}^{TE} \bar{\mathbf{M}}_{n',m'}^{(2)}(\bar{\mathbf{r}}') + {}_2b_{n',m'}^{TM} \bar{\mathbf{N}}_{n',m'}^{(2)}(\bar{\mathbf{r}}') \right], \quad (2.34)$$

$$\bar{\mathbf{H}}_2 = j \frac{k}{\omega\mu} \sum_{n'=1}^{\infty} \sum_{m'=-n'}^{n'} \left[ {}_2b_{n',m'}^{TE} \bar{\mathbf{N}}_{n',m'}^{(2)}(\bar{\mathbf{r}}') + {}_2b_{n',m'}^{TM} \bar{\mathbf{M}}_{n',m'}^{(2)}(\bar{\mathbf{r}}') \right], \quad (2.35)$$

where

$${}_2b_{n,m}^{TE} = b_{n,m}^{TE} \quad \text{and} \quad {}_2b_{n,m}^{TM} = b_{n,m}^{TM}. \quad (2.36)$$

## Mutual impedance

The fields for antenna 1 and antenna 2 have been expanded in vector spherical harmonics as follows:

$$\bar{\mathbf{E}}_1 = \sum_{n'=1}^{\infty} \sum_{m'=-n'}^{n'} \left[ {}_1b_{n',m'}^{TE} \bar{\mathbf{M}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') + {}_1b_{n',m'}^{TM} \bar{\mathbf{N}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') \right], \quad (2.37)$$

$$\bar{\mathbf{H}}_1 = j \frac{k}{\omega\mu} \sum_{n'=1}^{\infty} \sum_{m'=-n'}^{n'} \left[ {}_1b_{n',m'}^{TE} \bar{\mathbf{N}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') + {}_1b_{n',m'}^{TM} \bar{\mathbf{M}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') \right], \quad (2.38)$$

$$\bar{\mathbf{E}}_2 = \sum_{n=1}^{\infty} \sum_{m=-n}^n \left[ {}_2b_{n,m}^{TE} \bar{\mathbf{M}}_{n,m}^{(2)}(\bar{\mathbf{r}}') + {}_2b_{n,m}^{TM} \bar{\mathbf{N}}_{n,m}^{(2)}(\bar{\mathbf{r}}') \right], \quad (2.39)$$

$$\bar{\mathbf{H}}_2 = j \frac{k}{\omega\mu} \sum_{n=1}^{\infty} \sum_{m=-n}^n \left[ {}_2b_{n,m}^{TE} \bar{\mathbf{N}}_{n,m}^{(2)}(\bar{\mathbf{r}}') + {}_2b_{n,m}^{TM} \bar{\mathbf{M}}_{n,m}^{(2)}(\bar{\mathbf{r}}') \right]. \quad (2.40)$$

The pre-subscripts on the coefficients correspond to the antenna designation. The superscript showing the type of spherical Bessel's function used in the vector harmonic is designated with the superscript  $(i)$  in the expressions for  $\bar{\mathbf{E}}_1$  and  $\bar{\mathbf{H}}_1$ . It is assumed that the harmonics have already been translated using the translation theorem, so the coefficients  ${}_1b_{n',m'}^{TE}$  and  ${}_1b_{n',m'}^{TM}$  are the translated coefficients. As noted above, the types of Bessel's functions contained in (2.37) and (2.38) depend on the relative size of the distance from the old coordinate origin to the new coordinate origin,  $r''$ , and the distance from the new origin to the field point,  $r'$ .

The mutual impedance between the antennas is obtained from (1.46), where the integration is performed over the sphere  $|\bar{\mathbf{r}}'| = \text{constant}$  around antenna 2 (Figure 1). It is necessary that the sphere of integration contain only antenna 2, and this requires

$$|\bar{\mathbf{r}}'| < |\bar{\mathbf{r}}''|. \quad (2.41)$$

Since  $d\bar{\mathbf{s}}' = r'^2 \hat{\mathbf{r}}' \sin \theta' d\theta' d\phi'$ , the mutual impedance (1.46)

$$z_{12} = z_{21} = -\frac{1}{i_1 i_2} \int_0^{2\pi} d\phi' \int_0^\pi r'^2 \sin \theta' d\theta' (\bar{\mathbf{E}}_1 \times \bar{\mathbf{H}}_2 - \bar{\mathbf{E}}_2 \times \bar{\mathbf{H}}_1) \cdot \hat{\mathbf{r}}'. \quad (2.42)$$

Substituting the fields, expanded in vector spherical harmonics, into (2.42) leads to

$$z_{12} = z_{21} = -\frac{1}{i_1 i_2} j \frac{k}{\omega \mu} \sum_{n'=1}^{\infty} \sum_{m'=-n'}^{n'} \sum_{n=1}^{\infty} \sum_{m=-n}^n \left\{ \begin{aligned} & \left( {}_1 b_{n',m'}^{TE} {}_2 b_{n,m}^{TE} + {}_2 b_{n,m}^{TM} {}_1 b_{n',m'}^{TM} \right) \int_0^{2\pi} d\phi' \int_0^\pi r'^2 \sin \theta' \bar{\mathbf{M}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') \times \bar{\mathbf{N}}_{n,m}^{(2)}(\bar{\mathbf{r}}') \cdot \hat{\mathbf{r}}' d\theta' \\ & + \left( {}_1 b_{n',m'}^{TE} {}_2 b_{n,m}^{TM} + {}_2 b_{n,m}^{TE} {}_1 b_{n',m'}^{TM} \right) \int_0^{2\pi} d\phi' \int_0^\pi r'^2 \sin \theta' \bar{\mathbf{M}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') \times \bar{\mathbf{M}}_{n,m}^{(2)}(\bar{\mathbf{r}}') \cdot \hat{\mathbf{r}}' d\theta' \\ & + \left( {}_1 b_{n',m'}^{TM} {}_2 b_{n,m}^{TE} + {}_2 b_{n,m}^{TM} {}_1 b_{n',m'}^{TE} \right) \int_0^{2\pi} d\phi' \int_0^\pi r'^2 \sin \theta' \bar{\mathbf{N}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') \times \bar{\mathbf{N}}_{n,m}^{(2)}(\bar{\mathbf{r}}') \cdot \hat{\mathbf{r}}' d\theta' \\ & + \left( {}_1 b_{n',m'}^{TM} {}_2 b_{n,m}^{TM} + {}_2 b_{n,m}^{TE} {}_1 b_{n',m'}^{TE} \right) \int_0^{2\pi} d\phi' \int_0^\pi r'^2 \sin \theta' \bar{\mathbf{N}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') \times \bar{\mathbf{M}}_{n,m}^{(2)}(\bar{\mathbf{r}}') \cdot \hat{\mathbf{r}}' d\theta' \end{aligned} \right\}. \quad (2.43)$$

Each of these integrals contains the factor

$$\int_0^{2\pi} e^{j(m+m')\phi'} d\phi' = 2\pi \delta_{m,-m'}, \quad (2.44)$$

so each integral is zero unless  $m' = -m$ , eliminating the summation over  $m'$ . Writing (2.43) in terms of integrals evaluated in Appendix IV, the mutual impedance is

$$z_{12} = z_{21} = -\frac{1}{i_1 i_2} j \frac{k}{\omega \mu} \sum_{n'=1}^{\infty} \sum_{n=1}^{\infty} \sum_{m=-\min(n,n')}^{\min(n,n')} C_{n',-m} C_{n,m} \left( \begin{aligned} & A_{n,n',m} G_{n,n'}(kr') [I_1(n,n',m) - I_3(n,n',m)] \\ & + j B_{n,n',m} H_{n,n'}(kr') [I_2(n,n',m) - I_2(n',n,-m)] \end{aligned} \right) \quad (2.45)$$

where

$$I_1(n,n',m) = \int_{-1}^1 (1-x^2) \frac{d}{dx} P_n^m(x) \frac{d}{dx} P_{n'}^{-m}(x) dx, \quad (2.46)$$

$$I_2(n,n',m) = m \int_{-1}^1 P_n^m(x) \frac{d}{dx} P_{n'}^{-m}(x) dx, \quad (2.47)$$

$$I_3(n,n',m) = -m^2 \int_{-1}^1 \frac{1}{1-x^2} P_n^m(x) P_{n'}^{-m}(x) dx, \quad (2.48)$$

$$B_{n,n',m} \equiv \frac{{}_1 b_{n',-m}^{TE} {}_2 b_{n,m}^{TM} + {}_2 b_{n,m}^{TE} {}_1 b_{n',-m}^{TM}}{2}, \quad (2.50)$$

$$C_{n,m} \equiv \sqrt{\frac{(2n+1)(n-m)!}{n(n+1)(n+m)!}}, \quad (2.4)$$

$$G_{n,n'}(kr') \equiv -r' \frac{z_n^{(i)}(kr')}{k} \frac{\partial}{\partial r'} [r' h_n^{(2)}(kr')] + r' \frac{h_n^{(2)}(kr')}{k} \frac{\partial}{\partial r'} [r' z_n^{(i)}(kr')], \quad (2.51)$$

$$H_{n,n'}(kr') \equiv r'^2 h_n^{(2)}(kr') z_n^{(i)}(kr') + \frac{\frac{\partial}{\partial r'} [r' h_n^{(2)}(kr')]}{k^2} \frac{\partial}{\partial r'} [r' z_n^{(i)}(kr')], \quad (2.52)$$

and, since the integration sphere must not contain antenna 1,  $r' < r''$ ,

$$z_n^{(i)}(kr') = j_n(kr'). \quad (2.53)$$

These equations look rather unwieldy, but fortunately, additional simplification is possible. In addition, it appears from (2.51) and (2.52) that the impedance depends on the radius of the integration sphere, contrary to expectation. However, it will be shown subsequently that, as long as the integration sphere does not contain antenna 1 so  $r' < r''$ , the mutual impedance is independent of the radius.

It is shown in Appendix IV that

$$I_2(n, n', m) - I_2(n', n, -m) = 0, \quad (2.54)$$

and using (IV-77) and (IV-79)

$$I_1(n, n', m) - I_3(n, n', m) = \begin{cases} 0; & n' \neq n \\ (-1)^m \frac{2n(n+1)}{(2n+1)}; & n' = n \end{cases}. \quad (2.55)$$

Thus, the only non-zero terms in (2.45) occur when  $n' = n$ . The mutual impedance can now be written

$$z_{12} = z_{21} = -\frac{1}{i_1 i_2} j \frac{k}{\omega \mu} \sum_{n=1}^{\infty} \sum_{m=-n}^n (-1)^m \frac{2n(n+1)}{(2n+1)} C_{n,-m} C_{n,m} A_{n,n,m} G_{n,n}(kr'). \quad (2.56)$$

An alternate method of arriving at (2.56) from (2.43) is contained in Appendix VI.

Substitution of (2.4) further simplifies (2.56)

$$z_{12} = z_{21} = -\frac{2}{i_1 i_2} j \frac{k}{\omega \mu} \sum_{n=1}^{\infty} \sum_{m=-n}^n (-1)^m A_{n,n,m} G_{n,n}(kr'). \quad (2.57)$$

Using (2.53) in (2.51)

$$\begin{aligned} G_{n,n}(kr') &= \frac{r'}{k} \left[ -j_n(kr') \frac{\partial}{\partial r'} [r' h_n^{(2)}(kr')] + h_n^{(2)}(kr') \frac{\partial}{\partial r'} [r' j_n(kr')] \right] \\ &= j \frac{r'}{k} \left[ j_n(kr') \frac{\partial}{\partial r'} [r' y_n(kr')] - y_n(kr') \frac{\partial}{\partial r'} [r' j_n(kr')] \right]. \end{aligned} \quad (2.58)$$

The spherical Bessel's functions satisfy the recursion [9],

$$x \frac{\partial}{\partial x} z_n^{(i)}(x) = n z_n^{(i)}(x) - x z_{n+1}^{(i)}(x), \quad (2.59)$$

so

$$\frac{\partial}{\partial r'} r' z_n^{(i)}(kr') = z_n^{(i)}(kr') + kr' \frac{\partial}{\partial (kr')} z_n^{(i)}(kr') = (n+1) z_n^{(i)}(kr') - kr' z_{n+1}^{(i)}(kr'). \quad (2.60)$$

Substituting (2.60) into (2.58)

$$G_{n,n}(kr') = j \frac{r'}{k} kr' [y_n(kr') j_{n+1}(kr') - j_n(kr') y_{n+1}(kr')]. \quad (2.61)$$

The cross-product relation between the spherical Bessel's functions is [9]

$$y_n(kr') j_{n+1}(kr') - j_n(kr') y_{n+1}(kr') = \frac{1}{k^2 r'^2}, \quad (2.62)$$

so

$$G_{n,n}(kr') = j \frac{1}{k^2}. \quad (2.63)$$

Using (2.49) and (2.63) in (2.57), the mutual impedance is simplified to

$$z_{12} = z_{21} = \frac{1}{i_1 i_2} \frac{1}{k^2} \sqrt{\frac{\epsilon}{\mu}} \sum_{n=1}^{\infty} \sum_{m=-n}^n (-1)^m \left( {}_1 b_{n,-m}^{TE} {}_2 b_{n,m}^{TE} + {}_2 b_{n,m}^{TM} {}_1 b_{n,-m}^{TM} \right). \quad (2.64)$$

The coefficients  ${}_1 b_{n,m}^{TE}$ ,  ${}_1 b_{n,m}^{TM}$ ,  ${}_2 b_{n,m}^{TE}$ ,  ${}_2 b_{n,m}^{TM}$  have units of volts/meter, and the currents  $i_1$ ,  $i_2$  are measured in amperes.

Suppose the integration sphere had been allowed to contain both antennas so  $r' > r''$ . In this case, the proper choice of  $z_n^{(i)}(kr')$  is

$$z_n^{(i)}(kr') = h_n^{(2)}(kr'). \quad (2.65)$$

Using (2.65) in (2.51)

$$\begin{aligned} G_{n,n}(kr') &= r' \frac{h_n^{(2)}(kr')}{k} \frac{\partial}{\partial r'} [r' h_n^{(2)}(kr')] - r' \frac{h_n^{(2)}(kr')}{k} \frac{\partial}{\partial r'} [r' h_n^{(2)}(kr')] \\ &= 0 \end{aligned} \quad (2.66)$$

In effect, the reaction of antenna 1 on antenna 2 is cancelled by the reaction of antenna 2 on antenna 1, and the integration gives zero. Thus, the requirement that the sphere of integration not contain both antennas is necessary.

### **Mutual impedance from radiation pattern measurement**

The total time-averaged power radiated by an element is

$$P = \frac{1}{2} \operatorname{Re} \left( \int_0^{2\pi} \int_0^\pi \mathbf{E} \times \mathbf{H}^* r^2 \sin \theta d\theta d\phi \right), \quad (2.67)$$

and substituting the vector-spherical-harmonic expansions for the fields, it can be shown that the total time-averaged power radiated by a single antenna element is

$$P = \frac{1}{2k^2} \sqrt{\frac{\epsilon_0}{\mu_0}} \sum_{n=1}^{\infty} \sum_{m=-n}^n \left( |b_{n,m}^{TE}|^2 + |b_{n,m}^{TM}|^2 \right). \quad (2.68)$$

The total time-averaged radiated power is also given by

$$P = \frac{1}{2} i^2 \operatorname{Re}(Z_{11}) \eta_{rad}, \quad (2.69)$$

where  $Z_{11}$  is the input impedance, and  $\eta_{rad}$  is the radiation efficiency of the element.

The mutual impedance can be written in terms of the radiation efficiency and the resistive part of the input impedance,

$$z_{12} = z_{21} = \frac{\operatorname{Re}(Z_{11}) \eta_{rad}}{\sum_{n=1}^{\infty} \sum_{m=-n}^n \left( |b_{n,m}^{TE}|^2 + |b_{n,m}^{TM}|^2 \right)} \sum_{n=1}^{\infty} \sum_{m=-n}^n (-1)^m \left( {}_1 b_{n,-m}^{TE} {}_2 b_{n,m}^{TE} + {}_2 b_{n,m}^{TM} {}_1 b_{n,-m}^{TM} \right). \quad (2.70)$$

When fully polarimetric radiation gain and phase patterns are measured, the quantity measured is a complex vector gain,

$$\begin{aligned} \bar{\mathbf{g}}(\theta, \phi) &= g_{cal} \frac{kr}{e^{-jkr}} \bar{\mathbf{E}}(\bar{\mathbf{r}}) = g_{cal} \sum_{n=1}^{\infty} \sum_{m=-n}^n \left[ j^{n+1} b_{n,m}^{TE} \bar{\mathbf{X}}_{n,m}(\theta, \phi) + j^n b_{n,m}^{TM} \hat{\mathbf{r}} \times \bar{\mathbf{X}}_{n,m}(\theta, \phi) \right] \\ &= \sum_{n=1}^{\infty} \sum_{m=-n}^n \left[ c_{n,m}^{TE} \bar{\mathbf{X}}_{n,m}(\theta, \phi) + c_{n,m}^{TM} \hat{\mathbf{r}} \times \bar{\mathbf{X}}_{n,m}(\theta, \phi) \right] \end{aligned} \quad (2.71)$$

where the far-field vector spherical harmonics (2.7) and (2.8) are used, and  $g_{cal}$  is the calibration factor, chosen so that the gain pattern is

$$G(\bar{\mathbf{r}}) = \left| \sum_{n=1}^{\infty} \sum_{m=-n}^n \left[ c_{n,m}^{TE} \bar{\mathbf{X}}_{n,m}(\bar{\mathbf{r}}) + c_{n,m}^{TM} \hat{\mathbf{r}} \times \bar{\mathbf{X}}_{n,m}(\bar{\mathbf{r}}) \right] \right|^2. \quad (2.72)$$

The field-expansion coefficients are related to the complex gain coefficients as

$$b_{n,m}^{TE} = \frac{1}{j^{n+1} g_{cal}} c_{n,m}^{TE}, \quad (2.73)$$

and

$$b_{n,m}^{TM} = \frac{1}{j^n g_{cal}} c_{n,m}^{TM}. \quad (2.74)$$

The efficiency is

$$\eta_{rad} = \frac{1}{4\pi} \sum_{n=1}^{\infty} \sum_{m=-n}^n \left( |c_{n,m}^{TE}|^2 + |c_{n,m}^{TM}|^2 \right). \quad (2.75)$$

Thus, when the expansion coefficients represent the gain pattern (typically the case with measured patterns), then the mutual impedance is simply

$$z_{12} = z_{21} = \frac{1}{4\pi} \operatorname{Re}(Z_{11}) \sum_{n=1}^{\infty} \sum_{m=-n}^n (-1)^{m+n} \left( -{}_1 c_{n,-m}^{TE} {}_2 c_{n,m}^{TE} + {}_2 c_{n,m}^{TM} {}_1 c_{n,-m}^{TM} \right). \quad (2.76)$$

However, care must be maintained when using measured complex-gain patterns. It is also common to use expansion coefficients defined so the far-field gain is

$$G(\bar{\mathbf{r}}) = \left| \sum_{n=1}^{\infty} \sum_{m=-n}^n j^n \left[ j d_{n,m}^{TE} \bar{\mathbf{X}}_{n,m}(\bar{\mathbf{r}}) + d_{n,m}^{TM} \hat{\mathbf{r}} \times \bar{\mathbf{X}}_{n,m}(\bar{\mathbf{r}}) \right] \right|^2. \quad (2.77)$$

In this case

$$b_{n,m}^{TE} = \frac{1}{g_{cal}} d_{n,m}^{TE}, \quad (2.78)$$

$$b_{n,m}^{TM} = \frac{1}{g_{cal}} d_{n,m}^{TM}, \quad (2.79)$$

and the mutual impedance is

$$z_{12} = z_{21} = \frac{1}{4\pi} \operatorname{Re}(Z_{11}) \sum_{n=1}^{\infty} \sum_{m=-n}^n (-1)^m \left( {}_1d_{n,-m}^{TE} {}_2d_{n,m}^{TE} + {}_2d_{n,m}^{TM} {}_1d_{n,-m}^{TM} \right). \quad (2.80)$$

This Page Intentionally Contains No Relevant Information

## Appendix I — Vector Spherical Harmonics

### General vector harmonics

The vector wave equation is

$$\nabla^2 \bar{\mathbf{F}} + k^2 \bar{\mathbf{F}} = \nabla \nabla \cdot \bar{\mathbf{F}} - \nabla \times \nabla \times \bar{\mathbf{F}} + k^2 \bar{\mathbf{F}} = 0. \quad (\text{I-1})$$

Three independent solutions of (I-1) are [15]

$$\bar{\mathbf{F}} = \bar{\mathbf{L}} = \nabla \psi, \quad (\text{I-2})$$

$$\bar{\mathbf{F}} = \bar{\mathbf{M}} = \nabla \times \hat{\mathbf{a}} \psi, \quad (\text{I-3})$$

and

$$\bar{\mathbf{F}} = \bar{\mathbf{N}} = \frac{1}{k} \nabla \times \bar{\mathbf{M}} = \frac{1}{k} \nabla \times \nabla \times \hat{\mathbf{a}} \psi, \quad (\text{I-4})$$

where  $\psi$  is a solution of the scalar wave equation

$$\nabla^2 \psi + k^2 \psi = 0, \quad (\text{I-5})$$

and  $\hat{\mathbf{a}}$  is a unit vector. That  $\bar{\mathbf{L}}$  is a solution of (I-1) is easily demonstrated. We have

$$\nabla^2 \bar{\mathbf{L}} + k^2 \bar{\mathbf{L}} = \nabla \nabla \cdot (\nabla \psi) - \nabla \times \nabla \times (\nabla \psi) + \nabla k^2 \psi. \quad (\text{I-6})$$

Since  $\nabla \times \nabla \psi \equiv 0$ , we have

$$\nabla (\nabla^2 \psi + k^2 \psi) = 0. \quad (\text{I-7})$$

Substituting  $\bar{\mathbf{M}}$  into (I-1)

$$\nabla^2 \bar{\mathbf{M}} + k^2 \bar{\mathbf{M}} = \nabla^2 (\nabla \times \hat{\mathbf{a}} \psi) + k^2 (\nabla \times \hat{\mathbf{a}} \psi). \quad (\text{I-8})$$

Since  $\nabla^2$  commutes with  $\nabla \times$  (see Appendix II), we see that

$$\nabla \times (\nabla^2 \hat{\mathbf{a}} \psi + k^2 \hat{\mathbf{a}} \psi) = \nabla \times \hat{\mathbf{a}} (\nabla^2 \psi + k^2 \psi) = 0. \quad (\text{I-9})$$

Similarly, substituting  $\bar{\mathbf{N}}$  into (I-1)

$$\nabla^2 \bar{\mathbf{N}} + k^2 \bar{\mathbf{N}} = \nabla^2 \left( \frac{1}{k} \nabla \times \bar{\mathbf{M}} \right) + k^2 \left( \frac{1}{k} \nabla \times \bar{\mathbf{M}} \right). \quad (\text{I-10})$$

Again, using the commutation property of  $\nabla^2$  and  $\nabla \times$

$$\nabla^2 \bar{\mathbf{N}} + k^2 \bar{\mathbf{N}} = \frac{1}{k} \nabla \times (\nabla^2 \bar{\mathbf{M}} + k^2 \bar{\mathbf{M}}) = 0. \quad (\text{I-11})$$

From (I-3) and (I-4)  $\bar{\mathbf{M}}$  and  $\bar{\mathbf{N}}$  have no divergence,

$$\nabla \cdot \bar{\mathbf{M}} = 0, \quad (\text{I-12})$$

and

$$\nabla \cdot \bar{\mathbf{N}} = 0. \quad (\text{I-13})$$

Also, from (I-2)  $\bar{\mathbf{L}}$  has no curl,

$$\nabla \times \bar{\mathbf{L}} = 0, \quad (\text{I-14})$$

and since  $\psi$  is a solution of the wave equation,

$$\nabla \cdot \bar{\mathbf{L}} = \nabla^2 \psi = -k^2 \psi.$$

When the curl operator,  $\nabla \times$ , and  $\hat{\mathbf{a}}$  are anti-commutative (which is true when  $\hat{\mathbf{a}} = \text{constant}$  or  $\hat{\mathbf{a}} = \hat{\mathbf{r}}$ , see Appendix II), the vector harmonics are also related by

$$\bar{\mathbf{M}} = \nabla \times \hat{\mathbf{a}}\psi = -\hat{\mathbf{a}} \times \nabla \psi = -\hat{\mathbf{a}} \times \bar{\mathbf{L}} = \frac{1}{k} \nabla \times \bar{\mathbf{N}}. \quad (\text{I-15})$$

## Vector harmonics in the spherical coordinate system

In the spherical coordinate system, the requirement that  $\hat{\mathbf{a}}$  be a constant can be relaxed to the extent that it can be replaced by the radial unit vector  $\hat{\mathbf{r}}$  [15], because the curl operator,  $\nabla \times$ , and  $\hat{\mathbf{r}}$  are anti-commutative.

The set of vector spherical harmonics used here based on the normalized radially-independent vector spherical harmonic and notation as defined in Jackson [8]

$$\bar{\mathbf{X}}_{n,m}(\theta, \phi) \equiv \frac{1}{j\sqrt{n(n+1)}} \bar{\mathbf{r}} \times \nabla Y_{n,m}(\theta, \phi), \quad (\text{I-16})$$

where  $Y_{n,m}(\theta, \phi)$  are the scalar spherical harmonics given by

$$Y_{n,m}(\theta, \phi) \equiv \sqrt{\frac{2n+1}{4\pi} \frac{(n-m)!}{(n+m)!}} P_n^m(\cos\theta) e^{jm\phi}, \quad (\text{I-17})$$

and  $P_n^m(x)$  is the associated Legendre function given by <sup>†</sup>

$$P_n^m(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_n(x), \quad (\text{I-18})$$

where  $P_n(x)$  is the Legendre function

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n. \quad (\text{I-19})$$

For negative order,  $m$ , [8]

$$P_n^{-m}(x) = (-1)^m \frac{(n-m)!}{(n+m)!} P_n^m(x), \quad (\text{I-20})$$

so that

$$Y_{n,-m}(\theta, \phi) = (-1)^m Y_{n,m}^*(\theta, \phi). \quad (\text{I-21})$$

The scalar spherical harmonics,  $Y_{n,m}(\theta, \phi)$  are orthonormal [8], so that

---

<sup>†</sup> Hanson [13], Stratton [15], Arfken [16], and Mathews and Walker [18] omit the factor  $(-1)^m$ , but it is included by Jackson [8], Abramowitz and Stegun [9], Chew [10], Lebedev [11], and Balanis [3].

$$\iint_{\Omega} Y_{n,m}(\theta, \phi) Y_{n',m'}^*(\theta, \phi) d\Omega = \delta_{n,n'} \delta_{m,m'}, \quad (\text{I-22})$$

where  $\iint_{\Omega} d\Omega$  is  $\int_0^{\pi} \sin \theta d\theta \int_0^{2\pi} d\phi$ . The radially independent vector spherical harmonic,

$\bar{\mathbf{X}}_{n,m}$ , satisfies the orthogonality relation [8]

$$\iint_{\Omega} \bar{\mathbf{X}}_{n,m} \cdot \bar{\mathbf{X}}_{n',m'}^* d\Omega = \delta_{n,n'} \delta_{m,m'}. \quad (\text{I-23})$$

The explicit form for  $\bar{\mathbf{X}}_{n,m}$  is

$$\bar{\mathbf{X}}_{n,m}(\bar{\mathbf{r}}) = j \sqrt{\frac{(2n+1)(n-m)!}{4\pi n(n+1)(n+m)!}} \left[ \begin{array}{c} \frac{jm}{\sin \theta} P_n^m(\cos \theta) e^{jm\phi} \hat{\boldsymbol{\theta}} \\ + \sin \theta \left. \frac{d}{dx} P_n^m(x) \right|_{x=\cos \theta} e^{jm\phi} \hat{\boldsymbol{\phi}} \end{array} \right], \quad (\text{I-24})$$

or

$$\bar{\mathbf{X}}_{n,m}(\bar{\mathbf{r}}) = \frac{j}{\sqrt{n(n+1)}} \left[ \frac{jm}{\sin \theta} Y_{n,m}(\theta, \phi) \hat{\boldsymbol{\theta}} + \sin \theta \frac{\partial}{\partial \cos \theta} Y_{n,m}(\theta, \phi) \hat{\boldsymbol{\phi}} \right]. \quad (\text{I-25})$$

From (I-21), we see that

$$\bar{\mathbf{X}}_{n,-m}(\bar{\mathbf{r}}) = (-1)^m \frac{j}{\sqrt{n(n+1)}} \left[ \frac{-jm}{\sin \theta} Y_{n,m}^*(\theta, \phi) \hat{\boldsymbol{\theta}} + \sin \theta \frac{\partial}{\partial \cos \theta} Y_{n,m}^*(\theta, \phi) \hat{\boldsymbol{\phi}} \right]. \quad (\text{I-26})$$

The usefulness of the vector spherical harmonic derives from the fact it forms a solution of the vector wave equation as

$$\nabla^2 z_n^{(i)}(kr) \bar{\mathbf{X}}_{n,m}(\theta, \phi) + k^2 z_n^{(i)}(kr) \bar{\mathbf{X}}_{n,m}(\theta, \phi) \equiv 0, \quad (\text{I-27})$$

where  $z_n^{(i)}(kr)$  represents any of the spherical Bessel's functions

$$z_n^{(i)}(kr) = \begin{cases} h_n^{(1)}(kr); & i = 1 \\ h_n^{(2)}(kr); & i = 2 \\ j_n(kr); & i = 3 \\ y_n(kr); & i = 4 \end{cases}. \quad (\text{I-28})$$

$\nabla \times z_n^{(i)}(kr) \bar{\mathbf{X}}_{n,m}(\theta, \phi)$  is also a solution of the vector wave equation, since the operators  $\nabla^2$  and  $\nabla \times$  commute (see Appendix II),

$$\nabla^2 \left( \nabla \times z_n^{(i)}(kr) \bar{\mathbf{X}}_{n,m}(\theta, \phi) \right) + k^2 \left( \nabla \times z_n^{(i)}(kr) \bar{\mathbf{X}}_{n,m}(\theta, \phi) \right) \equiv 0. \quad (\text{I-29})$$

Thus,  $z_n^{(i)}(kr) \bar{\mathbf{X}}_{n,m}(\theta, \phi)$  and  $\nabla \times z_n^{(i)}(kr) \bar{\mathbf{X}}_{n,m}(\theta, \phi)$  are both harmonic solutions of the wave equation. The two types of vector spherical harmonics are defined in terms of  $\bar{\mathbf{X}}_{n,m}$  as

$$\bar{\mathbf{M}}_{n,m}^{(i)} \equiv z_n^{(i)}(kr) \bar{\mathbf{X}}_{n,m}(\theta, \phi), \quad (\text{I-30})$$

and

$$\bar{\mathbf{N}}_{n,m}^{(i)} \equiv \frac{1}{k} \nabla \times z_n^{(i)}(kr) \bar{\mathbf{X}}_{n,m}(\theta, \phi) = \frac{1}{k} \nabla \times \bar{\mathbf{M}}_{n,m}^{(i)}. \quad (\text{I-31})$$

It is obvious that  $\bar{\mathbf{M}}_{n,m}^{(i)}$  and  $\bar{\mathbf{N}}_{n,m}^{(i)}$  correspond to the general  $\bar{\mathbf{M}}$  and  $\bar{\mathbf{N}}$  harmonics defined in (I-3) and (I-4) for the special case

$$\psi = -\frac{z_n^{(i)}(kr)}{j\sqrt{n(n+1)}} Y_{n,m}(\theta, \phi), \quad (\text{I-32})$$

since

$$\hat{\mathbf{r}} \times \nabla z_n^{(i)}(kr) = 0. \quad (\text{I-33})$$

The  $\bar{\mathbf{M}}_{n,m}^{(i)}$  and  $\bar{\mathbf{N}}_{n,m}^{(i)}$  have no divergence, and the corresponding harmonic with no curl is

$$\bar{\mathbf{L}}_n^{(i)} \equiv \nabla \psi = -\nabla \frac{z_n^{(i)}(kr)}{j\sqrt{n(n+1)}} Y_{n,m}(\theta, \phi). \quad (\text{I-34})$$

Only the harmonics without divergence are of interest outside the source region, since the magnetic field has no divergence anywhere, and electric field has no divergence away from the sources.

It will be useful to note that

$$\begin{aligned} \nabla \times \bar{\mathbf{N}}_{l,m}^{(i)} &= \nabla \times \frac{1}{k} \nabla \times z_l^{(i)}(kr) \bar{\mathbf{X}}_{l,m}(\theta, \phi) = \nabla \times \frac{1}{k} \nabla \times \bar{\mathbf{M}}_{l,m}^{(i)} \\ &= \frac{1}{k} \nabla \nabla \cdot \bar{\mathbf{M}}_{l,m}^{(i)} - \frac{1}{k} \nabla^2 \bar{\mathbf{M}}_{l,m}^{(i)}, \end{aligned} \quad (\text{I-35})$$

and since  $\nabla \cdot \bar{\mathbf{M}}_{l,m}^{(i)} = 0$ , and  $\bar{\mathbf{M}}_{l,m}^{(i)}$  is a solution of the wave equation, we see that

$$\bar{\mathbf{M}}_{l,m}^{(i)} = \frac{1}{k} \nabla \times \bar{\mathbf{N}}_{l,m}^{(i)}. \quad (\text{I-36})$$

This is the same result as (I-15) for the general vector-harmonic case.

The explicit forms for the normalized vector spherical harmonics are

$$\bar{\mathbf{L}}_n^{(i)} = j \sqrt{\frac{2n+1}{4\pi n(n+1)} \frac{(n-m)!}{(n+m)!}} \left[ \begin{aligned} &\frac{d}{dr} z_n^{(i)}(kr) P_n^m(\cos\theta) e^{jm\phi} \hat{\mathbf{r}} - \frac{\sin\theta}{r} z_n^{(i)}(kr) \frac{d}{dx} P_n^m(x) \Big|_{x=\cos\theta} e^{jm\phi} \hat{\boldsymbol{\theta}} \\ &+ \frac{jm}{r \sin\theta} z_n^{(i)}(kr) P_n^m(\cos\theta) e^{jm\phi} \hat{\boldsymbol{\phi}} \end{aligned} \right] \quad (\text{I-37})$$

$$\bar{\mathbf{M}}_{n,m}^{(i)}(\bar{\mathbf{r}}) = j \sqrt{\frac{(2n+1)(n-m)!}{4\pi n(n+1)(n+m)!}} \left[ \begin{array}{l} \frac{jm}{\sin\theta} z_n^{(i)}(kr) P_n^m(\cos\theta) e^{jm\phi} \hat{\boldsymbol{\theta}} \\ + \sin\theta z_n^{(i)}(kr) \frac{d}{dx} P_n^m(x) \Big|_{x=\cos\theta} e^{jm\phi} \hat{\boldsymbol{\phi}} \end{array} \right], \quad (\text{I-38})$$

and

$$\bar{\mathbf{N}}_{n,m}^{(i)}(\bar{\mathbf{r}}) = j \sqrt{\frac{(2n+1)(n-m)!}{4\pi n(n+1)(n+m)!}} \left\{ \begin{array}{l} \frac{z_n^{(i)}(kr)}{kr} n(n+1) P_n^m(\cos\theta) e^{jm\phi} \hat{\mathbf{r}} \\ - \frac{1}{kr} \frac{\partial}{\partial r} [rz_n^{(i)}(kr)] \sin\theta \frac{d}{dx} P_n^m(x) \Big|_{x=\cos\theta} e^{jm\phi} \hat{\boldsymbol{\theta}} \\ + \frac{1}{kr} \frac{\partial}{\partial r} [rz_n^{(i)}(kr)] \frac{jm}{\sin\theta} P_n^m(\cos\theta) e^{jm\phi} \hat{\boldsymbol{\phi}} \end{array} \right\}. \quad (\text{I-39})$$

Since

$$\hat{\mathbf{r}} \times \bar{\mathbf{X}}_{n,m}(\theta, \phi) = j \sqrt{\frac{2n+1}{4\pi n(n+1)}} \frac{(n-m)!}{(n+m)!} \left[ -\sin\theta \frac{\partial P_n^m(x)}{\partial x} \Big|_{x=\cos\theta} \hat{\boldsymbol{\theta}} + \frac{jm}{\sin\theta} P_n^m(\cos\theta) \hat{\boldsymbol{\phi}} \right] e^{jm\phi} \quad (\text{I-40})$$

we see that

$$\begin{aligned} \bar{\mathbf{N}}_{n,m}^{(i)}(\bar{\mathbf{r}}) &= j \sqrt{\frac{(2n+1)(n-m)!}{4\pi n(n+1)(n+m)!}} \frac{z_n^{(i)}(kr)}{kr} n(n+1) P_n^m(\cos\theta) e^{jm\phi} \hat{\mathbf{r}} \\ &\quad + \frac{1}{kr} \frac{\partial}{\partial r} [rz_n^{(i)}(kr)] \hat{\mathbf{r}} \times \bar{\mathbf{X}}_{n,m}(\theta, \phi) \end{aligned}, \quad (\text{I-41})$$

or

$$\bar{\mathbf{N}}_{n,m}^{(i)}(\bar{\mathbf{r}}) = j \sqrt{n(n+1)} \frac{z_n^{(i)}(kr)}{kr} Y_n^m(\theta, \phi) \hat{\mathbf{r}} + \frac{1}{kr} \frac{\partial}{\partial r} [rz_n^{(i)}(kr)] \hat{\mathbf{r}} \times \bar{\mathbf{X}}_{n,m}(\theta, \phi), \quad (\text{I-42})$$

and the transverse part of  $\bar{\mathbf{N}}_{n,m}^{(i)}(\bar{\mathbf{r}})$  is simply

$$\bar{\mathbf{N}}_{n,m}^{(i)}(\bar{\mathbf{r}}) - \bar{\mathbf{N}}_{n,m}^{(i)}(\bar{\mathbf{r}}) \cdot \hat{\mathbf{r}} \hat{\mathbf{r}} = \frac{1}{kr} \frac{\partial}{\partial r} [rz_n^{(i)}(kr)] \hat{\mathbf{r}} \times \bar{\mathbf{X}}_{n,m}(\theta, \phi). \quad (\text{I-43})$$

These normalized vector spherical harmonics differ from those defined by Chew [10] and Stratton [15]. The harmonics used by Chew must be multiplied by a factor of  $j/\sqrt{n(n+1)}$  to produce the normalized harmonics, and from those defined by Stratton

must be multiplied by a factor of  $j(-1)^m \sqrt{\frac{(2n+1)(n-m)!}{4\pi n(n+1)(n+m)!}}$  to yield the normalized

harmonics. Hanson [13] uses normalized harmonics almost identical to those used here. However, Hanson's harmonics, which are written with a different notation, must be multiplied by a factor of  $j(m/|m|)^m$  to produce the ones defined here. The relation between Hanson's notation and that used here is

$$\bar{\mathbf{M}}_{n,m}^{(i)}(\bar{\mathbf{r}}) = j \left( \frac{m}{|m|} \right)^m \bar{\mathbf{F}}_{1,m,n}^{(i)}(\bar{\mathbf{r}}), \quad (\text{I-44})$$

and

$$\bar{\mathbf{N}}_{n,m}^{(i)}(\bar{\mathbf{r}}) = j \left( \frac{m}{|m|} \right)^m \bar{\mathbf{F}}_{2,m,n}^{(i)}(\bar{\mathbf{r}}). \quad (\text{I-45})$$

A far-field representation of the vector spherical harmonics is obtained for out-going waves, where the  $z_n^i(kr)$  become  $h_n^{(2)}(kr)$ . Since

$$h_n^{(2)}(kr) = j^{n+1} \frac{e^{-jkr}}{kr} \sum_{p=0}^n \frac{\Gamma(n+p+1)}{p! \Gamma(n-p+1)} (2jkr)^{-p}, \quad (\text{I-46})$$

the approximation for the far field region is,

$$h_n^{(2)}(kr) \cong j^{n+1} \frac{e^{-jkr}}{kr} \text{ as } kr \rightarrow \infty, \quad (\text{I-47})$$

and

$$\left. \frac{\partial}{\partial z} h_n^{(2)}(z) \right|_{z=kr} \cong j^n \frac{e^{-jkr}}{kr} \text{ as } kr \rightarrow \infty. \quad (\text{I-48})$$

Thus, in the far field region,

$$\bar{\mathbf{M}}_{n,m}^{(2)}(\bar{\mathbf{r}}) = -\frac{j^n e^{-jkr}}{kr} \sqrt{\frac{(2n+1)(n-m)!}{4\pi n(n+1)(n+m)!}} \left[ \frac{jm}{\sin \theta} P_n^m(\cos \theta) \hat{\boldsymbol{\theta}} + \sin \theta \frac{d}{dx} P_n^m(x) \Big|_{x=\cos \theta} \hat{\boldsymbol{\phi}} \right] e^{jm\phi}, \quad (\text{I-49})$$

and

$$\bar{\mathbf{N}}_{n,m}^{(2)}(\bar{\mathbf{r}}) = \frac{j^{n-1} e^{-jkr}}{kr} \sqrt{\frac{2n+1}{4\pi n(n+1)(n+m)!}} \left[ \sin \theta \frac{d}{dx} P_n^m(x) \Big|_{x=\cos \theta} \hat{\boldsymbol{\theta}} - \frac{jm}{\sin \theta} P_n^m(\cos \theta) \hat{\boldsymbol{\phi}} \right] e^{jm\phi}, \quad (\text{I-50})$$

or simply

$$\bar{\mathbf{M}}_{n,m}^{(2)}(\bar{\mathbf{r}}) = \frac{j^{n+1} e^{-jkr}}{kr} \bar{\mathbf{X}}_{n,m}(\theta, \phi), \quad (\text{I-51})$$

and

$$\bar{\mathbf{N}}_{n,m}^{(2)}(\bar{\mathbf{r}}) = \frac{j^n e^{-jkr}}{kr} \hat{\mathbf{r}} \times \bar{\mathbf{X}}_{n,m}(\theta, \phi), \quad (\text{I-52})$$

for the normalized vector spherical harmonics.

The orthogonality integrals for the normalized vector harmonics are [8]

$$\begin{aligned} \mathcal{I}_{\bar{\mathbf{M}}\bar{\mathbf{M}}}(m, n; m', n'; r) &= \int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta \bar{\mathbf{M}}_{n,m}^{(i)} \cdot (\bar{\mathbf{M}}_{n',m'}^{(i)})^* \\ &= \left| z_n^{(i)}(kr) \right|^2 \delta_{m'm} \delta_{nn'}, \end{aligned} \quad (\text{I-53})$$

and

$$\begin{aligned}
\mathcal{J}_{\bar{\mathbf{N}}\cdot\bar{\mathbf{N}}}(m, n; m', n'; r) &\equiv \int_0^{2\pi} d\phi \int_0^\pi \sin\theta d\theta \bar{\mathbf{N}}_{n,m}^{(i)} \cdot (\bar{\mathbf{N}}_{n',m'}^{(i)})^* \\
&= \left[ \frac{|z_n^{(i)}(kr)|^2}{k^2 r^2} n(n+1) + \frac{1}{k^2 r^2} \left| \frac{\partial}{\partial r} [rz_n^{(i)}(kr)] \right|^2 \right] \delta_{m'} \delta_{mm'} \\
&= \left[ |z_n^{(i)}(kr)|^2 + \frac{1}{k^2 r^2} \frac{\partial}{\partial r} [rz_n^{(i)}(kr)] \frac{\partial}{\partial r} [rz_n^{(i)*}(kr)] \right] \delta_{m'} \delta_{mm'}
\end{aligned} \tag{I-54}$$

or, expanding (I-54),

$$\mathcal{J}_{\bar{\mathbf{N}}\cdot\bar{\mathbf{N}}}(m, n; m', n'; r) = \left[ \begin{aligned} & \left| z_n^{(i)}(kr) \right|^2 \left( 1 + \frac{1}{k^2 r^2} \right) + \left| \frac{d}{dx} z_n^{(i)}(x) \Big|_{x=kr} \right|^2 \\ & + \frac{z_n^{(i)*}(kr)}{kr} \frac{d}{dx} z_n^{(i)}(x) \Big|_{x=kr} + 3 \frac{z_n^{(i)}(kr)}{kr} \frac{d}{dx} z_n^{(i)*}(x) \Big|_{x=kr} \\ & + z_n^{(i)}(kr) \frac{d^2}{dx^2} z_n^{(i)*}(x) \Big|_{x=kr} \end{aligned} \right] \delta_{m'} \delta_{mm'} \tag{I-55}$$

and

$$\begin{aligned}
\mathcal{J}_{\bar{\mathbf{M}}\cdot\bar{\mathbf{N}}}(m, n, m', n'; r) &\equiv \int_0^{2\pi} d\phi \int_0^\pi \sin\theta d\theta \bar{\mathbf{M}}_{n,m}^{(i)} \cdot (\bar{\mathbf{N}}_{n',m'}^{(i)})^* \\
&= 0
\end{aligned} \tag{I-56}$$

When  $r$  is small, the  $\hat{\mathbf{r}}$  component of  $\bar{\mathbf{N}}_{n,m}^{(i)}$  can be significant. However, it will be necessary to expand a measured field in terms of the vector spherical harmonics, and the  $\hat{\mathbf{r}}$  component of the field is typically not measured. Thus, for small  $r$ , the use of (I-55) for the normalization can produce an error. Instead, we need the orthogonality integral for the transverse part of  $\bar{\mathbf{N}}_{n,m}^{(i)}$

$$\begin{aligned}
\mathcal{J}_{(\bar{\mathbf{N}}\cdot\bar{\mathbf{N}})\cdot\hat{\mathbf{r}}\cdot\bar{\mathbf{N}}}(m, n; m', n'; r) &\equiv \int_0^{2\pi} d\phi \int_0^\pi \sin\theta d\theta (\bar{\mathbf{N}}_{n,m}^{(i)} - \bar{\mathbf{N}}_{n,m}^{(i)} \cdot \hat{\mathbf{r}} \hat{\mathbf{r}}) \cdot (\bar{\mathbf{N}}_{n',m'}^{(i)})^* \\
&= \frac{1}{k^2 r^2} \left| \frac{\partial}{\partial r} [rz_n^{(i)}(kr)] \right|^2 \delta_{m'} \delta_{mm'}
\end{aligned} \tag{I-57}$$

### **Special combinations of the vector spherical harmonics**

Consider the far-field representation of  $\bar{\mathbf{M}}_{1,\pm 1}^{(2)}(\bar{\mathbf{r}})$  obtained from (I-51)

$$\bar{\mathbf{M}}_{1,-1}^{(2)}(\bar{\mathbf{r}}) = j \frac{e^{-jkr}}{kr} \frac{1}{2} \sqrt{\frac{3}{4\pi}} [\hat{\boldsymbol{\theta}} - j \cos\theta \hat{\boldsymbol{\phi}}] e^{-j\phi}, \tag{I-58}$$

and

$$\bar{\mathbf{M}}_{1,1}^{(2)}(\bar{\mathbf{r}}) = j \frac{e^{-jkr}}{kr} \frac{1}{2} \sqrt{\frac{3}{4\pi}} [\hat{\boldsymbol{\theta}} + j \cos\theta \hat{\boldsymbol{\phi}}] e^{j\phi}. \tag{I-59}$$

For a TE field,  $\bar{\mathbf{M}}_{1,-1}^{(2)}(\bar{\mathbf{r}})$  represents a right-hand circularly polarized electric field at  $\theta = 0^\circ$ , while  $\bar{\mathbf{M}}_{1,1}^{(2)}(\bar{\mathbf{r}})$  is the left-hand circularly polarized electric field. Now consider the far-field representation of  $\bar{\mathbf{N}}_{1,\pm 1}^{(2)}(\bar{\mathbf{r}})$ , from (I-52)

$$\bar{\mathbf{N}}_{1,-1}^{(2)}(\bar{\mathbf{r}}) = \frac{e^{-jkr}}{kr} \frac{1}{2} \sqrt{\frac{3}{4\pi}} \left[ -\cos\theta \hat{\boldsymbol{\theta}} + j\hat{\boldsymbol{\phi}} \right] e^{-j\phi}, \quad (\text{I-60})$$

and,

$$\bar{\mathbf{N}}_{1,1}^{(2)}(\bar{\mathbf{r}}) = \frac{e^{-jkr}}{kr} \frac{1}{2} \sqrt{\frac{3}{4\pi}} \left[ \cos\theta \hat{\boldsymbol{\theta}} + j\hat{\boldsymbol{\phi}} \right] e^{j\phi}. \quad (\text{I-61})$$

Similarly, for the TM field,  $\bar{\mathbf{N}}_{1,-1}^{(2)}(\bar{\mathbf{r}})$  is the right-hand circularly polarized electric field when  $\theta = 0^\circ$ , and  $\bar{\mathbf{N}}_{1,1}^{(2)}(\bar{\mathbf{r}})$  is left-hand circularly polarized. At  $\theta \neq 0^\circ$ , these fields are elliptical.

For a TE field,

$$\bar{\mathbf{M}}_{1,1}^{(2)}(\bar{\mathbf{r}}) + \bar{\mathbf{M}}_{1,-1}^{(2)}(\bar{\mathbf{r}}) = j \frac{e^{-jkr}}{kr} \sqrt{\frac{3}{4\pi}} \left[ \cos\phi \hat{\boldsymbol{\theta}} - \cos\theta \sin\phi \hat{\boldsymbol{\phi}} \right], \quad (\text{I-62})$$

$$\bar{\mathbf{M}}_{1,1}^{(2)}(\bar{\mathbf{r}}) - \bar{\mathbf{M}}_{1,-1}^{(2)}(\bar{\mathbf{r}}) = -\frac{e^{-jkr}}{kr} \sqrt{\frac{3}{4\pi}} \left[ \sin\phi \hat{\boldsymbol{\theta}} + \cos\phi \cos\theta \hat{\boldsymbol{\phi}} \right], \quad (\text{I-63})$$

which represent magnetic dipoles.

For the TM field

$$\bar{\mathbf{N}}_{1,1}^{(2)}(\bar{\mathbf{r}}) + \bar{\mathbf{N}}_{1,-1}^{(2)}(\bar{\mathbf{r}}) = j \frac{e^{-jkr}}{kr} \sqrt{\frac{3}{4\pi}} \left[ \cos\theta \sin\phi \hat{\boldsymbol{\theta}} + \cos\phi \hat{\boldsymbol{\phi}} \right], \quad (\text{I-64})$$

$$\bar{\mathbf{N}}_{1,1}^{(2)}(\bar{\mathbf{r}}) - \bar{\mathbf{N}}_{1,-1}^{(2)}(\bar{\mathbf{r}}) = \frac{e^{-jkr}}{kr} \sqrt{\frac{3}{4\pi}} \left[ \cos\theta \cos\phi \hat{\boldsymbol{\theta}} - \sin\phi \hat{\boldsymbol{\phi}} \right], \quad (\text{I-65})$$

$$(1-j)\bar{\mathbf{N}}_{1,1}^{(2)}(\bar{\mathbf{r}}) - (1+j)\bar{\mathbf{N}}_{1,-1}^{(2)}(\bar{\mathbf{r}}) = \frac{e^{-jkr}}{kr} \sqrt{\frac{3}{4\pi}} \left[ \cos\theta (\sin\phi + \cos\phi) \hat{\boldsymbol{\theta}} - (\sin\phi - \cos\phi) \hat{\boldsymbol{\phi}} \right], \quad (\text{I-66})$$

and

$$(1-j)\bar{\mathbf{N}}_{1,-1}^{(2)}(\bar{\mathbf{r}}) - (1+j)\bar{\mathbf{N}}_{1,1}^{(2)}(\bar{\mathbf{r}}) = \frac{e^{-jkr}}{kr} \sqrt{\frac{3}{4\pi}} \left[ \cos\theta (\sin\phi - \cos\phi) \hat{\boldsymbol{\theta}} + (\sin\phi + \cos\phi) \hat{\boldsymbol{\phi}} \right]. \quad (\text{I-67})$$

Equations (I-64), (I-65), (I-66), and (I-67) describe small linear dipoles oriented as  $E_\phi$ ,  $E_\theta$ ,  $E_{45^\circ}$ , and  $E_{135^\circ}$  dipoles, respectively.

**Explicit forms for the associated Legendre function and its derivative**

Explicit Forms for $P_n^m(x)$						
$n$	$m$					
	0	1	2	3	4	5
0	1	0	0	0	0	0
1	$x$	$-\sqrt{1-x^2}$	0	0	0	0
2	$\frac{3x^2-1}{2}$	$-3x\sqrt{1-x^2}$	$3(1-x^2)$	0	0	0
3	$\frac{5x^3-3x}{2}$	$-3\frac{5x^2-1}{2}\sqrt{1-x^2}$	$15x(1-x^2)$	$-15(1-x^2)^{3/2}$	0	0
4	$\frac{35x^4-30x^2+3}{8}$	$-5\frac{7x^3-3x}{2}\sqrt{1-x^2}$	$15\frac{7x^2-1}{2}(1-x^2)$	$-105x(1-x^2)^{3/2}$	$105(1-x^2)^2$	0
5	$\frac{63x^5-70x^3+15x}{8}$	$-15\frac{21x^4-14x^2+1}{8}\sqrt{1-x^2}$	$105\frac{3x^3-x}{2}(1-x^2)$	$-105\frac{9x^2-1}{2}(1-x^2)^{3/2}$	$945x(1-x^2)^2$	$-945(1-x^2)^{5/2}$

Explicit Forms for $P_n^m(\cos\theta)$						
$n$	$m$					
	0	1	2	3	4	5
0	1	0	0	0	0	0
1	$\cos\theta$	$-\sin\theta$	0	0	0	0
2	$\frac{3\cos^2\theta-1}{2}$	$-3\sin\theta\cos\theta$	$3\sin^2\theta$	0	0	0
3	$\frac{5\cos^3\theta-3\cos\theta}{2}$	$-3\frac{5\cos^2\theta-1}{2}\sin\theta$	$15\cos\theta\sin^2\theta$	$-15\sin^3\theta$	0	0
4	$\frac{35\cos^4\theta-30\cos^2\theta+3}{8}$	$-5\frac{7\cos^3\theta-3\cos\theta}{2}\sin\theta$	$15\frac{7\cos^2\theta-1}{2}\sin^2\theta$	$-105\cos\theta\sin^3\theta$	$105\sin^4\theta$	0
5	$\frac{63\cos^5\theta-70\cos^3\theta+15\cos\theta}{8}$	$-15\frac{21\cos^4\theta-14\cos^2\theta+1}{8}\sin\theta$	$105\frac{3\cos^3\theta-\cos\theta}{2}\sin^2\theta$	$-105\frac{9\cos^2\theta-1}{2}\sin^3\theta$	$945\cos\theta\sin^4\theta$	$-945\sin^5\theta$

Explicit Forms for $\frac{d}{dx} P_n^m(x)$						
$n$	$m$					
	0	1	2	3	4	5
0	0	0	0	0	0	0
1	1	$\frac{x}{\sqrt{1-x^2}}$	0	0	0	0
2	$3x$	$3\frac{2x^2-1}{\sqrt{1-x^2}}$	$-6x$	0	0	0
3	$3\frac{5x^2-1}{2}$	$3\frac{15x^3-11x}{2\sqrt{1-x^2}}$	$15(1-3x^2)$	$45x\sqrt{1-x^2}$	0	0
4	$5\frac{7x^3-3x}{2}$	$5\frac{28x^4-27x^2+3}{2\sqrt{1-x^2}}$	$30(4x-7x^3)$	$105(4x^2-1)\sqrt{1-x^2}$	$-420x(1-x^2)$	0
5	$15\frac{21x^4-14x^2+1}{8}$	$15\frac{105x^5-126x^3+29x}{8\sqrt{1-x^2}}$	$-105\frac{15x^4-12x^2+1}{2}$	$315\frac{15x^3-7x}{2}\sqrt{1-x^2}$	$945(1-x^2)(1-5x^2)$	$4725x(1-x^2)^{3/2}$

Explicit Forms for $\frac{d}{dx} P_n^m(\cos\theta)$						
$n$	$m$					
	0	1	2	3	4	5
0	0	0	0	0	0	0
1	1	$\frac{\cos\theta}{\sin\theta}$	0	0	0	0
2	$3\cos\theta$	$3\frac{2\cos^2\theta-1}{\sin\theta}$	$-6\cos\theta$	0	0	0
3	$3\frac{5\cos^2\theta-1}{2}$	$3\frac{15\cos^3\theta-11\cos\theta}{2\sin\theta}$	$15(1-3\cos^2\theta)$	$45\cos\theta\sin\theta$	0	0
4	$5\frac{7\cos^3\theta-3\cos\theta}{2}$	$5\frac{28\cos^4\theta-27\cos^2\theta+3}{2\sin\theta}$	$30(4\cos\theta-7\cos^3\theta)$	$105(4\cos^2\theta-1)\sin\theta$	$-420\cos\theta\sin^2\theta$	0
5	$15\frac{21\cos^4\theta-14\cos^2\theta+1}{8}$	$15\frac{105\cos^5\theta-126\cos^3\theta+29\cos\theta}{8\sin\theta}$	$-105\frac{15\cos^4\theta-12\cos^2\theta+1}{2}$	$315\frac{15\cos^3\theta-7\cos\theta}{2}\sin\theta$	$945\sin^2\theta(1-5\cos^2\theta)$	$4725\cos\theta\sin^3\theta$

**Explicit forms for the scalar spherical harmonic and its derivative**

Explicit Forms for $Y_{n,m}(\cos\theta)$					
$n$	$m$				
	0	$\pm 1$	$\pm 2$	$\pm 3$	$\pm 4$
0	$\sqrt{\frac{1}{4\pi}}$	0	0	0	0
1	$\sqrt{\frac{3}{4\pi}} \cos\theta$	$\mp \frac{1}{2} \sqrt{\frac{3}{2\pi}} e^{\pm j\phi} \sin\theta$	0	0	0
2	$\sqrt{\frac{5}{4\pi}} \frac{3\cos^2\theta - 1}{2}$	$\mp \frac{3}{2} \sqrt{\frac{5}{6\pi}} e^{\pm j\phi} \sin\theta \cos\theta$	$\frac{3}{4} \sqrt{\frac{5}{6\pi}} e^{\pm j2\phi} \sin^2\theta$	0	0
3	$\sqrt{\frac{7}{4\pi}} \frac{5\cos^3\theta - 3\cos\theta}{2}$	$\mp \frac{3}{4} \sqrt{\frac{7}{3\pi}} e^{\pm j\phi} \frac{5\cos^2\theta - 1}{2} \sin\theta$	$\frac{15}{4} \sqrt{\frac{7}{30\pi}} e^{\pm j2\phi} \cos\theta \sin^2\theta$	$\mp \frac{5}{8} \sqrt{\frac{7}{5\pi}} e^{\pm j3\phi} \sin^3\theta$	0
4	$3\sqrt{\frac{1}{4\pi}} \frac{35\cos^4\theta - 30\cos^2\theta + 3}{8}$	$\mp \frac{15}{4} \sqrt{\frac{1}{5\pi}} e^{\pm j\phi} \frac{7\cos^3\theta - 3\cos\theta}{2} \sin\theta$	$\frac{15}{4} \sqrt{\frac{1}{10\pi}} e^{\pm j2\phi} \frac{7\cos^2\theta - 1}{2} \sin^2\theta$	$\mp \frac{105}{8} \sqrt{\frac{1}{35\pi}} e^{\pm j3\phi} \cos\theta \sin^3\theta$	$\frac{105}{16} \sqrt{\frac{1}{70\pi}} e^{\pm j4\phi} \sin^4\theta$
Explicit Forms for $\frac{dY_{n,m}(\cos\theta)}{d\cos\theta}$					
$n$	$m$				
	0	$\pm 1$	$\pm 2$	$\pm 3$	$\pm 4$
0	0	0	0	0	0
1	$\sqrt{\frac{3}{4\pi}}$	$\pm \frac{1}{2} \sqrt{\frac{3}{2\pi}} e^{\pm j\phi} \frac{\cos\theta}{\sin\theta}$	0	0	0
2	$3\sqrt{\frac{3}{4\pi}} \cos\theta$	$\pm \frac{3}{2} \sqrt{\frac{5}{6\pi}} e^{\pm j\phi} \frac{2\cos^2\theta - 1}{\sin\theta}$	$-\frac{3}{2} \sqrt{\frac{5}{6\pi}} e^{\pm j2\phi} \cos\theta$	0	0
3	$3\sqrt{\frac{7}{4\pi}} \frac{5\cos^2\theta - 1}{2}$	$\pm \frac{3}{4} \sqrt{\frac{7}{3\pi}} e^{\pm j\phi} \frac{15\cos^3\theta - 11\cos\theta}{2\sin\theta}$	$\frac{15}{4} \sqrt{\frac{7}{30\pi}} e^{\pm j2\phi} (1 - 3\cos^2\theta)$	$\pm \frac{15}{8} \sqrt{\frac{7}{5\pi}} e^{\pm j3\phi} \cos\theta \sin\theta$	0
4	$3\sqrt{\frac{1}{4\pi}} \frac{35\cos^3\theta - 15\cos\theta}{2}$	$\pm \frac{15}{4} \sqrt{\frac{1}{5\pi}} e^{\pm j\phi} \frac{28\cos^4\theta - 27\cos^2\theta + 3}{2\sin\theta}$	$\frac{15}{2} \sqrt{\frac{1}{10\pi}} e^{\pm j2\phi} (4\cos\theta - 7\cos^3\theta)$	$\pm \frac{105}{8} \sqrt{\frac{1}{35\pi}} e^{\pm j3\phi} (4\cos^2\theta - 1) \sin\theta$	$-\frac{105}{4} \sqrt{\frac{1}{70\pi}} e^{\pm j4\phi} \cos\theta \sin^2\theta$

**Explicit expressions for the  $\bar{X}_{n,m}(\theta, \phi)$  vector spherical harmonic:**

$\bar{X}_{1,0}(\theta, \phi) = j \frac{1}{2} \sqrt{\frac{3}{2\pi}} \sin \theta \hat{\phi}$
$\bar{X}_{1,-1}(\theta, \phi) = \frac{1}{2} \sqrt{\frac{3}{4\pi}} e^{-j\phi} [\hat{\theta} - j \cos \theta \hat{\phi}]$
$\bar{X}_{1,1}(\theta, \phi) = \frac{1}{2} \sqrt{\frac{3}{4\pi}} e^{j\phi} [\hat{\theta} + j \cos \theta \hat{\phi}]$
$\bar{X}_{2,0}(\theta, \phi) = j \sqrt{\frac{5}{6\pi}} \frac{3 \cos^2 \theta - 1}{4} \sin \theta \hat{\phi}$
$\bar{X}_{2,-1}(\theta, \phi) = \frac{1}{4} \sqrt{\frac{5}{\pi}} e^{-j\phi} [\cos \theta \hat{\theta} - j(2 \cos^2 \theta - 1) \hat{\phi}]$
$\bar{X}_{2,1}(\theta, \phi) = \frac{1}{4} \sqrt{\frac{5}{\pi}} e^{j\phi} [\cos \theta \hat{\theta} + j(2 \cos^2 \theta - 1) \hat{\phi}]$
$\bar{X}_{2,-2}(\theta, \phi) = \frac{1}{4} \sqrt{\frac{5}{\pi}} e^{-j2\phi} [\sin \theta \hat{\theta} - j \sin \theta \cos \theta \hat{\phi}]$
$\bar{X}_{2,2}(\theta, \phi) = -\frac{1}{4} \sqrt{\frac{5}{\pi}} e^{j2\phi} [\sin \theta \hat{\theta} + j \sin \theta \cos \theta \hat{\phi}]$

**Explicit expressions for the  $\hat{\mathbf{r}} \times \bar{\mathbf{X}}_{n,m}(\theta, \phi)$  vector spherical harmonic:**

$\hat{\mathbf{r}} \times \bar{\mathbf{X}}_{1,0}(\theta, \phi) = -j \frac{1}{2} \sqrt{\frac{3}{2\pi}} \sin \theta \hat{\boldsymbol{\theta}}$
$\hat{\mathbf{r}} \times \bar{\mathbf{X}}_{1,-1}(\theta, \phi) = \frac{1}{2} \sqrt{\frac{3}{4\pi}} e^{-j\phi} [j \cos \theta \hat{\boldsymbol{\theta}} + \hat{\boldsymbol{\phi}}]$
$\hat{\mathbf{r}} \times \bar{\mathbf{X}}_{1,1}(\theta, \phi) = \frac{1}{2} \sqrt{\frac{3}{4\pi}} e^{j\phi} [-j \cos \theta \hat{\boldsymbol{\theta}} + \hat{\boldsymbol{\phi}}]$
$\hat{\mathbf{r}} \times \bar{\mathbf{X}}_{2,0}(\theta, \phi) = -j \sqrt{\frac{5}{6\pi}} \frac{3 \cos^2 \theta - 1}{4} \sin \theta \hat{\boldsymbol{\theta}}$
$\hat{\mathbf{r}} \times \bar{\mathbf{X}}_{2,-1}(\theta, \phi) = \frac{1}{4} \sqrt{\frac{5}{\pi}} e^{-j\phi} [j(2 \cos^2 \theta - 1) \hat{\boldsymbol{\theta}} + \cos \theta \hat{\boldsymbol{\phi}}]$
$\hat{\mathbf{r}} \times \bar{\mathbf{X}}_{2,1}(\theta, \phi) = \frac{1}{4} \sqrt{\frac{5}{\pi}} e^{j\phi} [-j(2 \cos^2 \theta - 1) \hat{\boldsymbol{\theta}} + \cos \theta \hat{\boldsymbol{\phi}}]$
$\hat{\mathbf{r}} \times \bar{\mathbf{X}}_{2,-2}(\theta, \phi) = \frac{1}{4} \sqrt{\frac{5}{\pi}} e^{-j2\phi} [j \sin \theta \cos \theta \hat{\boldsymbol{\theta}} + \sin \theta \hat{\boldsymbol{\phi}}]$
$\hat{\mathbf{r}} \times \bar{\mathbf{X}}_{2,2}(\theta, \phi) = \frac{1}{4} \sqrt{\frac{5}{\pi}} e^{j2\phi} [j \sin \theta \cos \theta \hat{\boldsymbol{\theta}} - \sin \theta \hat{\boldsymbol{\phi}}]$

**Explicit expressions for the  $\bar{\mathbf{M}}_{n,m}^{(i)}(\bar{\mathbf{r}})$  vector spherical harmonics:**

$\bar{\mathbf{M}}_{1,0}^{(i)}(\bar{\mathbf{r}}) = jz_1^{(i)}(kr) \frac{1}{2} \sqrt{\frac{3}{2\pi}} \sin \theta \hat{\phi}$
$\bar{\mathbf{M}}_{1,-1}^{(i)}(\bar{\mathbf{r}}) = z_1^{(i)}(kr) \frac{1}{2} \sqrt{\frac{3}{4\pi}} e^{-j\phi} [\hat{\theta} - j \cos \theta \hat{\phi}]$
$\bar{\mathbf{M}}_{1,1}^{(i)}(\bar{\mathbf{r}}) = z_1^{(i)}(kr) \frac{1}{2} \sqrt{\frac{3}{4\pi}} e^{j\phi} [\hat{\theta} + j \cos \theta \hat{\phi}]$
$\bar{\mathbf{M}}_{2,0}^{(i)}(\bar{\mathbf{r}}) = jz_2^{(i)}(kr) \sqrt{\frac{5}{6\pi}} \frac{3 \cos^2 \theta - 1}{4} \sin \theta \hat{\phi}$
$\bar{\mathbf{M}}_{2,-1}^{(i)}(\bar{\mathbf{r}}) = z_2^{(i)}(kr) \frac{1}{4} \sqrt{\frac{5}{\pi}} e^{-j\phi} [\cos \theta \hat{\theta} - j(2 \cos^2 \theta - 1) \hat{\phi}]$
$\bar{\mathbf{M}}_{2,1}^{(i)}(\bar{\mathbf{r}}) = z_2^{(i)}(kr) \frac{1}{4} \sqrt{\frac{5}{\pi}} e^{j\phi} [\cos \theta \hat{\theta} + j(2 \cos^2 \theta - 1) \hat{\phi}]$
$\bar{\mathbf{M}}_{2,-2}^{(i)}(\bar{\mathbf{r}}) = z_2^{(i)}(kr) \frac{1}{4} \sqrt{\frac{5}{\pi}} e^{-j2\phi} [\sin \theta \hat{\theta} - j \sin \theta \cos \theta \hat{\phi}]$
$\bar{\mathbf{M}}_{2,2}^{(i)}(\bar{\mathbf{r}}) = -z_2^{(i)}(kr) \frac{1}{4} \sqrt{\frac{5}{\pi}} e^{j2\phi} [\sin \theta \hat{\theta} + j \sin \theta \cos \theta \hat{\phi}]$

**Explicit expressions for the  $\bar{\mathbf{N}}_{n,m}^{(i)}(\bar{\mathbf{r}})$  vector spherical harmonics:**

$\bar{\mathbf{N}}_{1,0}^{(i)}(\bar{\mathbf{r}}) = j \frac{z_1^{(i)}(kr)}{kr} \sqrt{\frac{3}{2\pi}} \cos \theta \hat{\mathbf{r}} - j \frac{1}{kr} \frac{\partial}{\partial r} [rz_1^{(i)}(kr)] \frac{1}{2} \sqrt{\frac{3}{2\pi}} \sin \theta \hat{\boldsymbol{\theta}}$
$\bar{\mathbf{N}}_{1,-1}^{(i)}(\bar{\mathbf{r}}) = j \frac{z_1^{(i)}(kr)}{kr} \sqrt{\frac{3}{4\pi}} e^{-j\phi} \sin \theta \hat{\mathbf{r}} + \frac{1}{kr} \frac{\partial}{\partial r} [rz_1^{(i)}(kr)] \frac{1}{2} \sqrt{\frac{3}{4\pi}} e^{-j\phi} [j \cos \theta \hat{\boldsymbol{\theta}} + \hat{\boldsymbol{\phi}}]$
$\bar{\mathbf{N}}_{1,1}^{(i)}(\bar{\mathbf{r}}) = -j \frac{z_1^{(i)}(kr)}{kr} \sqrt{\frac{3}{4\pi}} e^{j\phi} \sin \theta \hat{\mathbf{r}} + \frac{1}{kr} \frac{\partial}{\partial r} [rz_1^{(i)}(kr)] \frac{1}{2} \sqrt{\frac{3}{4\pi}} e^{j\phi} [-j \cos \theta \hat{\boldsymbol{\theta}} + \hat{\boldsymbol{\phi}}]$
$\bar{\mathbf{N}}_{2,0}^{(i)}(\bar{\mathbf{r}}) = j \frac{z_2^{(i)}(kr)}{kr} \frac{1}{2} \sqrt{\frac{15}{2\pi}} (3 \cos^2 \theta - 1) \hat{\mathbf{r}} - j \frac{1}{kr} \frac{\partial}{\partial r} [rz_2^{(i)}(kr)] \sqrt{\frac{5}{6\pi}} \frac{3 \cos^2 \theta - 1}{4} \sin \theta \hat{\boldsymbol{\theta}}$
$\bar{\mathbf{N}}_{2,-1}^{(i)}(\bar{\mathbf{r}}) = j \frac{z_2^{(i)}(kr)}{kr} \frac{3}{2} \sqrt{\frac{5}{\pi}} e^{-j\phi} \sin \theta \cos \theta \hat{\mathbf{r}} + \frac{1}{kr} \frac{\partial}{\partial r} [rz_2^{(i)}(kr)] \frac{1}{4} \sqrt{\frac{5}{\pi}} e^{-j\phi} [j(2 \cos^2 \theta - 1) \hat{\boldsymbol{\theta}} + \cos \theta \hat{\boldsymbol{\phi}}]$
$\bar{\mathbf{N}}_{2,1}^{(i)}(\bar{\mathbf{r}}) = -j \frac{z_2^{(i)}(kr)}{kr} \frac{3}{2} \sqrt{\frac{5}{\pi}} e^{j\phi} \sin \theta \cos \theta \hat{\mathbf{r}} + \frac{1}{kr} \frac{\partial}{\partial r} [rz_2^{(i)}(kr)] \frac{1}{4} \sqrt{\frac{5}{\pi}} e^{j\phi} [-j(2 \cos^2 \theta - 1) \hat{\boldsymbol{\theta}} + \cos \theta \hat{\boldsymbol{\phi}}]$
$\bar{\mathbf{N}}_{2,-2}^{(i)}(\bar{\mathbf{r}}) = j \frac{z_2^{(i)}(kr)}{kr} \frac{3}{4} \sqrt{\frac{5}{\pi}} e^{-j2\phi} \sin^2 \theta \hat{\mathbf{r}} + \frac{1}{kr} \frac{\partial}{\partial r} [rz_2^{(i)}(kr)] \frac{1}{4} \sqrt{\frac{5}{\pi}} e^{-j2\phi} [j \sin \theta \cos \theta \hat{\boldsymbol{\theta}} + \sin \theta \hat{\boldsymbol{\phi}}]$
$\bar{\mathbf{N}}_{2,2}^{(i)}(\bar{\mathbf{r}}) = j \frac{z_2^{(i)}(kr)}{kr} \frac{3}{4} \sqrt{\frac{5}{\pi}} e^{j2\phi} \sin^2 \theta \hat{\mathbf{r}} + \frac{1}{kr} \frac{\partial}{\partial r} [rz_2^{(i)}(kr)] \frac{1}{4} \sqrt{\frac{5}{\pi}} e^{j2\phi} [j \sin \theta \cos \theta \hat{\boldsymbol{\theta}} - \sin \theta \hat{\boldsymbol{\phi}}]$

This Page Intentionally Contains No Relevant Information

## Appendix II — Commutation Relations

Consider the operators  $\nabla^2$  and  $\nabla \times$ . The commutator is  $\nabla^2 (\nabla \times) - (\nabla \times) \nabla^2$ . Using the representation of the operators in Cartesian coordinates, we see that

$$\begin{aligned}
 \nabla^2 (\nabla \times) &= \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \left( \frac{\partial}{\partial x} \hat{\mathbf{x}} + \frac{\partial}{\partial y} \hat{\mathbf{y}} + \frac{\partial}{\partial z} \hat{\mathbf{z}} \right) \times \\
 &= \left[ \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \frac{\partial}{\partial x} \hat{\mathbf{x}} + \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \frac{\partial}{\partial y} \hat{\mathbf{y}} + \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \frac{\partial}{\partial z} \hat{\mathbf{z}} \right] \times \\
 &= \left[ \frac{\partial}{\partial x} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \hat{\mathbf{x}} + \frac{\partial}{\partial y} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \hat{\mathbf{y}} + \frac{\partial}{\partial z} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \hat{\mathbf{z}} \right] \times \quad (\text{II-1}) \\
 &= \nabla \nabla^2 \times \\
 &= \left( \left[ \frac{\partial}{\partial x} \hat{\mathbf{x}} + \frac{\partial}{\partial y} \hat{\mathbf{y}} + \frac{\partial}{\partial z} \hat{\mathbf{z}} \right] \times \right) \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \\
 &= (\nabla \times) \nabla^2
 \end{aligned}$$

so that

$$\nabla^2 (\nabla \times) - (\nabla \times) \nabla^2 = 0. \quad (\text{II-2})$$

The commutator is zero, so the operators  $\nabla^2$  and  $\nabla \times$  commute.

Let  $\bar{\mathbf{a}}$  be a constant vector. Consider  $\nabla \times \bar{\mathbf{a}}\psi$ ,

$$\begin{aligned}
 \nabla \times &= \left( \frac{\partial}{\partial x} \hat{\mathbf{x}} + \frac{\partial}{\partial y} \hat{\mathbf{y}} + \frac{\partial}{\partial z} \hat{\mathbf{z}} \right) \times \bar{\mathbf{a}}\psi \\
 &= \left( \frac{\partial}{\partial x} \hat{\mathbf{x}} + \frac{\partial}{\partial y} \hat{\mathbf{y}} + \frac{\partial}{\partial z} \hat{\mathbf{z}} \right) \psi \times \bar{\mathbf{a}} \\
 &= -\bar{\mathbf{a}} \times \left( \frac{\partial}{\partial x} \hat{\mathbf{x}} + \frac{\partial}{\partial y} \hat{\mathbf{y}} + \frac{\partial}{\partial z} \hat{\mathbf{z}} \right) \psi \\
 &= -\bar{\mathbf{a}} \times \nabla \psi
 \end{aligned} \quad (\text{II-3})$$

Thus, we have the operator anti-commutative relation,

$$\nabla \times \bar{\mathbf{a}} + \bar{\mathbf{a}} \times \nabla = 0. \quad (\text{II-4})$$

Consider  $\nabla \times \bar{\mathbf{r}}$  when there is no additional function to the right. We have

$$\begin{aligned}
 \nabla \times \bar{\mathbf{r}} &= \left( \frac{\partial}{\partial x} \hat{\mathbf{x}} + \frac{\partial}{\partial y} \hat{\mathbf{y}} + \frac{\partial}{\partial z} \hat{\mathbf{z}} \right) \times (x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}) \\
 &= \hat{\mathbf{x}} \times \hat{\mathbf{x}} + \hat{\mathbf{y}} \times \hat{\mathbf{y}} + \hat{\mathbf{z}} \times \hat{\mathbf{z}} \\
 &= 0
 \end{aligned} \quad (\text{II-5})$$

Next consider the *operators*  $\nabla \times \bar{\mathbf{r}}$  and  $\bar{\mathbf{r}} \times \nabla$ . Note that  $\nabla$  in  $\nabla \times \bar{\mathbf{r}}$  is expected to operate on whatever function is immediately to the right of  $\nabla \times \bar{\mathbf{r}}$ , in addition to  $\bar{\mathbf{r}}$ , unlike in (II-5). Representing the operators in Cartesian coordinates, and explicitly including the right-hand function,  $\psi$ , on which the operators operate,

$$\begin{aligned}
\nabla \times \bar{\mathbf{r}}\psi &= \left( \frac{\partial}{\partial x} \hat{\mathbf{x}} + \frac{\partial}{\partial y} \hat{\mathbf{y}} + \frac{\partial}{\partial z} \hat{\mathbf{z}} \right) \times (x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}) \psi \\
&= \frac{\partial}{\partial x} y\psi\hat{\mathbf{z}} - \frac{\partial}{\partial x} z\psi\hat{\mathbf{y}} - \frac{\partial}{\partial y} x\psi\hat{\mathbf{z}} + \frac{\partial}{\partial y} z\psi\hat{\mathbf{x}} + \frac{\partial}{\partial z} x\psi\hat{\mathbf{y}} - \frac{\partial}{\partial z} y\psi\hat{\mathbf{x}} \\
&= -x \frac{\partial}{\partial y} \psi\hat{\mathbf{z}} + x \frac{\partial}{\partial z} \psi\hat{\mathbf{y}} + y \frac{\partial}{\partial x} \psi\hat{\mathbf{z}} - y \frac{\partial}{\partial z} \psi\hat{\mathbf{x}} - z \frac{\partial}{\partial x} \psi\hat{\mathbf{y}} + z \frac{\partial}{\partial y} \psi\hat{\mathbf{x}}, \\
&= -x \left( \frac{\partial}{\partial y} \hat{\mathbf{z}} - \frac{\partial}{\partial z} \hat{\mathbf{y}} \right) \psi - y \left( \frac{\partial}{\partial z} \hat{\mathbf{x}} - \frac{\partial}{\partial x} \hat{\mathbf{z}} \right) \psi - z \left( \frac{\partial}{\partial x} \hat{\mathbf{y}} - \frac{\partial}{\partial y} \hat{\mathbf{x}} \right) \psi \\
&= -(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}) \times \left( \frac{\partial}{\partial x} \hat{\mathbf{x}} + \frac{\partial}{\partial y} \hat{\mathbf{y}} + \frac{\partial}{\partial z} \hat{\mathbf{z}} \right) \psi \\
&= -\bar{\mathbf{r}} \times \nabla \psi
\end{aligned} \tag{II-6}$$

so that

$$\nabla \times \bar{\mathbf{r}}\psi + \bar{\mathbf{r}} \times \nabla \psi = 0. \tag{II-7}$$

Thus, the anti-commutator of the operators  $\nabla \times$  and  $\bar{\mathbf{r}}$  is zero.

Now consider  $\nabla \times \hat{\mathbf{r}}\psi$ , where  $\hat{\mathbf{r}} \equiv \bar{\mathbf{r}}/|\bar{\mathbf{r}}|$ . We have

$$\begin{aligned}
\nabla \times \hat{\mathbf{r}}\psi &= \nabla \times \frac{\bar{\mathbf{r}}}{|\bar{\mathbf{r}}|} \psi \\
&= -\bar{\mathbf{r}} \times \nabla \left( \frac{\psi}{|\bar{\mathbf{r}}|} \right) + \frac{\psi}{|\bar{\mathbf{r}}|} \nabla \times \bar{\mathbf{r}} \\
&= -\hat{\mathbf{r}} \times \nabla \psi - \psi \hat{\mathbf{r}} \times \nabla \left( \frac{1}{|\bar{\mathbf{r}}|} \right). \\
&= -\hat{\mathbf{r}} \times \nabla \psi + \psi \frac{1}{|\bar{\mathbf{r}}|^2} \hat{\mathbf{r}} \times \hat{\mathbf{r}} \\
&= -\hat{\mathbf{r}} \times \nabla \psi
\end{aligned} \tag{II-8}$$

Thus, we also have the operator anti-commutative relation

$$\nabla \times \hat{\mathbf{r}} + \hat{\mathbf{r}} \times \nabla = 0. \tag{II-9}$$

## Appendix III Addition Theorem for Vector Spherical Harmonics

The vector-spherical-harmonic addition theorem allows a vector harmonic referenced to one coordinate system to be expanded in terms of vector harmonics referenced to another coordinate system, which has been translated with respect to the first. The derivation of this theorem is outlined well by Weng Cho Chew [10]. For a translation such that  $\bar{\mathbf{r}} = \bar{\mathbf{r}}' + \bar{\mathbf{r}}''$ , as indicated in Figure III - 1, the addition theorem says [10]

$$\bar{\mathbf{M}}_{n,m}^{(2)}(\bar{\mathbf{r}}) = \sum_{n',m'} A_{n',m';n,m} \bar{\mathbf{M}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') + B_{n',m';n,m} \bar{\mathbf{N}}_{n',m'}^{(i)}(\bar{\mathbf{r}}'), \quad (\text{III-1})$$

and

$$\bar{\mathbf{N}}_{n,m}^{(2)}(\bar{\mathbf{r}}) = \sum_{n',m'} A_{n',m';n,m} \bar{\mathbf{N}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') + B_{n',m';n,m} \bar{\mathbf{M}}_{n',m'}^{(i)}(\bar{\mathbf{r}}'), \quad (\text{III-2})$$

where

$$A_{n',m';n,m} = \frac{2\pi j^{n'-n}}{\sqrt{n(n+1)n'(n'+1)}} \sum_{n''} \left( j^{n''} \left[ n(n+1) + n'(n'+1) - n''(n''+1) \right] \cdot A(m, n, -m', n', n'') z_{n''}^{(i)}(kr_w'') Y_{n'', m-m'}(\theta_w'', \phi_w'') \right), \quad (\text{III-3})$$

and

$$B_{n',m';n,m} = \frac{2\pi j^{n'-n}}{\sqrt{n(n+1)n'(n'+1)}} \sum_{n''} \left( j^{n''} B(m, n, -m', n', n'') z_{n''}^{(i)}(kr_w'') Y_{n'', m-m'}(\theta_w'', \phi_w'') \right), \quad (\text{III-4})$$

and the difference in scaling factors between the definitions of the vector spherical harmonics defined by Chew [10] and those used here has been accounted for.

In (III-1)–(III-4), the  $Y_{n,m}(\theta, \phi)$  is the scalar spherical harmonic (see Appendix I),

$$Y_{n,m}(\theta, \phi) = \sqrt{\frac{(2n+1)(n-m)!}{4\pi(n+m)!}} P_n^m(\cos \theta) e^{jm\phi}. \quad (\text{III-5})$$

The choice of which spherical Bessel's function,  $z_n^{(i)}(kr)$ , to use in (III-1)–(III-4) depends the relative sizes of  $r'$  and  $r''$ :

$$\text{in } \bar{\mathbf{M}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') \text{ and } \bar{\mathbf{N}}_{n',m'}^{(i)}(\bar{\mathbf{r}}'): z_{n'}^{(i)}(kr') = \begin{cases} j_{n'}(kr'), & r' < r'' \\ h_{n'}^{(2)}(kr'), & r' > r'' \end{cases} \quad (\text{III-6})$$

and

$$\text{in } A_{n',m';n,m} \text{ and } B_{n',m';n,m}: z_{n''}^{(i)}(kr'') = \begin{cases} h_{n''}^{(2)}(kr''), & r' < r'' \\ j_{n''}(kr''), & r' > r'' \end{cases} \quad (\text{III-7})$$

Also, in (III-1)–(III-4)

$$A(m, n, m', n', n'') = (-1)^m \sqrt{\frac{(2n+1)(2n'+1)(2n''+1)}{4\pi}} \begin{pmatrix} n & n' & n'' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} n & n' & n'' \\ -m & -m' & m+m' \end{pmatrix}, \quad (\text{III-8})$$

$$B(m, n, m', n', n'') = \sqrt{\frac{2n''+1}{2n''-1}} \left\{ \begin{array}{l} -\sqrt{(n'+m')(n'-m'+1)(n''+m+m')(n''+m+m'-1)} A(m, n, m'-1, n', n''-1) \\ +\sqrt{(n'-m')(n'+m'+1)(n''-m-m')(n''-m-m'-1)} A(m, n, m'+1, n', n''-1) \\ -2m'\sqrt{(n''-m-m')(n''+m+m')} A(m, n, m', n', n''-1) \end{array} \right\}, \quad (\text{III-9})$$

and  $\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$  is the Wigner 3j symbol. There is a slight difference in the notation used here for  $B(m, n, m', n', n'')$ , and that used by Chew [10], who inserts an additional argument,  $n''-1$ , after  $n''$ . Since that argument is redundant for the usage here, it has been deleted. A typographical error contained in [10] has been corrected in (III-9). This correction has been verified by symmetry arguments, comparison with [19], and the application of the addition to translate vector spherical harmonics. Note that Tsang and Kong [19] use a different normalization for the vector harmonics than either Chew [10] or this report. These different normalizations must be considered when comparing formulas from different sources.

The Wigner 3j symbol is related to the Clebsch-Gordon coefficients as [9, 10, 20]

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \frac{(-1)^{j_1-j_2-m_3}}{\sqrt{2j_3+1}} (j_1 m_1 j_2 m_2 | j_1 j_2 j_3, -m_3). \quad (\text{III-10})$$

The Wigner 3j symbol and the Clebsch-Gordon coefficient will be described in more detail, and an accurate method for computing them will be given.

The Wigner 3j symbol,  $\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$ , is nonzero only if  $m_3 = -m_1 - m_2$  and if

$j_1 + j_2 \geq j_3 \geq |j_1 - j_2|$ . The special case  $\begin{pmatrix} n & n' & n'' \\ 0 & 0 & 0 \end{pmatrix}$  is nonzero only if  $n + n' + n''$  is an even integer.

From (III-8), we see that each term in the sum over  $n''$ , which is contained in  $A_{n', m'; n, m}$

(III-3), has a factor  $\begin{pmatrix} n & n' & n'' \\ 0 & 0 & 0 \end{pmatrix}$ . This means that the sum will only contain terms where

$n + n' + n''$  is even. In addition, each term contains the factor  $\begin{pmatrix} n & n' & n'' \\ -m & m' & m-m' \end{pmatrix}$ , which will be zero unless  $n''$  satisfies  $|n - n'| \leq n'' \leq n + n'$ . Note that when  $n'' = |n - n'|$ , then  $n + n' + n''$  is even for any  $n, n'$ . Thus, we can write

$$A_{n',m';n,m} = \frac{2\pi j^{n'-n}}{\sqrt{n(n+1)n'(n'+1)}} \sum_{\substack{n''=|n-n'|, \\ 2+|n-n'|, \dots}}^{n+n'} \left[ j^{n''} [n(n+1) + n'(n'+1) - n''(n''+1)] \cdot A(m, n, -m', n', n'') z_{n''}^{(i)}(kr_w'') Y_{n'', m-m'}(\theta_w'', \phi_w'') \right]. \quad (\text{III-11})$$

Similarly, from (III-9), we see that each term in the sum over  $n''$ , which is contained in

$B_{n',m';n,m}$  (III-4), has a factor  $\begin{pmatrix} n & n' & n''-1 \\ 0 & 0 & 0 \end{pmatrix}$ . This means that the sum will only contain

terms where  $n + n' + n''$  is odd. In addition, each term contains the factor

$\begin{pmatrix} n & n' & n''-1 \\ -m & m' & m-m' \end{pmatrix}$  or  $\begin{pmatrix} n & n' & n''-1 \\ -m & m' \pm 1 & m-m' \mp 1 \end{pmatrix}$ , which will be zero unless  $n''$  satisfies

$1 + |n - n'| \leq n'' \leq 1 + n + n'$ . Note that when  $n'' = 1 + |n - n'|$ , then  $n + n' + n''$  is odd for any  $n, n'$ . Thus, we can write

$$B_{n',m';n,m} = \frac{2\pi j^{n'-n}}{\sqrt{n(n+1)n'(n'+1)}} \sum_{\substack{n''=1+|n-n'|, \\ 3+|n-n'|, \dots}}^{1+n+n'} \left[ j^{n''} B(m, n, -m', n', n'') z_{n''}^{(i)}(kr_w'') Y_{n'', m-m'}(\theta_w'', \phi_w'') \right]. \quad (\text{III-12})$$

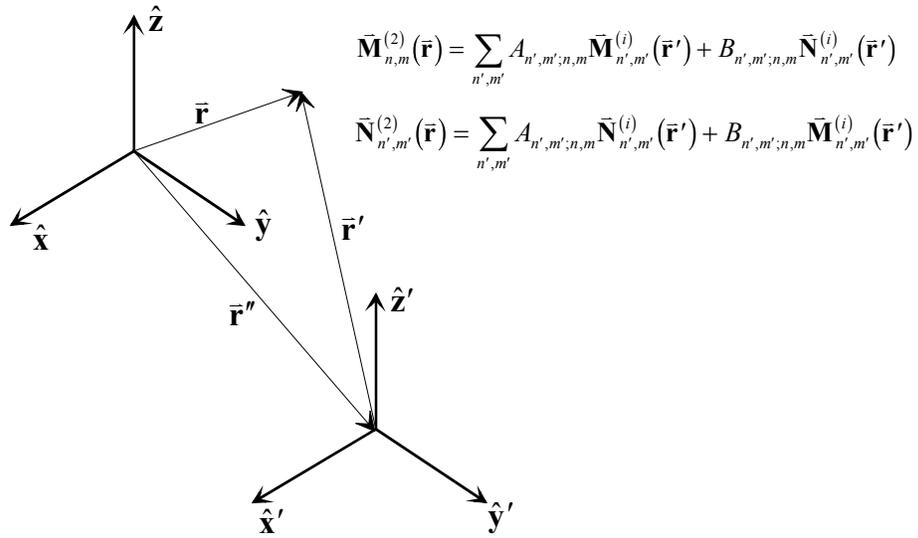


Figure III - 1 Relationship between the unprimed and primed coordinate systems for the vector-harmonic addition theorem.

### Application to translation of electromagnetic field

Suppose the expansion of the magnetic field for an antenna is known in terms of vector spherical harmonics referenced to the origin of the unprimed coordinate system, as illustrated Figure III - 1. The magnetic field in terms of harmonics referenced to the unprimed system is

$$\bar{\mathbf{H}}(\bar{\mathbf{r}}) = j \frac{k}{\omega \mu} \sum_{n=1}^{\infty} \sum_{m=-n}^n \left[ b_{n,m}^{TE} \bar{\mathbf{N}}_{n,m}^{(2)}(\bar{\mathbf{r}}) + b_{n,m}^{TM} \bar{\mathbf{M}}_{n,m}^{(2)}(\bar{\mathbf{r}}) \right]. \quad (\text{III-13})$$

In terms of harmonics referenced to the primed coordinate system, the same magnetic field is

$$\bar{\mathbf{H}}(\bar{\mathbf{r}}') = j \frac{k}{\omega\mu} \sum_{n=1}^{\infty} \sum_{m=-n}^n \sum_{n',m'} \left[ \left[ b_{n,m}^{TE} B_{n',m';n,m} + b_{n,m}^{TM} A_{n',m';n,m} \right] \bar{\mathbf{M}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') \right. \\ \left. + \left[ b_{n,m}^{TE} A_{n',m';n,m} + b_{n,m}^{TM} B_{n',m';n,m} \right] \bar{\mathbf{N}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') \right]. \quad (\text{III-14})$$

### Computing the Wigner 3j symbol and Clebsch-Gordon coefficient

The Wigner 3j symbol and Clebsch-Gordon are the vector coupling coefficients from quantum mechanics [20, 21]. In this context,  $j_1$  and  $j_2$  represent the angular momenta of systems 1 and 2, and  $j_3$  represents the angular momentum of the total system composed of 1 and 2 together. Here, they will be considered simply as a factor of the coefficients necessary to represent a vector spherical harmonic referenced to one coordinate system in terms of the harmonics referenced to a translated coordinate system.

The Wigner 3j symbol,  $\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$ , is given in closed form by the Racah formula

[20]

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1 - j_2 - m_3} \sqrt{\frac{(j_1 + j_2 - j_3)!(j_2 + j_3 - j_1)!(j_3 + j_1 - j_2)!}{(j_1 + j_2 + j_3 + 1)!}} \\ \cdot \sqrt{(j_1 + m_1)!(j_1 - m_1)!(j_2 + m_2)!(j_2 - m_2)!(j_3 + m_3)!(j_3 - m_3)!} \\ \cdot \sum_k (-1)^k \frac{1}{k!(j_3 - j_2 + m_1 + k)!(j_3 - j_1 - m_2 + k)!(j_1 + j_2 - j_3 - k)!(j_1 - m_1 - k)!(j_2 + m_2 - k)!} \quad , \quad (\text{III-15})$$

where the 3j symbol is zero unless

$$m_1 + m_2 + m_3 = 0, \quad (\text{III-16})$$

$$|m_1| \leq j_1, \quad |m_2| \leq j_2, \quad \text{and} \quad |m_3| \leq j_3, \quad (\text{III-17})$$

and

$$|j_1 - j_2| \leq j_3 \leq j_1 + j_2. \quad (\text{III-18})$$

In general [9, 20, 21],  $j_1, j_2, j_3$  can be positive integers or multiples of half integers,  $m_1, m_2, m_3$  can be positive or negative integers or multiples of half integers, and  $j_1 + j_2 + j_3$  must be an integer. However, for the purposes here,  $j_1, j_2, j_3$  will be restricted to positive integers and  $m_1, m_2, m_3$  will be restricted to positive or negative integers.

The Wigner 3j symbol satisfies certain symmetries [20, 21]

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} j_3 & j_1 & j_2 \\ m_3 & m_1 & m_3 \end{pmatrix} = \begin{pmatrix} j_2 & j_3 & j_1 \\ m_2 & m_3 & m_1 \end{pmatrix}, \quad (\text{III-19})$$

$$(-1)^{j_1+j_2+j_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} j_2 & j_1 & j_3 \\ m_2 & m_1 & m_3 \end{pmatrix} = \begin{pmatrix} j_1 & j_3 & j_2 \\ m_1 & m_3 & m_2 \end{pmatrix} = \begin{pmatrix} j_3 & j_2 & j_1 \\ m_3 & m_2 & m_1 \end{pmatrix}, \quad (\text{III-20})$$

and

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1+j_2+j_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix}. \quad (\text{III-21})$$

### Special values of the Wigner 3j symbol

The special case  $\begin{pmatrix} j_1 & j_2 & j_3 \\ 0 & 0 & 0 \end{pmatrix}$  is nonzero only if  $j_1 + j_2 + j_3$  is an even integer [20] and

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ 0 & 0 & 0 \end{pmatrix} = \begin{cases} 0, & j_1 + j_2 + j_3 \text{ is odd} \\ (-1)^{(j_1+j_2+j_3)/2} \sqrt{\frac{(j_1+j_2-j_3)!(j_2+j_3-j_1)!(j_3+j_1-j_2)!}{(j_1+j_2+j_3+1)!}} \left(\frac{j_1+j_2+j_3}{2}\right)! & \text{when } j_1 + j_2 + j_3 \text{ is even} \\ (-1)^{(j_1+j_2+j_3)/2} \frac{(j_1+j_2-j_3)! (j_2+j_3-j_1)! (j_3+j_1-j_2)!}{\left(\frac{j_1+j_2-j_3}{2}\right)! \left(\frac{j_2+j_3-j_1}{2}\right)! \left(\frac{j_3+j_1-j_2}{2}\right)!} & \end{cases} \quad (\text{III-22})$$

When  $m_1 = m_2 = 0$ , the following special values are obtained from (III-22)

$$\begin{pmatrix} j_1 & j_2 & j_1 + j_2 \\ 0 & 0 & 0 \end{pmatrix} = (-1)^{j_1+j_2} \sqrt{\frac{(2j_2)!(2j_1)!}{(2j_1+2j_2+1)!}} \frac{(j_1+j_2)!}{(j_2)!(j_1)!}, \quad (\text{III-23})$$

$$\begin{pmatrix} j_1 & j_2 & |j_1 - j_2| \\ 0 & 0 & 0 \end{pmatrix} = \begin{cases} (-1)^{j_1} \sqrt{\frac{(2j_2)!(2j_1-2j_2)!}{(2j_1+1)!}} \frac{(j_1)!}{(j_2)!(j_1-j_2)!} & \text{when } j_1 \geq j_2 \\ (-1)^{j_2} \sqrt{\frac{(2j_1)!(2j_2-2j_1)!}{(2j_2+1)!}} \frac{(j_2)!}{(j_1)!(j_2-j_1)!} & \text{when } j_2 \geq j_1 \end{cases}, \quad (\text{III-24})$$

and

$$\begin{pmatrix} j_1 & j_2 & j_1 + j_2 - 1 \\ 0 & 0 & 0 \end{pmatrix} = 0, \quad (\text{III-25})$$

$$\begin{pmatrix} j_1 & j_2 & |j_1 - j_2| + 1 \\ 0 & 0 & 0 \end{pmatrix} = 0. \quad (\text{III-26})$$

From (III-15), we see that for any  $|m| \leq j$ ,

$$\begin{pmatrix} j & j & 2j \\ m & m & -2m \end{pmatrix} = (-1)^{2m} (2j)! \sqrt{\frac{1}{(4j+1)!}} \frac{\sqrt{(2j-2m)!(2j+2m)!}}{(j+m)!(j-m)!}, \quad (\text{III-27})$$

and

$$\begin{pmatrix} j & j & 2|m| \\ m & m & -2m \end{pmatrix} = (-1)^{j+m} \frac{(j+|m|)!}{(j-|m|)!(2|m|)!} \sqrt{\frac{(2j-2|m|)!(4|m|)!}{(2j+2|m|+1)!}}. \quad (\text{III-28})$$

Applying the symmetry relation (III-20), we see that

$$\begin{pmatrix} j & j & j_3 \\ m & m & -2m \end{pmatrix} = 0 \text{ for } j_3 \text{ an odd integer.} \quad (\text{III-29})$$

Other special values, not containing factorials, are [20]

$$\begin{pmatrix} j & j & 0 \\ m & -m & 0 \end{pmatrix} = \begin{pmatrix} 0 & j & j \\ 0 & m & -m \end{pmatrix} = \begin{pmatrix} j & 0 & j \\ -m & 0 & m \end{pmatrix} = (-1)^{j-m} \frac{1}{\sqrt{2j+1}}, \quad (\text{III-30})$$

$$\begin{pmatrix} j_1 & j_2 & (j_1 + j_2) \\ j_1 & j_2 & -(j_1 + j_2) \end{pmatrix} = \frac{1}{\sqrt{2(j_1 + j_2) + 1}}, \quad (\text{III-31})$$

and from [21]

$$\begin{pmatrix} j & j & 1 \\ m & -m & 0 \end{pmatrix} = (-1)^{j-m} \frac{m}{\sqrt{j(j+1)(2j+1)}}. \quad (\text{III-32})$$

### Computing arbitrary values

Using the Racah formula (III-15) to compute the Wigner 3j symbol presents some potential difficulties, because it contains products of factorials. While the 3j symbol itself is well behaved when the arguments become large, intermediate products and factors can cause numerical overflow if appropriate precautions are not taken. The logarithm of the factorial function can be computed easily for very large arguments, and can be utilized to avoid overflow when factorials of large integers are needed. An approach that avoids overflow is to compute the 3j symbol as follows

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1-j_2-m_3} \sum_k (-1)^k \exp \left( \begin{array}{l} \frac{1}{2} \ln(j_1 + j_2 - j_3)! + \frac{1}{2} \ln(j_2 + j_3 - j_1)! \\ + \frac{1}{2} \ln(j_3 + j_1 - j_2)! + \frac{1}{2} \ln(j_1 + m_1)! \\ + \frac{1}{2} \ln(j_1 - m_1)! + \frac{1}{2} \ln(j_2 + m_2)! \\ + \frac{1}{2} \ln(j_2 - m_2)! + \frac{1}{2} \ln(j_3 + m_3)! + \frac{1}{2} \ln(j_3 - m_3)! \\ - \frac{1}{2} \ln(j_1 + j_2 + j_3 + 1)! - \ln k! - \ln(j_3 - j_2 + m_1 + k)! \\ - \ln(j_3 - j_1 - m_2 + k)! - \ln(j_1 + j_2 - j_3 - k)! \\ - \ln(j_1 - m_1 - k)! - \ln(j_2 + m_2 - k)! \end{array} \right). \quad (\text{III-33})$$

Unfortunately, this approach can loose accuracy for large degree because the terms of the sum alternate in sign, while some of the terms are many orders of magnitude larger than the final sum.

The Racah formula (III-15) can be used if the factorials are factored into their prime factors before the multiplication. By canceling common factors before multiplication, overflow is avoided. Since the factors are all integers, an exact expression can be

obtained in terms of ratios and square roots of ratios of prime factors, called a root-rational fraction. However, this approach is complicated to program and is not very efficient. However, it is attractive for exact computations, and the root-rational-fraction approach has been implemented by Stone and Wood [22].

As indicated above, some special values of the Wigner  $3j$  symbol can be computed with a single term. However, several of these special values, for example (III-23), (III-24), (III-27), and (III-28), still require computing products of factorials, leading to the possibility of numerical overflow.

### Stable recursive algorithm

Luscombe and Luban [23] give a recursive algorithm for computing the Wigner  $3j$  symbol. The algorithm uses two nonlinear, two-term recursions and a three-term recursion. No problems with overflow or rescaling occur, and the algorithm is very accurate. However, there are pathological cases where the algorithm cannot be used precisely as described. These cases will be addressed below. The algorithm is implemented as follows.

Let  $\psi_j$  represent  $\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & -m_1 - m_2 \end{pmatrix}$ . The  $\psi_j$  satisfy the recursion relation [23]

$$X_j \psi_{j+1} + Y_j \psi_j + Z_j \psi_{j-1} = 0, \quad j_{\min} \leq j \leq j_{\max}, \quad (\text{III-34})$$

where

$$X_j = j \sqrt{[(j+1)^2 - (j_1 - j_2)^2] [(j_1 + j_2 + 1)^2 - (j+1)^2] [(j+1)^2 - (m_1 + m_2)^2]}, \quad (\text{III-35})$$

$$Y_j = (2j+1) \{ (m_1 + m_2) [j_1(j_1+1) - j_2(j_2+1)] - (m_1 - m_2) j(j+1) \}, \quad (\text{III-36})$$

$$Z_j = (j+1) \sqrt{[j^2 - (j_1 - j_2)^2] [(j_1 + j_2 + 1)^2 - j^2] [j^2 - (m_1 + m_2)^2]}, \quad (\text{III-37})$$

$$j_{\min} = \max(|j_1 - j_2|, |m_1 + m_2|), \quad (\text{III-38})$$

and

$$j_{\max} = j_1 + j_2. \quad (\text{III-39})$$

The three-term recursion (III-34) can be converted to two two-term recursions by defining ratios of the  $\psi_j$  as follows

$$r_j = \frac{\psi_j}{\psi_{j-1}} = \frac{-Z_j}{Y_j + X_j r_{j+1}}, \quad j \leq j_{\max} - 1, \quad (\text{III-40})$$

and

$$s_j = \frac{\psi_j}{\psi_{j+1}} = \frac{-X_j}{Y_j + Z_j s_{j-1}}, \quad j \geq j_{\min} + 1. \quad (\text{III-41})$$

The three-term recursion (III-34) is stable in either direction over a center portion of the range of  $j$ . The two-term recursion (III-40) is used at the upper end of the range of  $j$  and

is stable only in the downward direction. Similarly, the other two-term recursion (III-41) is used at the lower end of the range of  $j$  and is stable only in the upward direction.

The downward iteration with (III-40) is started with

$$r_{j_{\max}} = \frac{-Z_{j_{\max}}}{Y_{j_{\max}}}, \quad (\text{III-42})$$

and  $r_j$  is iterated until

$$r_{j_+} > 1, \quad (\text{III-43})$$

which determines a value for  $j_+$ . Next, the upward recursion (III-41) is iterated, with a starting value of

$$s_{j_{\min}} = \frac{-X_{j_{\min}}}{Y_{j_{\min}}}. \quad (\text{III-44})$$

The  $s_j$  are iterated until

$$s_{j_-} > 1, \quad (\text{III-45})$$

which defines  $j_-$ .

A normalized version of  $\psi_j$  is defined

$$u_j^- \equiv \frac{\Psi_j}{\Psi_{j_-}},$$

with the obvious condition that

$$u_{j_-}^- = 1. \quad (\text{III-46})$$

We now compute

$$u_{j_- - k}^- = \prod_{p=1}^k s_{j_- - p}, \text{ for } 1 \leq k \leq j_- - j_{\min}. \quad (\text{III-47})$$

The ratio  $u_j^-$  satisfies also satisfies (III-34), since it differs from  $\psi_j$  by only a constant, so

$$X_j u_{j+1}^- + Y_j u_j^- + Z_j u_{j-1}^- = 0. \quad (\text{III-48})$$

From (III-47)

$$u_{j_- - 1}^- = s_{j_- - 1}, \quad (\text{III-49})$$

so, with (III-46) and (III-49) as starting values, the  $u_j^-$  are iterated with (III-48) until  $j = j_+$ . Now

$$u_{j_+}^- = \frac{\Psi_{j_+}}{\Psi_{j_-}}, \quad (\text{III-50})$$

so continuing,

$$u_{j_+ + k}^- = u_{j_+}^- \prod_{p=1}^k r_{j_+ + p}, \text{ for } 1 \leq k \leq j_{\max} - j_+. \quad (\text{III-51})$$

Since the  $u_j^-$  are un-normalized versions of the Wigner 3j symbols, the Wigner 3j symbol is

$$\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & -m_1 - m_2 \end{pmatrix} = \frac{S}{D} u_j^-, \quad (\text{III-52})$$

where the correct normalization is [23]

$$D = \sqrt{\sum_{j=j_{\min}}^{j_{\max}} (2j+1) u_j^2}, \quad (\text{III-53})$$

and

$$S = \text{sgn}(u_{j_{\max}}^-) (-1)^{j_1 - j_2 + m_1 + m_2}. \quad (\text{III-54})$$

This finishes the Luscombe and Luban recursive algorithm.

Note that it is possible for  $Y_j + X_j r_{j+1}$  to be zero in (III-40) and  $Y_j + Z_j s_{j-1}$  to be zero in (III-41). Obviously, the two-term recursion cannot proceed when this occurs. When (III-40) or (III-41) fails in this way, the implication is that  $\psi_{j-1} = 0$  or  $\psi_{j+1} = 0$ , respectively. It is also possible that  $Y_{j_{\max}} = 0$  in (III-42) or  $Y_{j_{\min}} = 0$  in (III-44), or both. In this case, one or both of the two-term recursions cannot be started. The following situations require special procedures:

1. Neither of the two-term recursions can be started with (III-42) and (III-44), because  $Y_{j_{\min}} = Y_{j_{\max}} = 0$ , when any of the following occur:

$$j_1 = m_1 = 0,$$

$$j_2 = m_2 = 0,$$

$$m_1 = m_2 = 0 \text{ for any } j_1, j_2,$$

or

$$j_1 = j_2 \text{ and } m_1 = m_2.$$

2. The downward two-term recursion cannot be started with (III-42) because

$$Y_{j_{\max}} = 0 \text{ when } j_2 = j_1 \frac{m_2}{m_1}. \quad (\text{III-55})$$

3. The upward two-term recursion cannot be started with (III-44) because

$$Y_{j_{\min}} = 0 \text{ when } \begin{cases} j_2 = j_1 \text{ and } m_2 = -m_1 \\ j_1 > j_2 \text{ and } j_2 = j_1 + \frac{1}{2} \text{ or } j_2 = -(j_1 + 1) \frac{m_2}{m_1} \\ j_1 < j_2 \text{ and } j_2 = j_1 - \frac{1}{2} \text{ or } j_2 = -\left(j_1 \frac{m_2}{m_1} + 1\right) \end{cases} . \quad (\text{III-56})$$

*Case 1*

When determining the method of computing the  $3j$  symbols, Case 1 should be addressed first.

When  $j_1 = m_1 = 0$ , or  $j_2 = m_2 = 0$ , (III-30) can be used without difficulty, so

$$\begin{pmatrix} 0 & j & j \\ 0 & m & -m \end{pmatrix} = \begin{pmatrix} 0 & j & j \\ 0 & -m & m \end{pmatrix} = \begin{pmatrix} j & 0 & j \\ -m & 0 & m \end{pmatrix} = \begin{pmatrix} j & 0 & j \\ m & 0 & -m \end{pmatrix} = (-1)^{j+m} \frac{1}{\sqrt{2j+1}} . \quad (\text{III-57})$$

When  $j_1 = j_2$  and  $m_1 = m_2$ , (III-28) and (III-29) can be used to start the iteration of the three-term recursion (III-34) in the upward direction. Alternatively, (III-27) and (III-29) can be used to start the downward iteration of the three-term recursion. When  $m_1 = m_2 = 0$ , (III-23) and (III-25) can be used to start the downward three-term recursion, or (III-24) and (III-26) can be used to start the upward three-term recursion. However,

since  $\begin{pmatrix} j & j & j_3 \\ m & m & -2m \end{pmatrix} = 0$  when  $j_3$  is an odd integer and  $\begin{pmatrix} j_1 & j_2 & j_3 \\ 0 & 0 & 0 \end{pmatrix} = 0$  when

$j_1 + j_2 + j_3$  is an odd integer, the three-term recursion (III-34) can actually be written as a two-term recursion as follows

$$u_{2k} = -\frac{Z_{2k-1}}{X_{2k-1}} u_{2(k-1)}, \quad \frac{j_{\min}}{2} + 1 \leq k \leq \frac{j_{\max}}{2} . \quad (\text{III-58})$$

When  $j_1 = j_2$  and  $m_1 = m_2$ , the coefficients in (III-58) are

$$X_{2k-1} = 2k(2k-1) \sqrt{[(2j_1+1)^2 - (2k)^2]} \sqrt{[(2k)^2 - 4m_1^2]} , \quad (\text{III-59})$$

and

$$Z_{2k-1} = 2k(2k-1) \sqrt{[(2j_1+1)^2 - (2k-1)^2]} \sqrt{[(2k-1)^2 - 4m_1^2]} . \quad (\text{III-60})$$

When  $m_1 = m_2 = 0$ , the coefficients in (III-58)

$$X_{2k-1} = 2k(2k-1) \sqrt{[(2k)^2 - (j_1 - j_2)^2]} \sqrt{[(j_1 + j_2 + 1)^2 - (2k)^2]} , \quad (\text{III-61})$$

and

$$Z_{2k-1} = 2k(2k-1) \sqrt{[(2k-1)^2 - (j_1 - j_2)^2]} \sqrt{[(j_1 + j_2 + 1)^2 - (2k-1)^2]} . \quad (\text{III-62})$$

In this case, the starting value can be arbitrary, and the sequence of values is normalized with

$$D = \sqrt{\sum_{j=|j_1-j_2|}^{(j_1+j_2)} (2j+1)u_j^2}, \quad (\text{III-63})$$

and

$$S = (-1)^{j_1+j_2} \text{sgn}[u_{j_1+j_2}], \quad (\text{III-64})$$

so

$$\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & -m_1 - m_2 \end{pmatrix} = S \frac{u_j}{D}. \quad (\text{III-65})$$

By starting the recursion with an arbitrary value and applying the normalization (III-65), the evaluation of large factorials can be avoided. The normalization ensures that the values are accurate, and since a two-term recursion is used, loss of significance cannot occur.

#### Case 2

When Case 2 occurs, and it is certain that Case 1 has not occurred, then the upward two-term recursion (III-41) can be started using (III-44). Upward recursion is performed until  $s_{j_-} > 1$ , at which time the three-term recursion is applied in the upward direction to

$j = j_{\max}$ . From (III-40), (III-42) and (III-55), we can deduce that

$$\begin{pmatrix} j_1 & j_1 \frac{m_2}{m_1} & j_{\max-1} \\ m_1 & m_2 & -m_1 - m_2 \end{pmatrix} = 0. \quad (\text{III-66})$$

#### Case 3

When Case 3 occurs, and it is certain that Case 1 has not occurred, then the downward two-term recursion (III-40) can be started using (III-42). Downward recursion is performed until  $r_{j_+} > 1$ , and then the three-term recursion is applied in the downward direction to  $j = j_{\min}$ . From (III-41), (III-44), and (III-56), we deduce

$$\begin{pmatrix} j_1 & -(j_1+1) \frac{m_2}{m_1} & j_{\min+1} \\ m_1 & m_2 & -m_1 - m_2 \end{pmatrix} = 0 \text{ when } \frac{m_2}{m_1} < 0 \text{ and } j_1 > -(j_1+1) \frac{m_2}{m_1}, \quad (\text{III-67})$$

and

$$\begin{pmatrix} j_1 & -\left(j_1 \frac{m_2}{m_1} + 1\right) & j_{\min+1} \\ m_1 & m_2 & -m_1 - m_2 \end{pmatrix} = 0 \text{ when } \frac{m_2}{m_1} < 0 \text{ and } j_1 < -\left(j_1 \frac{m_2}{m_1} + 1\right). \quad (\text{III-68})$$

### Example families of Wigner 3j symbols

Few references tabulate or plot the Wigner 3j symbol for large degree and order, but it is useful to examine its behavior by considering a few numerical examples. When considered as families of values associated with  $(j_1, m_1)$  and  $(j_2, m_2)$ , and indexed by  $j_3$ , the Wigner 3j symbol exhibits a variety of behaviors. Several of these families will be illustrated.

For example, the Luscombe-Luban algorithm can be applied without any of the special considerations for the family defined by  $\begin{pmatrix} 100 & 60 & j_3 \\ 70 & -55 & -15 \end{pmatrix}$ , as illustrated in [23] and in

Figure III - 1. The values oscillate (change sign) over much of the range for which the 3j symbol is not trivially zero. If the orders  $m_1$  and  $m_2$  are changed to appropriate values, the oscillation disappears, as illustrated in Figure III - 2, where the family generated by

$\begin{pmatrix} 100 & 60 & j_3 \\ 59 & -60 & 1 \end{pmatrix}$  is plotted. However, if the first two columns of the 3j symbol are

interchanged, the symmetry relation (III-20) is invoked, and the oscillations return, but at a more rapid rate, as illustrated in Figure III - 3. In this case, the values alternate sign as  $j_3$  is incremented by one.

The algorithm is applied under the conditions of Case 1 for  $m_1 = m_2 = 0$ , where neither two-term recursion can be started. Figure III - 4 illustrates the family of values defined by  $\begin{pmatrix} 100 & 60 & j_3 \\ 0 & 0 & 0 \end{pmatrix}$ . Similar results obtain when  $j_1 = j_2$  and  $m_1 = m_2$ , illustrated in

Figure III - 5 for the family generated by  $\begin{pmatrix} 80 & 80 & j_3 \\ 35 & 35 & -70 \end{pmatrix}$ . In both examples, every other value is zero, and the three-term recursion reduces to a two-term recursion.

When  $j_2 = j_1 m_2 / m_1$ , the downward two-term recursion cannot be started, and the algorithm is applied with the constraints of Case 2. An example is the family of values

associated with  $\begin{pmatrix} 112 & 60 & j_3 \\ 56 & 30 & -86 \end{pmatrix}$ , illustrated in Figure III - 6. The values oscillate

rapidly in this case. However, since consecutive values have the same sign in pairs over much of the range, interchanging the first two columns will not eliminate the oscillations. However, doing so does eliminate the some of oscillations at the low end of the range of

$j_3$ , as illustrated in Figure III - 7, but  $\begin{pmatrix} 60 & 112 & j_3 \\ 30 & 56 & -86 \end{pmatrix}$  is no longer a Case 2 situation.

Case 3 occurs when  $j_1 > j_2$  and  $j_2 = -(j_1 + 1)m_2 / m_1$  or when  $j_1 < j_2$  and

$j_2 = -(j_1 m_2 / m_1 + 1)$ . In this situation, the upward two-term recursion cannot be started.

This case is illustrated in Figure III - 8 with the family obtained from

$$\begin{pmatrix} 99 & 60 & j_3 \\ -25 & 15 & 10 \end{pmatrix} = \begin{pmatrix} 60 & 99 & j_3 \\ -15 & 25 & -10 \end{pmatrix}.$$

Another Case 3 situation occurs when  $j_2 = j_1$  and  $m_2 = -m_1$ , and the upward recursion cannot be started. This situation is illustrated with the family of values generated by  $\begin{pmatrix} 60 & 60 & j_3 \\ 58 & -58 & 0 \end{pmatrix}$  in Figure III - 9. The values change smoothly as  $j_3$  is incremented.

However, interchanging the first two columns to obtain  $\begin{pmatrix} 60 & 60 & j_3 \\ -58 & 58 & 0 \end{pmatrix}$  invokes the symmetry of (III-20), with the result illustrated in Figure III - 10. In this Case 3 situation, the values oscillate rapidly, changing sign with each increment of  $j_3$ .

One final family is illustrated with large values of  $j_1, j_2$ . The family  $\begin{pmatrix} 200 & 179 & j_3 \\ -30 & 34 & -4 \end{pmatrix}$ , illustrated in Figure III - 11, generates the interesting "baseball" pattern.

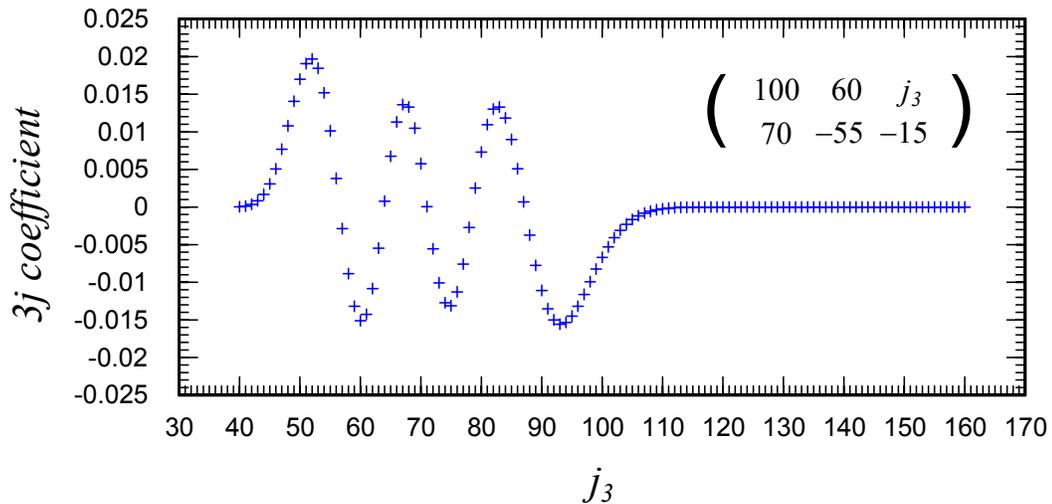


Figure III - 1 Family of points defined by  $\begin{pmatrix} 100 & 60 & j_3 \\ 70 & -55 & -15 \end{pmatrix}$ , computed with the Luscombe-Luban algorithm without any of the special considerations.

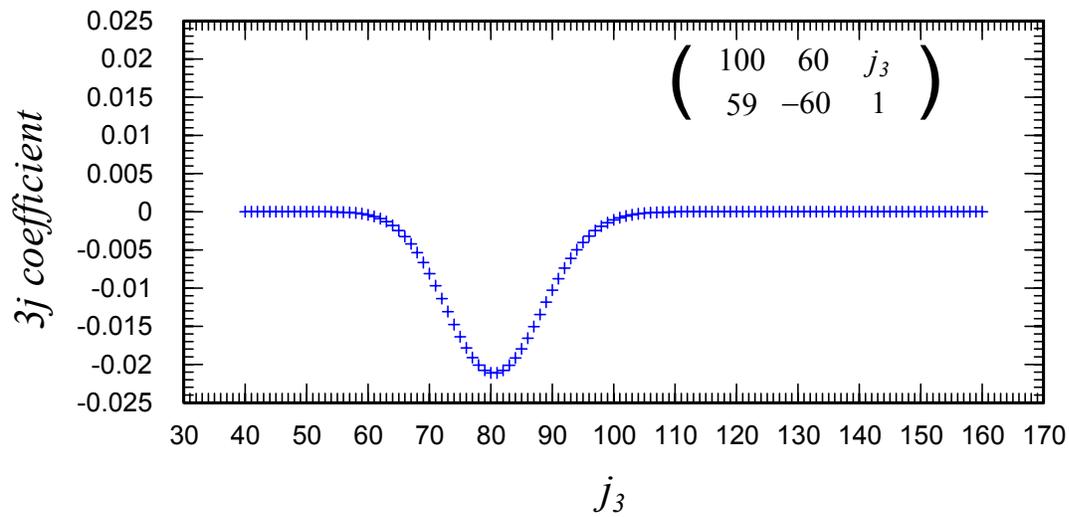


Figure III - 2 Family of points defined by  $\begin{pmatrix} 100 & 60 & j_3 \\ 59 & -60 & 1 \end{pmatrix}$ , computed with the Luscombe-Luban algorithm without any of the special considerations.

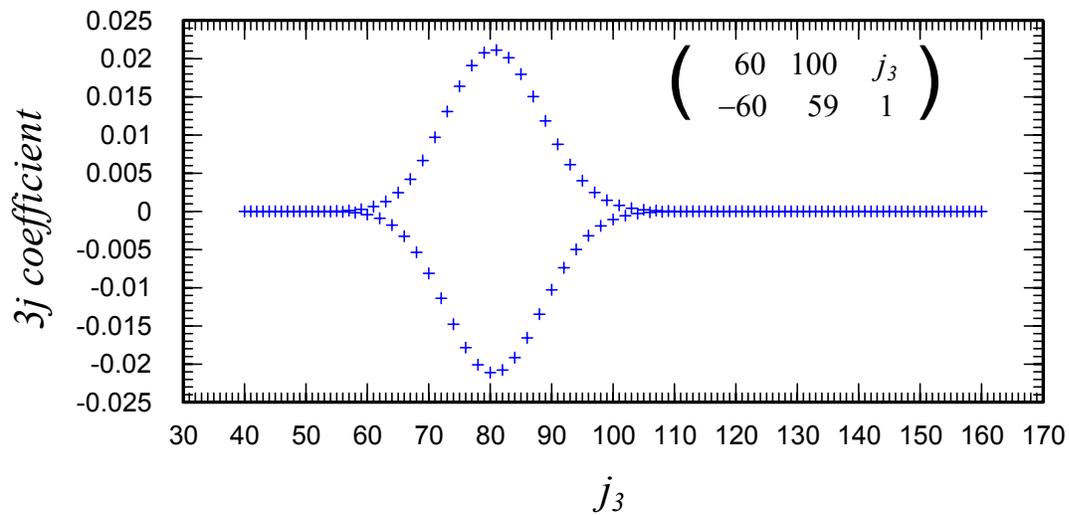


Figure III - 3 Family of points defined by  $\begin{pmatrix} 60 & 100 & j_3 \\ -60 & 59 & 1 \end{pmatrix}$ , is related to the family of points plotted in Figure III - 2 by the factor  $(-1)^{j_3}$ .

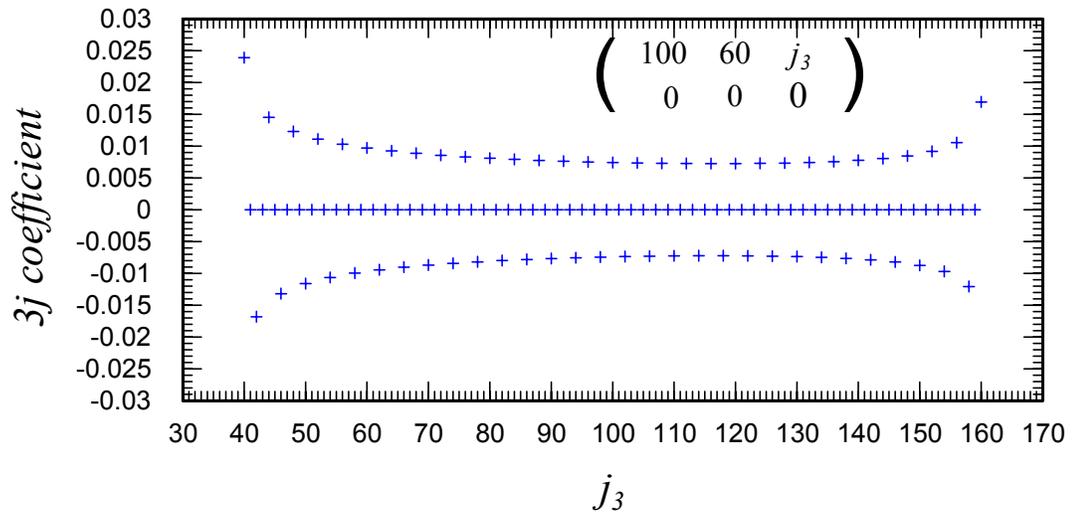


Figure III - 4 Family of points defined by  $\begin{pmatrix} 100 & 60 & j_3 \\ 0 & 0 & 0 \end{pmatrix}$ , a typical example of Case 1, with  $m_1 = m_2 = 0$ .

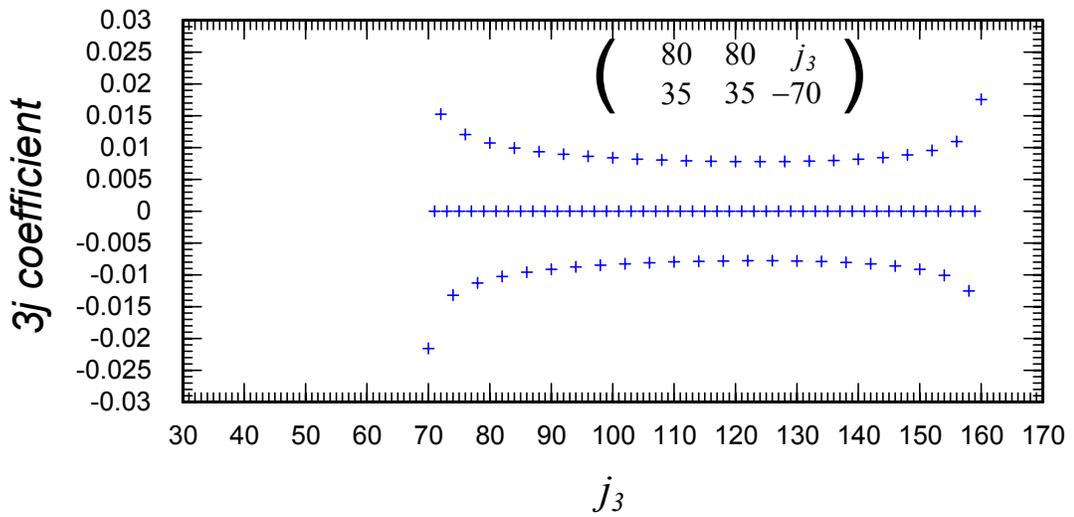


Figure III - 5 Family of points defined by  $\begin{pmatrix} 80 & 80 & j_3 \\ 35 & 35 & -70 \end{pmatrix}$ , a typical example of Case 1, with  $j_1 = j_2$  and  $m_1 = m_2$ .

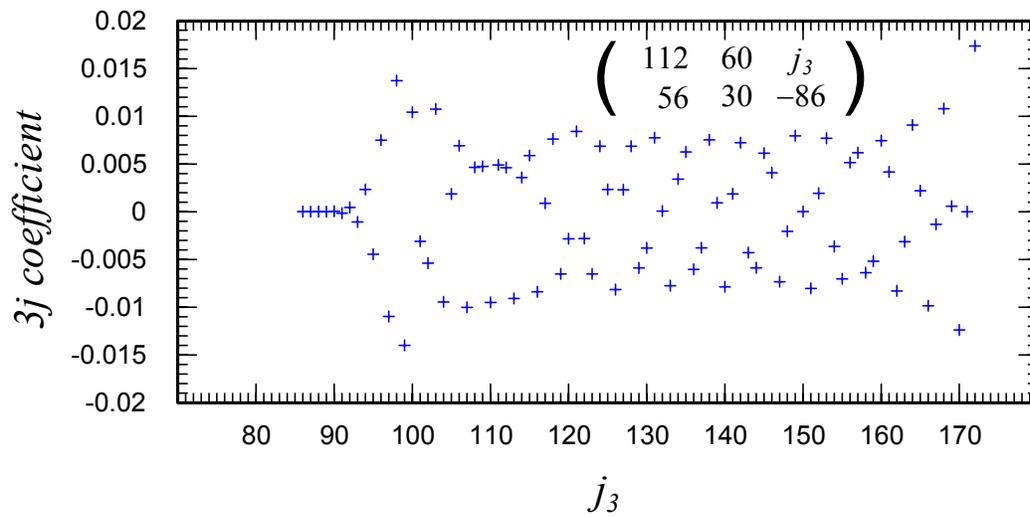


Figure III - 6 Family of points defined by  $\begin{pmatrix} 112 & 60 & j_3 \\ 56 & 30 & -86 \end{pmatrix}$ , an example of Case 2, with  $j_2 = j_1 m_2 / m_1$ .

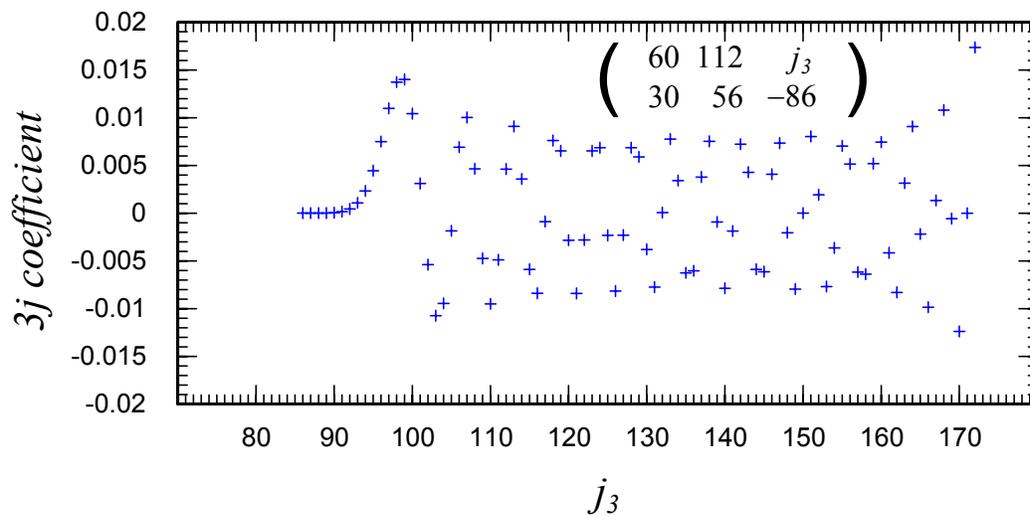


Figure III - 7 Family of points defined by  $\begin{pmatrix} 60 & 112 & j_3 \\ 30 & 56 & -86 \end{pmatrix}$ , related to the family plotted in Figure III - 6 by the factor  $(-1)^{j_3}$ , but not a Case 2 situation.

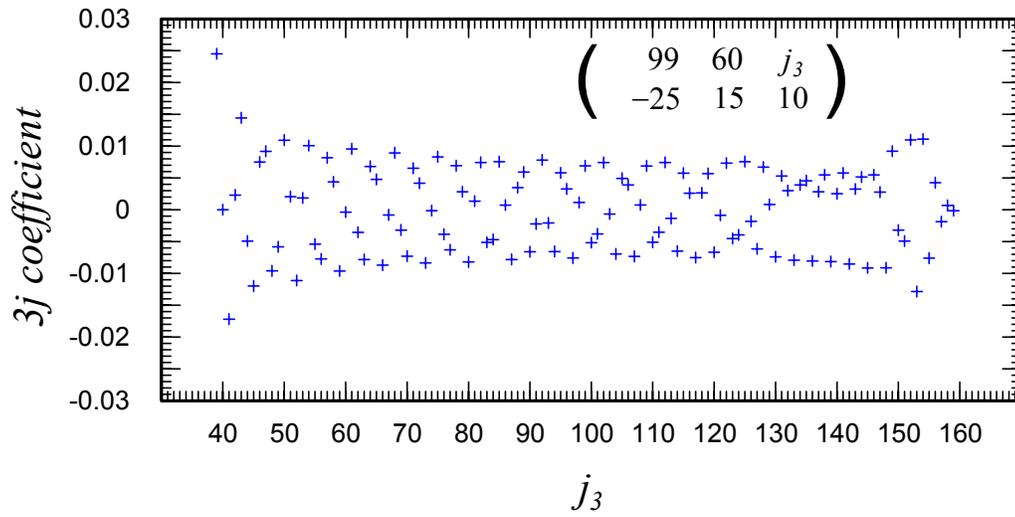


Figure III - 8 Family of points defined by  $\begin{pmatrix} 99 & 60 & j_3 \\ -25 & 15 & 10 \end{pmatrix} = \begin{pmatrix} 60 & 99 & j_3 \\ 15 & -25 & -10 \end{pmatrix}$ , an example of Case 3, with  $j_1 > j_2$  and  $j_2 = -(j_1 + 1)m_2/m_1$  or  $j_1 < j_2$  and  $j_2 = -(j_1 m_2/m_1 + 1)$ .

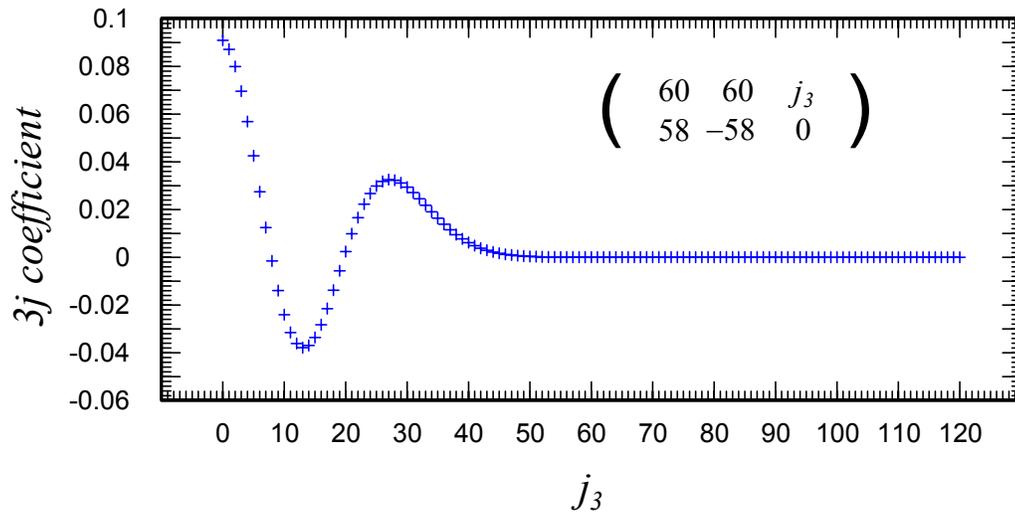


Figure III - 9 Family of points defined by  $\begin{pmatrix} 60 & 60 & j_3 \\ 58 & -58 & 0 \end{pmatrix}$ , an example of Case 3, with  $j_2 = j_1$  and  $m_2 = -m_1$ .

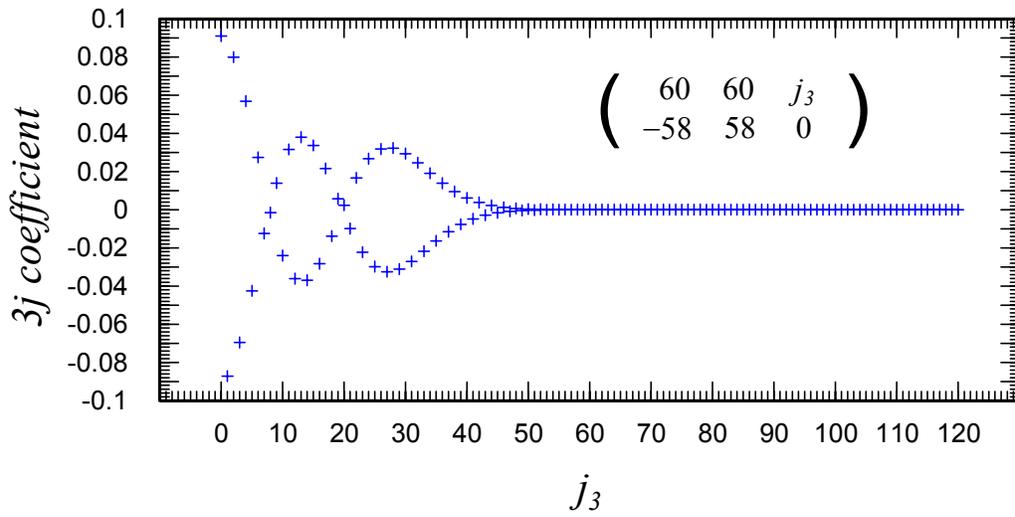


Figure III - 10 Family of points defined by  $\begin{pmatrix} 60 & 60 & j_3 \\ -58 & 58 & 0 \end{pmatrix}$ , another example of Case 3, where the points are related to those in Figure III - 9 by the factor  $(-1)^{j_3}$ .

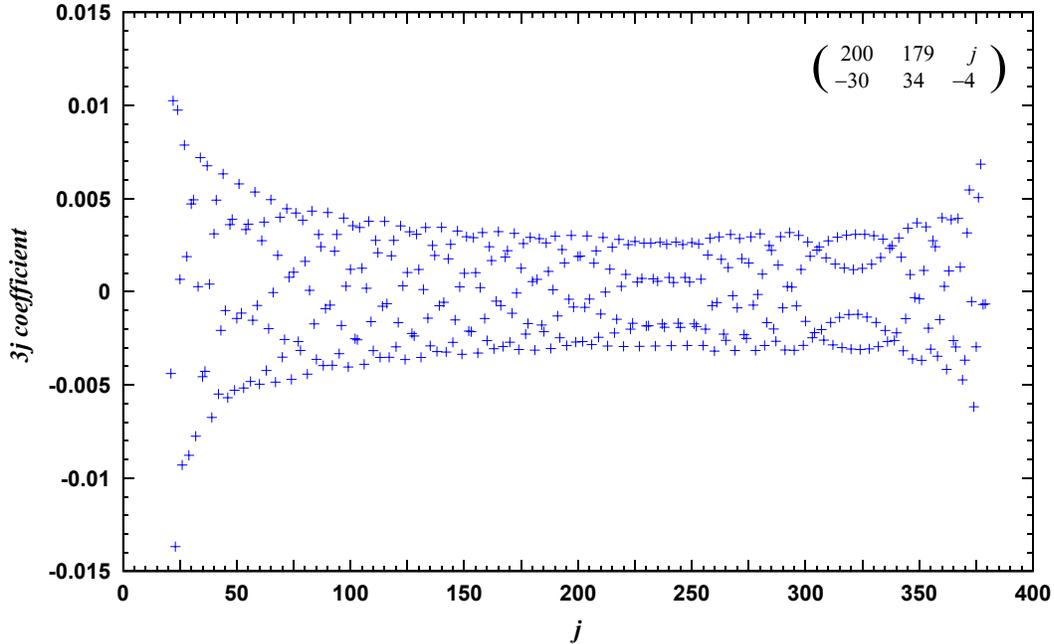


Figure III - 11 Family of points defined by  $\begin{pmatrix} 200 & 179 & j_3 \\ -30 & 34 & -4 \end{pmatrix}$ , the "baseball" family.

## Appendix IV — Certain Integrals Containing Associated Legendre Functions

In the evaluation of the mutual impedance, certain integrals containing associated Legendre functions are required. The integrals in (2.43), after substituting the vector spherical harmonics, are

$$\int_0^{2\pi} d\phi' \int_0^\pi r'^2 \sin \theta' d\theta' \left( \bar{\mathbf{M}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') \times \bar{\mathbf{M}}_{n,m}^{(2)}(\bar{\mathbf{r}}') \right) \cdot \hat{\mathbf{r}}' = j C_{n',m'} C_{n,m} \frac{r'^2 h_n^{(2)}(kr') z_{n'}^{(i)}(kr')}{4\pi} \int_0^{2\pi} e^{j(m+m')\phi'} d\phi' \int_{-1}^1 \left( m P_n^m(x) \frac{d}{dx} P_{n'}^{m'}(x) - m' P_{n'}^{m'}(x) \frac{d}{dx} P_n^m(x) \right) dx \quad (\text{IV-1})$$

$$\int_0^{2\pi} d\phi' \int_0^\pi r'^2 \sin \theta' d\theta' \left( \bar{\mathbf{M}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') \times \bar{\mathbf{N}}_{n,m}^{(2)}(\bar{\mathbf{r}}') \right) \cdot \hat{\mathbf{r}}' = -C_{n,m} C_{n',m'} \frac{r' z_{n'}^{(i)}(kr')}{4\pi k} \frac{\partial}{\partial r'} \left[ r' h_n^{(2)}(kr') \right] \int_0^{2\pi} e^{j(m+m')\phi'} d\phi' \int_{-1}^1 \left( (1-x^2) \frac{d}{dx} P_{n'}^{m'}(x) \frac{d}{dx} P_n^m(x) - \frac{mm'}{1-x^2} P_{n'}^{m'}(x) P_n^m(x) \right) dx \quad (\text{IV-2})$$

$$\int_0^{2\pi} d\phi' \int_0^\pi r'^2 \sin \theta' d\theta' \left( \bar{\mathbf{N}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') \times \bar{\mathbf{M}}_{n,m}^{(2)}(\bar{\mathbf{r}}') \right) \cdot \hat{\mathbf{r}}' = C_{n,m} C_{n',m'} \frac{r' h_n^{(2)}(kr')}{4\pi k} \frac{\partial}{\partial r'} \left[ r' z_{n'}^{(i)}(kr') \right] \int_0^{2\pi} e^{j(m+m')\phi'} d\phi' \int_{-1}^1 \left( (1-x^2) \frac{d}{dx} P_{n'}^{m'}(x) \frac{d}{dx} P_n^m(x) - \frac{mm'}{1-x^2} P_{n'}^{m'}(x) P_n^m(x) \right) dx \quad (\text{IV-3})$$

and

$$\int_0^{2\pi} d\phi' \int_0^\pi r'^2 \sin \theta' d\theta' \left( \bar{\mathbf{N}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') \times \bar{\mathbf{N}}_{n,m}^{(2)}(\bar{\mathbf{r}}') \right) \cdot \hat{\mathbf{r}}' = j C_{n',m'} C_{n,m} \frac{\frac{\partial}{\partial r'} \left[ r' h_n^{(2)}(kr') \right] \frac{\partial}{\partial r'} \left[ r' z_{n'}^{(i)}(kr') \right]}{4\pi k^2} \int_0^{2\pi} e^{j(m+m')\phi'} d\phi' \int_{-1}^1 \left[ m P_n^m(x) \frac{d}{dx} P_{n'}^{m'}(x) - m' P_{n'}^{m'}(x) \frac{d}{dx} P_n^m(x) \right] dx \quad (\text{IV-4})$$

Examination of (IV-1) through (IV-4) indicates that we only need to evaluate three integrals:

$$I_1(n, n', m) = \int_{-1}^1 (1-x^2) \frac{d}{dx} P_n^m(x) \frac{d}{dx} P_{n'}^{-m}(x) dx, \quad (\text{IV-5})$$

$$I_2(n, n', m) = m \int_{-1}^1 P_n^m(x) \frac{d}{dx} P_{n'}^{-m}(x) dx, \quad (\text{IV-6})$$

and

$$I_3(n, n', m) = -m^2 \int_{-1}^1 \frac{1}{1-x^2} P_n^m(x) P_{n'}^{-m}(x) dx. \quad (\text{IV-7})$$

The integrals (IV-5) and (IV-7) possess obvious symmetry properties:

$$\begin{aligned} I_1(n, n', m) &= I_1(n', n, -m) \\ I_1(n, n', -m) &= I_1(n', n, m) \end{aligned} \quad (\text{IV-8})$$

and

$$\begin{aligned} I_3(n, n', m) &= I_3(n', n, -m) \\ I_3(n, n', -m) &= I_3(n', n, m) \end{aligned} \quad (\text{IV-9})$$

It is convenient to express the integrals (IV-5)–(IV-7) in terms of integrals containing associated Legendre functions of only positive order,  $m \geq 0$ . The definition of the associated Legendre functions follows the convention of [3, 8, 9, 10, 11] and Appendix I. Thus, for negative order, we use

$$P_n^{-m}(x) = (-1)^m \frac{(n-m)!}{(n+m)!} P_n^m(x). \quad (\text{IV-10})$$

Using (IV-10) in (IV-5) to (IV-7), we have

$$\begin{aligned} I_1(n, n', m) &= (-1)^m \frac{(n'-m)!}{(n'+m)!} \int_{-1}^1 (1-x^2) \frac{d}{dx} P_n^m(x) \frac{d}{dx} P_{n'}^m(x) dx \\ &= (-1)^m \frac{(n'-m)!}{(n'+m)!} \mathcal{I}_1(n, n', m) \end{aligned} \quad (\text{IV-11})$$

$$\begin{aligned} I_2(n, n', m) &= (-1)^m \frac{(n'-m)!}{(n'+m)!} m \int_{-1}^1 P_n^m(x) \frac{d}{dx} P_{n'}^m(x) dx \\ &= (-1)^m \frac{(n'-m)!}{(n'+m)!} m \mathcal{I}_2(n, n', m) \end{aligned} \quad (\text{IV-12})$$

and

$$\begin{aligned} I_3(n, n', m) &= (-1)^{m+1} \frac{(n'-m)!}{(n'+m)!} m^2 \int_{-1}^1 \frac{1}{1-x^2} P_n^m(x) P_{n'}^m(x) dx \\ &= (-1)^{m+1} \frac{(n'-m)!}{(n'+m)!} m^2 \mathcal{I}_3(n, n', m) \end{aligned} \quad (\text{IV-13})$$

Thus, the required integrals are

$$\mathcal{I}_1(n, n', m) = \int_{-1}^1 (1-x^2) \frac{d}{dx} P_n^m(x) \frac{d}{dx} P_{n'}^m(x) dx, \quad (\text{IV-14})$$

$$\mathcal{I}_2(n, n', m) = \int_{-1}^1 P_n^m(x) \frac{d}{dx} P_{n'}^m(x) dx, \quad (\text{IV-15})$$

$$\mathcal{I}_3(n, n', m) = \int_{-1}^1 \frac{1}{1-x^2} P_n^m(x) P_{n'}^m(x) dx. \quad (\text{IV-16})$$

In addition, it will prove convenient to evaluate

$$\mathcal{I}_4(n, n', m, m') = \int_{-1}^1 \left( P_n^m(x) \frac{d}{dx} P_{n'}^{m'}(x) + P_{n'}^{m'}(x) \frac{d}{dx} P_n^m(x) \right) dx. \quad (\text{IV-17})$$

and

$$\mathcal{I}_0(n, n', m) = \int_{-1}^1 P_n^m(x) P_{n'}^m(x) dx. \quad (\text{IV-18})$$

## Establishing Orthogonality

The associated Legendre functions possess well-known orthogonality properties. The orthogonality relations are

$$\int_{-1}^1 P_n^m(x) P_{n'}^m(x) dx = \begin{cases} 0, & n \neq n' \\ \neq 0, & n = n' \end{cases} \quad (\text{IV-19})$$

and

$$\int_{-1}^1 \frac{1}{1-x^2} P_n^m(x) P_{n'}^m(x) dx = \begin{cases} 0, & |m| \neq |m'| \\ \neq 0, & |m| = |m'| \end{cases} \quad (\text{IV-20})$$

It will be useful to derive these orthogonality relations, in order to illuminate the relationship between the various integrals.

### Application of the differential equation

The associated Legendre differential equation for integer degree and order is

$$\frac{d}{dx} \left[ (1-x^2) \frac{d}{dx} P_n^m(x) \right] + \left[ n(n+1) - \frac{m^2}{1-x^2} \right] P_n^m(x) = 0. \quad (\text{IV-21})$$

Multiply (IV-21) by  $P_{n'}^{m'}(x)$

$$P_{n'}^{m'}(x) \frac{d}{dx} \left[ (1-x^2) \frac{d}{dx} P_n^m(x) \right] + \left[ n(n+1) - \frac{m^2}{1-x^2} \right] P_{n'}^{m'}(x) P_n^m(x) = 0. \quad (\text{IV-22})$$

Interchange  $n$  with  $n'$  and  $m$  with  $m'$  in (IV-22), subtract the new equation from (IV-22), and integrate over  $-1 \leq x \leq 1$  to obtain

$$\begin{aligned} & \int_{-1}^1 P_{n'}^{m'}(x) \frac{d}{dx} \left[ (1-x^2) \frac{d}{dx} P_n^m(x) \right] dx - \int_{-1}^1 P_n^m(x) \frac{d}{dx} \left[ (1-x^2) \frac{d}{dx} P_{n'}^{m'}(x) \right] dx \\ & + [n(n+1) - n'(n'+1)] \int_{-1}^1 P_{n'}^{m'}(x) P_n^m(x) dx - [m^2 - m'^2] \int_{-1}^1 \frac{1}{1-x^2} P_{n'}^{m'}(x) P_n^m(x) dx = 0 \end{aligned} \quad (\text{IV-23})$$

Applying the integration-by-parts procedure to the first two integrals

$$\begin{aligned}
& (1-x^2)P_n^{m'}(x)\frac{d}{dx}P_n^m(x)\Big|_{-1}^1 - \int_{-1}^1 (1-x^2)\frac{d}{dx}P_n^m(x)\frac{d}{dx}P_n^{m'}(x)dx \\
& - (1-x^2)P_n^m(x)\frac{d}{dx}P_n^{m'}(x)\Big|_{-1}^1 + \int_{-1}^1 (1-x^2)\frac{d}{dx}P_n^{m'}(x)\frac{d}{dx}P_n^m(x)dx \\
& + [n(n+1) - n'(n'+1)]\int_{-1}^1 P_n^{m'}(x)P_n^m(x)dx - [m^2 - m'^2]\int_{-1}^1 \frac{1}{1-x^2}P_n^{m'}(x)P_n^m(x)dx = 0
\end{aligned} \tag{IV-24}$$

Since  $P_n^m(x)$  is finite at  $x = \pm 1$  and  $\frac{d}{dx}P_n^m(x)$  has no worse than a  $\sqrt{1-x^2}$  singularity at  $x = \pm 1$ , then

$$[n(n+1) - n'(n'+1)]\int_{-1}^1 P_n^{m'}(x)P_n^m(x)dx - [m^2 - m'^2]\int_{-1}^1 \frac{1}{1-x^2}P_n^{m'}(x)P_n^m(x)dx = 0. \tag{IV-25}$$

If  $m = m'$ , then

$$\int_{-1}^1 P_n^m(x)P_n^m(x)dx = \begin{cases} 0, & n \neq n' \\ \neq 0, & n = n' \end{cases}. \tag{IV-26}$$

Thus,  $P_n^m(x)$  and  $P_n^{m'}(x)$  are orthogonal with weight one over the interval  $-1 \leq x \leq 1$ .

Similarly, if  $n = n'$ , then

$$\int_{-1}^1 \frac{1}{1-x^2}P_n^m(x)P_n^{m'}(x)dx = \begin{cases} 0, & |m| \neq |m'| \\ \neq 0, & |m| = |m'| \end{cases}, \tag{IV-27}$$

and  $P_n^m(x)$  and  $P_n^{m'}(x)$  are orthogonal with weight  $1/(1-x^2)$  over the interval  $-1 \leq x \leq 1$ . Thus, the orthogonality relations have been proven. It remains to evaluate the nonzero integrals.

## Relationship between the various integrals

Initially the closed-form of the integrals will be obtained for positive order,  $m \geq 0$ .

Negative order,  $m < 0$ , will be handled by applying the convention (IV-10), so

$$\mathcal{I}_i(n, n', -m) = \frac{(n-m)!(n'-m)!}{(n+m)!(n'+m)!}\mathcal{I}_i(n, n', m). \tag{IV-28}$$

Integrating (IV-22) produces

$$\begin{aligned}
\int_{-1}^1 (1-x^2)\frac{d}{dx}P_n^m(x)\frac{d}{dx}P_n^{m'}(x)dx &= n(n+1)\int_{-1}^1 P_n^m(x)P_n^{m'}(x)dx \\
&- m^2\int_{-1}^1 \left(\frac{1}{1-x^2}\right)P_n^m(x)P_n^{m'}(x)dx
\end{aligned} \tag{IV-29}$$

Thus, with  $m = m'$ , we obtain

$$\mathcal{I}_1(n, n', m) = n(n+1)\mathcal{I}_0(n, n', m) - m^2\mathcal{I}_3(n, n', m). \tag{IV-30}$$

Note that the integrals  $\mathcal{I}_0$ ,  $\mathcal{I}_1$ , and  $\mathcal{I}_3$  are symmetric with respect to interchange of  $n$  and  $n'$ . This symmetry, along with (IV-30) shows once again that

$$\int_{-1}^1 P_n^m(x) P_{n'}^m(x) dx = 0, n \neq n'. \quad (\text{IV-31})$$

### **Application of recursion relation**

The associated Legendre functions satisfy the following recursion relations [11]

$$xP_n^m(x) = \frac{n-m+1}{2n+1} P_{n+1}^m(x) + \frac{n+m}{2n+1} P_{n-1}^m(x), \quad (\text{IV-32})$$

and

$$(x^2-1) \frac{d}{dx} P_n^m(x) = nxP_n^m(x) - (n+m)P_{n-1}^m(x), \quad (\text{IV-33})$$

so that

$$(x^2-1) \frac{d}{dx} P_n^m(x) = \frac{n(n-m+1)}{2n+1} P_{n+1}^m(x) - \frac{(n+m)(n+1)}{2n+1} P_{n-1}^m(x), \quad (\text{IV-34})$$

valid for  $m \geq 0$ .

Substituting  $n'$  for  $n$  in (IV-34), multiplying by  $P_n^m(x)/(1-x^2)$ , and integrating, we obtain

$$\begin{aligned} \mathcal{I}_2(n, n', m) &= \int_{-1}^1 P_n^m(x) \frac{d}{dx} P_{n'}^m(x) dx \\ &= -\frac{n'(n'-m+1)}{(2n'+1)} \int_{-1}^1 \frac{P_n^m(x) P_{n'+1}^m(x)}{(1-x^2)} dx + \frac{(n'+m)(n'+1)}{(2n'+1)} \int_{-1}^1 \frac{P_n^m(x) P_{n'-1}^m(x)}{(1-x^2)} dx, \end{aligned} \quad (\text{IV-35})$$

or

$$\mathcal{I}_2(n, n', m) = \frac{(n'+m)(n'+1)}{(2n'+1)} \mathcal{I}_3(n, n'-1, m) - \frac{n'(n'-m+1)}{(2n'+1)} \mathcal{I}_3(n, n'+1, m) \text{ when } m \geq 0. \quad (\text{IV-36})$$

If expressions for the integrals  $\mathcal{I}_0(n, n', m)$  and  $\mathcal{I}_3(n, n', m)$  can be found, then

$\mathcal{I}_1(n, n', m)$  can be obtained from (IV-30) and  $\mathcal{I}_2(n, n', m)$  can be obtained from (IV-36)

### **Evaluation of $\mathcal{I}_0(n, n', m) = \int_{-1}^1 P_n^m(x) P_{n'}^m(x) dx$**

The explicit form of the associated Legendre function given by <sup>†</sup>

---

<sup>†</sup> Hanson [13], Stratton [15], Arfken [16], and Mathews and Walker [18] omit the factor  $(-1)^m$ , but it is included by Jackson [8], Abramowitz and Stegun [9], Chew [10], Lebedev [11], and Balanis [3].

$$P_n^m(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_n(x) \quad (\text{IV-37})$$

where  $m \geq 0$ , and  $P_n(x)$  is the Legendre function

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n. \quad (\text{IV-38})$$

Using (IV-38) in (IV-37)

$$P_n^m(x) = (-1)^m (1-x^2)^{m/2} \frac{1}{2^n n!} \frac{d^{n+m}}{dx^{n+m}} (x^2 - 1)^n. \quad (\text{IV-39})$$

Thus, the integral becomes

$$\begin{aligned} \mathcal{I}_0(n, n', m) &= \int_{-1}^1 P_n^m(x) P_{n'}^m(x) dx \\ &= \frac{1}{2^{n+n'} n! n'!} \int_{-1}^1 (1-x^2)^m \frac{d^{n+m}}{dx^{n+m}} (x^2 - 1)^n \frac{d^{n'+m}}{dx^{n'+m}} (x^2 - 1)^{n'} dx \end{aligned} \quad (\text{IV-40})$$

Since we have already shown that  $\mathcal{I}_0(n, n', m) = 0$  when  $n \neq n'$  (IV-26), we need to evaluate

$$\int_{-1}^1 [P_n^m(x)]^2 dx = \frac{1}{2^{2n} (n!)^2} \int_{-1}^1 (1-x^2)^m \frac{d^{n+m}}{dx^{n+m}} (x^2 - 1)^n \frac{d^{n+m}}{dx^{n+m}} (x^2 - 1)^n dx. \quad (\text{IV-41})$$

Integrating by parts  $(n+m)$  times

$$\int_{-1}^1 [P_n^m(x)]^2 dx = \mathcal{I}_\Sigma(n, m) + \mathcal{I}_J(n, m), \quad (\text{IV-42})$$

where

$$\mathcal{I}_\Sigma(n, m) = \frac{1}{2^{2n} (n!)^2} \sum_{k=1}^{n+m} (-1)^{k-1} \frac{d^{n+m-k}}{dx^{n+m-k}} (x^2 - 1)^n \frac{d^{k-1}}{dx^{k-1}} \left( (1-x^2)^m \frac{d^{n+m}}{dx^{n+m}} (x^2 - 1)^n \right) \Big|_{-1}^1, \quad (\text{IV-43})$$

and

$$\mathcal{I}_J(n, m) = \frac{(-1)^{n+m}}{2^{2n} (n!)^2} \int_{-1}^1 (x^2 - 1)^n \frac{d^{n+m}}{dx^{n+m}} \left( (1-x^2)^m \frac{d^{n+m}}{dx^{n+m}} (x^2 - 1)^n \right) dx. \quad (\text{IV-44})$$

Applying Leibnitz' differentiation formula <sup>‡</sup> [16]

<sup>‡</sup> Leibnitz' formula for the  $n^{\text{th}}$  derivative of a product:

$$\frac{d^n}{dx^n} [A(x)B(x)] = \sum_{s=0}^n \frac{n!}{s!(n-s)!} \frac{d^{n-s}}{dx^{n-s}} A(x) \frac{d^s}{dx^s} B(x)$$

$$\mathcal{I}_\Sigma(n, m) \frac{1}{2^{2n} (n!)^2} \sum_{k=1}^{n+m} \left[ (-1)^{k-1} \left[ \frac{d^{n+m-k}}{dx^{n+m-k}} (x^2 - 1)^n \right] \cdot \sum_{t=0}^{k-1} \frac{(k-1)!}{t!(k-1-t)!} \left[ \frac{d^{k-1-t}}{dx^{k-1-t}} (1-x^2)^m \right] \left[ \frac{d^{n+m+t}}{dx^{n+m+t}} (x^2 - 1)^n \right] \right]_{-1}^1, \quad (\text{IV-45})$$

and

$$\mathcal{I}_J(n, m) = \frac{(-1)^{n+m}}{2^{2n} (n!)^2} \int_{-1}^1 (x^2 - 1)^n \sum_{s=0}^{n+m} \left[ \frac{(n+m)!}{s!(n+m-s)!} \left[ \frac{d^{n+m-s}}{dx^{n+m-s}} (1-x^2)^m \right] \cdot \left[ \frac{d^{n+m+s}}{dx^{n+m+s}} (x^2 - 1)^n \right] \right] dx. \quad (\text{IV-46})$$

Using the binomial expansion, we can obtain the derivatives

$$\begin{aligned} \frac{d^q}{dx^q} (1-x^2)^\alpha &= \frac{d^q}{dx^q} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(\alpha+1)}{k! \Gamma(\alpha+1-k)} x^{2k} \\ &= \sum_{k=q/2}^{\infty} (-1)^k \frac{\Gamma(\alpha+1)(2k)!}{k! \Gamma(\alpha+1-k)(2k-q)!} x^{2k-q}, \end{aligned} \quad (\text{IV-47})$$

and

$$\frac{d^q}{dx^q} (x^2 - 1)^\alpha = (-1)^\alpha \sum_{k=q/2}^{\infty} (-1)^k \frac{\Gamma(\alpha+1)(2k)!}{k! \Gamma(\alpha+1-k)(2k-q)!} x^{2k-q}, \quad (\text{IV-48})$$

where  $q$  is an integer and  $\alpha$  is a real number, not necessarily an integer. For convergence, we require  $|x| < 1$ . Note that the infinite sums in (IV-47) and (IV-48) will terminate when  $k = \alpha$ , for  $\alpha$  a positive integer. However, if  $\alpha$  is not a positive integer, an infinite number of terms will occur in the sum.

When  $\alpha = n$ , an integer, the derivatives also can be expanded into another useful form as follows (using equation 0.432-3 from Gradshteyn and Ryzhik [17]),

$$\frac{d^q}{dx^q} (x^2 - 1)^n = n! \sum_{k=0}^{\text{int}(q/2)} \frac{q!}{k!(q-2k)!(n-q+k)!} 2^{q-2k} x^{q-2k} (x^2 - 1)^{n-q+k}. \quad (\text{IV-49})$$

or

$$\frac{d^q}{dx^q} (1-x^2)^n = (-1)^q n! \sum_{k=0}^{\text{int}(q/2)} (-1)^k \frac{q!}{k!(q-2k)!(n-q+k)!} 2^{q-2k} x^{q-2k} (1-x^2)^{n-q+k}. \quad (\text{IV-50})$$

This form is useful for evaluating the derivative at the values  $x = \pm 1$ , but *only* when the exponent is an integer. For  $\alpha = n$ , an integer, if  $q > 2n$ , all the terms in (IV-49) and (IV-50) will be zero ( $n - q + k$  is always negative). Additionally, if  $q < n$ , there will always be a factor of  $(1-x^2)$  so that

$$\left. \frac{d^q}{dx^q} (1-x^2)^n \right|_{x=\pm 1} = 0 \text{ when } q < n, \quad (\text{IV-51})$$

and, when  $q = n$ , there will be a single term

$$\left. \frac{d^n}{dx^n} (1-x^2)^n \right|_{x=\pm 1} = (\mp 1)^n n! 2^n. \quad (\text{IV-52})$$

Note that the  $\pm 1$  on the left side of (IV-52) correlates with the  $\mp 1$  on the right side.

When  $q = 2n$ , the derivative has only a constant term

$$\frac{d^{2n}}{dx^{2n}} (1-x^2)^n = (-1)^n (2n)!. \quad (\text{IV-53})$$

We see from (IV-49) that  $\left. \frac{d^q}{dx^q} (x^2 - 1)^n \right|_{x=\pm 1}$  can only be nonzero if  $n \leq q \leq 2n$ . Thus, in

$\mathcal{G}_\Sigma(n, m)$ , the factor  $\left[ \frac{d^{n+m-k}}{dx^{n+m-k}} (x^2 - 1)^n \right]_{-1}^1$  is zero for  $k > m$ , while  $\left[ \frac{d^{k-1-t}}{dx^{k-1-t}} (1-x^2)^m \right]_{-1}^1$

will be zero for  $k < m + 1$ . Therefore,  $\mathcal{G}_\Sigma(n, m)$  will be zero and

$$\int_{-1}^1 [P_n^m(x)]^2 dx = \mathcal{G}_{o,f}(n, m, n, m) = \frac{(-1)^{n+m}}{2^{2n} (n!)^2} \int_{-1}^1 (x^2 - 1)^n \sum_{s=0}^{n+m} \frac{(n+m)!}{s!(n+m-s)!} \left[ \frac{d^{n+m-s}}{dx^{n+m-s}} (1-x^2)^m \right] \left[ \frac{d^{n+m+s}}{dx^{n+m+s}} (x^2 - 1)^n \right] dx. \quad (\text{IV-54})$$

The left derivative in (IV-54) is zero everywhere unless  $s \geq n - m$ , while the right derivative is zero everywhere unless  $s \leq n - m$ . Thus only the term with  $s = (n - m)$  will contribute to the integral, and

$$\int_{-1}^1 [P_n^m(x)]^2 dx = \frac{(-1)^n (2n)! (n+m)!}{2^{2n} (n!)^2 (n-m)!} \int_{-1}^1 (x^2 - 1)^n dx. \quad (\text{IV-55})$$

Letting  $x = \cos \theta$ , we obtain [17]

$$\int_{-1}^1 (x^2 - 1)^n dx = (-1)^n \int_0^\pi \sin^{2n+1} \theta d\theta = (-1)^n 2 \frac{(2n)!!}{(2n+1)!!} = (-1)^n \frac{n!}{\Gamma(n+3/2)} \Gamma(1/2). \quad (\text{IV-56})$$

Since  $(2n)!! = 2^n n!$  and  $(2n-1)!! = 2^n \Gamma(n + \frac{1}{2}) / \sqrt{\pi}$  [9], it follows that

$$\frac{(2n)!}{2^n n!} = \frac{(2n)!}{(2n)!!} = (2n-1)!!,$$

and we obtain the well-known result [9]

$$\mathcal{G}_0(n, n, m) = \int_{-1}^1 [P_n^m(x)]^2 dx = \frac{2}{(2n+1)} \frac{(n+m)!}{(n-m)!} \text{ when } m \geq 0. \quad (\text{IV-57})$$

Using the convention for negative order (IV-10)

$$\begin{aligned}
\mathcal{I}_0(n, n, -m) &= \int_{-1}^1 [P_n^{-m}(x)]^2 dx \\
&= \left( \frac{(n-m)!}{(n+m)!} \right)^2 \int_{-1}^1 [P_n^m(x)]^2 dx = \frac{2}{(2n+1)} \frac{(n-m)!}{(n+m)!} \text{ when } m \geq 0
\end{aligned} \tag{IV-58}$$

Thus, combining (IV-57) and (IV-58), we have

$$\boxed{\mathcal{I}_0(n, n', m) = \int_{-1}^1 P_n^m(x) P_{n'}^m(x) dx = \begin{cases} 0 & \text{when } n' \neq n \\ \frac{2}{(2n+1)} \frac{(n+m)!}{(n-m)!} & \text{when } n' = n \end{cases}} \tag{IV-59}$$

**Evaluation of**  $\mathcal{I}_3(n, n', m) = \int_{-1}^1 \frac{1}{1-x^2} P_n^m(x) P_{n'}^m(x) dx$

Inserting (IV-39) into the integrand of  $\mathcal{I}_3(n, n', m)$ ,

$$\begin{aligned}
\mathcal{I}_3(n, n', m) &= \int_{-1}^1 \left( \frac{1}{1-x^2} \right) P_{n'}^m(x) P_n^m(x) dx \\
&= \frac{1}{2^{n+n'} n'! n!} \int_{-1}^1 (1-x^2)^{m-1} \frac{d^{n'+m}}{dx^{n'+m}} (x^2-1)^{n'} \frac{d^{n+m}}{dx^{n+m}} (x^2-1)^n dx
\end{aligned} \tag{IV-60}$$

where we require  $m \geq 0$ . We integrate by parts  $p$  times

$$\begin{aligned}
&\int_{-1}^1 \left( \frac{1}{1-x^2} \right) P_{n'}^m(x) P_n^m(x) dx = \\
&= \frac{1}{2^{n+n'} n'! n!} \left\{ \sum_{k=1}^p (-1)^{k-1} \frac{d^{k-1}}{dx^{k-1}} \left[ (1-x^2)^{m-1} \frac{d^{n'+m}}{dx^{n'+m}} (x^2-1)^{n'} \right] \frac{d^{n+m-k}}{dx^{n+m-k}} (x^2-1)^n \right\} \Bigg|_{-1}^1 \\
&\quad \left. + (-1)^p \int_{-1}^1 \frac{d^{n+m-p}}{dx^{n+m-p}} (x^2-1)^n \frac{d^p}{dx^p} \left[ (1-x^2)^{m-1} \frac{d^{n'+m}}{dx^{n'+m}} (x^2-1)^{n'} \right] dx \right\}
\end{aligned} \tag{IV-61}$$

Applying Leibnitz' product differentiation formula to (IV-61) gives

$$\begin{aligned}
&\int_{-1}^1 \left( \frac{1}{1-x^2} \right) P_{n'}^m(x) P_n^m(x) dx = \\
&= \frac{1}{2^{n+n'} n'! n!} \left\{ \sum_{k=1}^p (-1)^{k-1} \left[ \frac{d^{n+m-k}}{dx^{n+m-k}} (x^2-1)^n \right] \right. \\
&\quad \left. + (-1)^p \int_{-1}^1 \frac{d^{n+m-p}}{dx^{n+m-p}} (x^2-1)^n \left[ \sum_{s=0}^{k-1} \frac{(k-1)!}{s!(k-1-s)!} \left[ \frac{d^{k-1-s}}{dx^{k-1-s}} (1-x^2)^{m-1} \right] \left[ \frac{d^{n'+m+s}}{dx^{n'+m+s}} (x^2-1)^{n'} \right] \right] \right. \\
&\quad \left. + (-1)^p \int_{-1}^1 \left[ \frac{d^{n+m-p}}{dx^{n+m-p}} (x^2-1)^n \right] \left[ \sum_{s=0}^p \frac{p!}{s!(p-s)!} \left[ \frac{d^{p-s}}{dx^{p-s}} (1-x^2)^{m-1} \right] \left[ \frac{d^{n'+m+s}}{dx^{n'+m+s}} (x^2-1)^{n'} \right] dx \right] \right\}
\end{aligned} \tag{IV-62}$$

We see that the right-hand factor in the sum outside the integral in (IV-62),

$$\left. \frac{d^{n+m-k}}{dx^{n+m-k}} (x^2-1)^n \right|_{x=\pm 1}, \text{ is nonzero only when } m-n \leq k \leq m, \text{ while } \left. \frac{d^{k-1-s}}{dx^{k-1-s}} (1-x^2)^{m-1} \right|_{x=\pm 1}$$

is nonzero only when  $m \leq k - s \leq 2m - 1$ . Thus, the nonzero contribution occurs for  $k = m$  and  $s = 0$ , so that for  $p \geq m$

$$\int_{-1}^1 \left( \frac{1}{1-x^2} \right) P_{n'}^m(x) P_n^m(x) dx = \frac{1}{2^{n+n'} n'! n!} \left\{ \begin{aligned} & (-1)^{m-1} \left[ \frac{d^n}{dx^n} (x^2 - 1)^n \right] \left[ \frac{d^{m-1}}{dx^{m-1}} (1-x^2)^{m-1} \right] \left[ \frac{d^{n'+m}}{dx^{n'+m}} (x^2 - 1)^{n'} \right] \Big|_{-1}^1 \\ & + (-1)^p \int_{-1}^1 \left[ \frac{d^{n+m-p}}{dx^{n+m-p}} (x^2 - 1)^n \right] \sum_{s=0}^p \frac{p!}{s!(p-s)!} \left[ \frac{d^{p-s}}{dx^{p-s}} (1-x^2)^{m-1} \right] \left[ \frac{d^{n'+m+s}}{dx^{n'+m+s}} (x^2 - 1)^{n'} \right] dx \end{aligned} \right\}. \quad (\text{IV-63})$$

The integrand in (IV-63) will be nonzero for  $s \geq p - 2(m - 1)$  and  $s \leq n' - m$ . If we take  $p = n + m$ , the integrand is nonzero only when  $n - m + 2 \leq s \leq n' - m$ .

At this point, we have not specified the relative size of  $n'$  and  $n$ . If we choose  $n \geq n'$ , we see that no value of  $s$  allows a nonzero integrand. We simply obtain

$$\mathcal{J}_3(n, n', m) = \int_{-1}^1 \left( \frac{1}{1-x^2} \right) P_{n'}^m(x) P_n^m(x) dx = \begin{cases} 0 & \text{when } n' + n \text{ odd} \\ \frac{(n' + m)!}{m(n' - m)!} & \text{when } n' + n \text{ even, } n \geq n', \text{ and } m > 0 \\ \text{undefined} & \text{when } m = 0 \end{cases}. \quad (\text{IV-64})$$

Noting the symmetry of  $\mathcal{J}_3(n, n', m)$  with respect to  $n$  and  $n'$ , and using the convention (IV-10) for negative order ( $m < 0$ ), we write

$$\mathcal{J}_3(n, n', m) = \int_{-1}^1 \left( \frac{1}{1-x^2} \right) P_{n'}^m(x) P_n^m(x) dx = \begin{cases} 0 & \text{when } n' + n \text{ odd} \\ \frac{(\min(n, n') + m)!}{m(\min(n, n') - m)!} & \text{when } n' + n \text{ even and } m > 0 \\ -\frac{(\max(n, n') + m)!}{m(\max(n, n') - m)!} & \text{when } n' + n \text{ even and } m < 0 \\ \text{undefined} & \text{when } m = 0 \end{cases} \quad (\text{IV-65})$$

Note that  $\mathcal{J}_3(n, n', m)$  is undefined when  $m = 0$ . However, in cases where the integral is multiplied by  $m$ , letting  $m$  go to zero prior to performing the integration takes care of the problem.

**Evaluation of**  $\mathcal{I}_1(n, n', m) = \int_{-1}^1 (1-x^2) \frac{d}{dx} P_n^m(x) \frac{d}{dx} P_{n'}^m(x) dx$

The relationship between  $\mathcal{I}_1(n, n', m)$ ,  $\mathcal{I}_0(n, n', m)$ , and  $\mathcal{I}_3(n, n', m)$  is given in (IV-30). Substitution of (IV-59) and (IV-65) into (IV-30) gives

$$\mathcal{I}_1(n, n', m) = \begin{cases} 0 & \text{when } n' + n \text{ odd} \\ \frac{2n(n+1)(n+m)!}{(2n+1)(n-m)!} \delta_{n,n'} - m \frac{(\min(n, n') + m)!}{(\min(n, n') - m)!} & \text{when } n' + n \text{ even and } m \geq 0 \\ \frac{2n(n+1)(n+m)!}{(2n+1)(n-m)!} \delta_{n,n'} + m \frac{(\max(n, n') + m)!}{(\max(n, n') - m)!} & \text{when } n' + n \text{ even and } m < 0 \end{cases}, \quad (\text{IV-66})$$

where

$$\delta_{n,n'} = \begin{cases} 0 & \text{when } n \neq n' \\ 1 & \text{when } n = n' \end{cases}$$

**Evaluation of**  $\mathcal{I}_2(n, n', m) = \int_{-1}^1 P_n^m(x) \frac{d}{dx} P_{n'}^m(x) dx$

The relationship between  $\mathcal{I}_2(n, n', m)$  and  $\mathcal{I}_3(n, n', m)$  is given by (IV-36). Substitution of (IV-65) into (IV-36) gives

$$\mathcal{I}_2(n, n', m) = \begin{cases} 0; & n' + n \text{ even} \\ \frac{1}{m(2n'+1)} \begin{pmatrix} (n'+1)(n'+m) \frac{(\min(n, n'-1) + m)!}{(\min(n, n'-1) - m)!} \\ -n'(n'+1-m) \frac{(\min(n, n'+1) + m)!}{(\min(n, n'+1) - m)!} \end{pmatrix}; & \begin{cases} n' + n \text{ odd} \\ \text{and } m > 0 \end{cases} \\ -\frac{1}{m(2n'+1)} \begin{pmatrix} (n'+1)(n'+m) \frac{(\max(n, n'-1) + m)!}{(\max(n, n'-1) - m)!} \\ -n'(n'+1-m) \frac{(\max(n, n'+1) + m)!}{(\max(n, n'+1) - m)!} \end{pmatrix}; & \begin{cases} n' + n \text{ odd} \\ \text{and } m < 0 \end{cases} \end{cases}, \quad (\text{IV-67})$$

Note that (IV-67) does not give  $\mathcal{J}_2(n, n', m)$  when  $m = 0$  if  $n' + n$  is odd. Because  $\mathcal{J}_3(n, n', m = 0)$  is undefined, (IV-36) cannot be used to obtain  $\mathcal{J}_2(n, n', m)$  when  $m = 0$ . Taking a direct approach, and integrating (IV-15) by parts, with  $m = 0$ ,

$$\begin{aligned}\mathcal{J}_2(n, n', 0) &= \int_{-1}^1 P_n^0(x) \frac{d}{dx} P_{n'}^0(x) dx \\ &= P_n^0(x) P_{n'}^0(x) \Big|_{-1}^1 - \int_{-1}^1 P_{n'}^0(x) \frac{d}{dx} P_n^0(x) dx\end{aligned}\tag{IV-68}$$

Thus, we have

$$\mathcal{J}_2(n, n', 0) + \mathcal{J}_2(n', n, 0) = P_n^0(1) P_{n'}^0(1) - P_n^0(-1) P_{n'}^0(-1) = \begin{cases} 0; & n + n' \text{ even} \\ 2; & n + n' \text{ odd} \end{cases}.\tag{IV-69}$$

Substituting (IV-39) for  $m = 0$  into (IV-15)

$$\begin{aligned}\mathcal{J}_2(n, n', 0) &= \int_{-1}^1 P_n^0(x) \frac{d}{dx} P_{n'}^0(x) dx \\ &= \frac{1}{2^{n+n'} n! n'!} \int_{-1}^1 \frac{d^n}{dx^n} (x^2 - 1)^n \frac{d^{n'+1}}{dx^{n'+1}} (x^2 - 1)^{n'} dx\end{aligned}\tag{IV-70}$$

Integrating (IV-70) by parts one time

$$\mathcal{J}_2(n, n', 0) = \frac{1}{2^{n+n'} n! n'!} \left[ \frac{d^{n-1}}{dx^{n-1}} (x^2 - 1)^n \frac{d^{n'+1}}{dx^{n'+1}} (x^2 - 1)^{n'} \Big|_{-1}^1 - \int_{-1}^1 \frac{d^{n-1}}{dx^{n-1}} (x^2 - 1)^n \frac{d^{n'+2}}{dx^{n'+2}} (x^2 - 1)^{n'} dx \right].\tag{IV-71}$$

From (IV-51), we see that the first term in the brackets is zero. The integral can be further evaluated by the integration-by-parts procedure. Applying this procedure a total of  $n$  times,

$$\mathcal{J}_2(n, n', 0) = \frac{(-1)^n}{2^{n+n'} n! n'!} \int_{-1}^1 (x^2 - 1)^n \frac{d^{n'+n+1}}{dx^{n'+n+1}} (x^2 - 1)^{n'} dx.\tag{IV-72}$$

From (IV-53), we see that

$$\frac{d^{n'+n+1}}{dx^{n'+n+1}} (x^2 - 1)^{n'} = 0 \text{ when } n \geq n'.\tag{IV-73}$$

Thus, we have

$$\mathcal{J}_2(n, n', 0) = 0 \text{ when } n \geq n'.\tag{IV-74}$$

Using this information with (IV-69)

$$\mathcal{J}_2(n, n', 0) = \begin{cases} 0; & n + n' \text{ even} \\ 0; & n + n' \text{ odd and } n > n'. \\ 2; & n + n' \text{ odd and } n' > n \end{cases}\tag{IV-75}$$

The complete integral is

$$\mathcal{I}_2(n, n', m) = \int_{-1}^1 P_n^m(x) \frac{d}{dx} P_{n'}^m(x) dx$$

$$= \begin{cases} 0; & \begin{cases} n' + n \text{ even, for all } m \\ n + n' \text{ odd, } n > n', \text{ and } m = 0 \end{cases} \\ 2; & n + n' \text{ odd, } n' > n, \text{ and } m = 0 \\ \frac{1}{m(2n' + 1)} \begin{pmatrix} (n' + 1)(n' + m) \frac{(\min(n, n' - 1) + m)!}{(\min(n, n' - 1) - m)!} \\ -n'(n' + 1 - m) \frac{(\min(n, n' + 1) + m)!}{(\min(n, n' + 1) - m)!} \end{pmatrix}, & \begin{cases} n' + n \text{ odd} \\ \text{and } m > 0 \end{cases} \\ -\frac{1}{m(2n' + 1)} \begin{pmatrix} (n' + 1)(n' + m) \frac{(\max(n, n' - 1) + m)!}{(\max(n, n' - 1) - m)!} \\ -n'(n' + 1 - m) \frac{(\max(n, n' + 1) + m)!}{(\max(n, n' + 1) - m)!} \end{pmatrix}, & \begin{cases} n' + n \text{ odd} \\ \text{and } m < 0 \end{cases} \end{cases}$$

(IV-76)

**Evaluation of**  $I_1(n, n', m) = \int_{-1}^1 (1 - x^2) \frac{d}{dx} P_n^m(x) \frac{d}{dx} P_{n'}^{-m}(x) dx$

The integral  $\mathcal{I}_1$  is evaluated above, so using (IV-11) and (IV-66),

$$I_1(n, n', m) = (-1)^m \begin{cases} 0; & n' + n \text{ odd} \\ \frac{2n(n+1)}{(2n+1)} \delta_{n,n'} - m \frac{(n' - m)! (\min(n, n') + m)!}{(n' + m)! (\min(n, n') - m)!}, & \begin{cases} n' + n \text{ even} \\ \text{and } m \geq 0 \end{cases} \\ \frac{2n(n+1)}{(2n+1)} \delta_{n,n'} + m \frac{(n' - m)! (\max(n, n') + m)!}{(n' + m)! (\max(n, n') - m)!}, & \begin{cases} n' + n \text{ even} \\ \text{and } m < 0 \end{cases} \end{cases}$$

(IV-77)

**Evaluation of**  $I_2(n, n', m) = m \int_{-1}^1 P_n^m(x) \frac{d}{dx} P_{n'}^{-m}(x) dx$

The integral  $\mathcal{I}_2$  has been evaluated above, so with (IV-12) and (IV-67),

$$I_2(n, n', m) = \begin{cases} 0; & n' + n \text{ even} \\ \frac{(-1)^m}{(2n' + 1)} \begin{pmatrix} (n' + 1) \frac{(n' - m)! (\min(n, n' - 1) + m)!}{(n' - 1 + m)! (\min(n, n' - 1) - m)!} \\ -n' \frac{(n' + 1 - m)! (\min(n, n' + 1) + m)!}{(n' + m)! (\min(n, n' + 1) - m)!} \end{pmatrix}; & \begin{cases} n' + n \text{ odd} \\ \text{and } m \geq 0 \end{cases} \\ \frac{(-1)^{m+1}}{(2n' + 1)} \begin{pmatrix} (n' + 1) \frac{(n' - m)! (\max(n, n' - 1) + m)!}{(n' - 1 + m)! (\max(n, n' - 1) - m)!} \\ -n' \frac{(n' + 1 - m)! (\max(n, n' + 1) + m)!}{(n' + m)! (\max(n, n' + 1) - m)!} \end{pmatrix}; & \begin{cases} n' + n \text{ odd} \\ \text{and } m < 0 \end{cases} \end{cases} \quad (\text{IV-78})$$

**Evaluation of**  $I_3(n, n', m) = -m^2 \int_{-1}^1 \frac{1}{1-x^2} P_n^m(x) P_{n'}^{-m}(x) dx$

The integral  $\mathcal{I}_3$  is evaluated above, and using (IV-13) and (IV-65),

$$I_3(n, n', m) = \begin{cases} 0; & n' + n \text{ odd} \\ (-1)^{m+1} m \frac{(n' - m)! (\min(n, n') + m)!}{(n' + m)! (\min(n, n') - m)!}; & n' + n \text{ even and } m \geq 0 \\ (-1)^m m \frac{(n' - m)! (\max(n, n') + m)!}{(n' + m)! (\max(n, n') - m)!}; & n' + n \text{ even and } m < 0 \end{cases} \quad (\text{IV-79})$$

**Evaluation of**  $\mathcal{I}_4(n, n', m, m') = \int_{-1}^1 \left( P_n^m(x) \frac{d}{dx} P_{n'}^{m'}(x) + P_{n'}^{m'}(x) \frac{d}{dx} P_n^m(x) \right) dx$

Although similar to (IV-15), it is useful to separately evaluate

$$\mathcal{I}_4(n, n', m, m') = \int_{-1}^1 \left( P_n^m(x) \frac{d}{dx} P_{n'}^{m'}(x) + P_{n'}^{m'}(x) \frac{d}{dx} P_n^m(x) \right) dx. \quad (\text{IV-17})$$

Because of the symmetry in  $\mathcal{I}_4(n, n', m, m')$ , it can be integrated easily with the procedure of integration by parts. Let

$$u(x) = P_n^m(x), \text{ so } du = \frac{d}{dx} P_n^m(x) dx, \quad (\text{IV-80})$$

and

$$dv = \frac{d}{dx} P_{n'}^{m'}(x) dx, \text{ so } v(x) = P_{n'}^{m'}(x). \quad (\text{IV-81})$$

Inserting (IV-80) and (IV-81) into (IV-17)

$$\mathcal{G}_4(n, n', m, m') = \int_{-1}^1 u dv + \int_{-1}^1 v du. \quad (\text{IV-82})$$

Applying the integration-by-parts procedure one time,

$$\begin{aligned} \mathcal{G}_4(n, n', m, m') &= 2u(x)v(x)\Big|_{x=-1}^{x=1} - \int_{-1}^1 u dv - \int_{-1}^1 v du \\ &= 2u(x)v(x)\Big|_{x=-1}^{x=1} - \mathcal{G}_4(n, n', m, m') \end{aligned} \quad (\text{IV-83})$$

Thus,

$$\boxed{\begin{aligned} \mathcal{G}_4(n, n', m, m') &= P_n^m(1)P_{n'}^{m'}(1) - P_n^m(-1)P_{n'}^{m'}(-1) \\ &= \begin{cases} 1 + (-1)^{n+n'+1}, & m = m' = 0 \\ 0, & m \neq 0 \text{ or } m' \neq 0 \end{cases} \end{aligned}} \quad (\text{IV-84})$$

since

$$P_n^m(1) = \begin{cases} 1, & m = 0 \\ 0, & m \neq 0 \end{cases}$$

and

$$P_n^m(-1) = \begin{cases} (-1)^n, & m = 0 \\ 0, & m \neq 0 \end{cases} \quad (\text{IV-85})$$

**Evaluation of**  $m \left( \int_{-1}^1 P_n^m(x) \frac{d}{dx} P_{n'}^{-m}(x) dx + \int_{-1}^1 P_{n'}^{-m}(x) \frac{d}{dx} P_n^m(x) dx \right)$

Using (IV-15), we have

$$m \left( \int_{-1}^1 P_n^m(x) \frac{d}{dx} P_{n'}^{-m}(x) dx + \int_{-1}^1 P_{n'}^{-m}(x) \frac{d}{dx} P_n^m(x) dx \right) = I_2(n, n', m) - I_2(n', n, -m) \quad (4.86)$$

However, simplification is obtained by observing that (IV-78) is not the best way to evaluate  $I_2(n, n', m) - I_2(n', n, -m)$ . Rather, consider that

$$\begin{aligned} I_2(n, n', m) - I_2(n', n, -m) &= m \int_{-1}^1 P_n^m(x) \frac{d}{dx} P_{n'}^{-m}(x) dx - \left( -m \int_{-1}^1 P_{n'}^{-m}(x) \frac{d}{dx} P_n^m(x) dx \right) \\ &= m \mathcal{G}_4(n, n', m, -m) \end{aligned} \quad (\text{IV-87})$$

Using (IV-84), we obtain

$$\boxed{I_2(n, n', m) - I_2(n', n, -m) = 0}. \quad (\text{IV-88})$$

**Evaluation of**  $\mathcal{I}_{\bar{\mathbf{x}}\bar{\mathbf{x}}}(n, m, n', m') = \int_0^{2\pi} d\phi' \int_0^\pi \bar{\mathbf{X}}_{n',m'}(\theta', \phi') \cdot \bar{\mathbf{X}}_{n,m}(\theta', \phi') \sin \theta' d\theta'$

Evaluate:

$$\mathcal{I}_{\bar{\mathbf{x}}\bar{\mathbf{x}}}(n, m, n', m') = \int_0^{2\pi} d\phi' \int_0^\pi \bar{\mathbf{X}}_{n',m'}(\theta', \phi') \cdot \bar{\mathbf{X}}_{n,m}(\theta', \phi') \sin \theta' d\theta', \quad (\text{IV-89})$$

The radially-independent vector spherical harmonic is (see Appendix I)

$$\bar{\mathbf{X}}_{n,m}(\theta, \phi) = jC_{n,m} \frac{e^{jm\phi}}{2\sqrt{\pi}} \left[ \frac{jm}{\sin \theta} P_n^m(\cos \theta) \hat{\boldsymbol{\theta}} + \sin \theta \frac{d}{dx} P_n^m(x) \Big|_{x=\cos \theta} \hat{\boldsymbol{\phi}} \right], \quad (\text{IV-90})$$

where

$$C_{n,m} = \sqrt{\frac{2n+1}{n(n+1)} \frac{(n-m)!}{(n+m)!}}. \quad (\text{IV-91})$$

Substitution of (IV-90) into (IV-89)

$$\mathcal{I}_{\bar{\mathbf{x}}\bar{\mathbf{x}}}(n, m, n', m') = -C_{n,m} C_{n',m'} \int_0^{2\pi} \frac{e^{j(m+m')\phi}}{4\pi} d\phi' \int_0^\pi \left[ \begin{array}{l} -\frac{mm'}{\sin^2 \theta'} P_{n'}^{m'}(\cos \theta') P_n^m(\cos \theta') \\ + \sin^2 \theta' \frac{d}{dx} P_{n'}^{m'}(x) \Big|_{x=\cos \theta'} \frac{d}{dx} P_n^m(x) \Big|_{x=\cos \theta'} \end{array} \right] \sin \theta' d\theta'. \quad (\text{IV-92})$$

Evaluating the integral over  $\phi'$  and making the change of variables  $x = \cos \theta'$ ,

$$\mathcal{I}_{\bar{\mathbf{x}}\bar{\mathbf{x}}}(n, m, n', m') = -C_{n,m} C_{n',m'} \frac{1}{2} \delta_{m,-m'} \int_{-1}^1 \left[ -\frac{mm'}{(1-x^2)} P_{n'}^{m'}(x) P_n^m(x) + (1-x^2) \frac{d}{dx} P_{n'}^{m'}(x) \frac{d}{dx} P_n^m(x) \right] dx. \quad (\text{IV-93})$$

With the identification with integrals previously evaluated, we have

$$\mathcal{I}_{\bar{\mathbf{x}}\bar{\mathbf{x}}}(n, m, n', m') = -\sqrt{\frac{2n+1}{n(n+1)} \frac{(n-m)!}{(n+m)!}} \sqrt{\frac{2n'+1}{n'(n'+1)} \frac{(n'-m')!}{(n'+m')!}} \frac{1}{2} \delta_{m,-m'} [I_1(n, n', m) - I_3(n, n', m)], \quad (\text{IV-94})$$

and substitution of (IV-77) and (IV-79) gives

$$\boxed{\mathcal{I}_{\bar{\mathbf{x}}\bar{\mathbf{x}}}(n, m, n', m') = (-1)^{m+1} \delta_{m,-m'} \delta_{n,n'}} \quad (\text{IV-95})$$

**Evaluation of**  $\mathcal{I}_{\hat{\mathbf{r}}\times\bar{\mathbf{x}}\bar{\mathbf{x}}}(n, m, n', m') = \int_0^{2\pi} d\phi' \int_0^\pi \hat{\mathbf{r}}' \times \bar{\mathbf{X}}_{n',m'}(\theta', \phi') \cdot \bar{\mathbf{X}}_{n,m}(\theta', \phi') \sin \theta' d\theta'$

Evaluate:

$$\mathcal{I}_{\hat{\mathbf{r}}\times\bar{\mathbf{x}}\bar{\mathbf{x}}}(n, m, n', m') = \int_0^{2\pi} d\phi' \int_0^\pi \hat{\mathbf{r}}' \times \bar{\mathbf{X}}_{n',m'}(\theta', \phi') \cdot \bar{\mathbf{X}}_{n,m}(\theta', \phi') \sin \theta' d\theta'. \quad (\text{IV-96})$$

Substituting (IV-90) into (IV-96) and performing the cross product,

$$\mathcal{I}_{\hat{\mathbf{r}}\times\bar{\mathbf{x}}\bar{\mathbf{x}}}(n, m, n', m') = -C_{n',m'} C_{n,m} \int_0^{2\pi} \frac{e^{j(m+m')\phi}}{4\pi} d\phi' \int_0^\pi \left[ jm' P_{n'}^{m'}(x) \frac{d}{dx} P_n^m(x) - jm P_n^m(x) \frac{d}{dx} P_{n'}^{m'}(x) \right] dx. \quad (\text{IV-97})$$

After performing the  $\phi'$  integration and applying the change of variables  $x = \cos \theta'$ , we have

$$\mathcal{J}_{\hat{r} \times \bar{X} \cdot \bar{X}}(n, m, n', m') = jC_{n', m'} C_{n, m} \frac{1}{2} \delta_{m, -m'} \int_0^\pi \left[ m P_{n'}^{-m}(x) \frac{d}{dx} P_n^m(x) + m P_n^m(x) \frac{d}{dx} P_{n'}^{-m}(x) \right] dx. \quad (\text{IV-98})$$

Identification of the integrals with previously evaluated integrals

$$\mathcal{J}_{\hat{r} \times \bar{X} \cdot \bar{X}}(n, m, n', m') = jC_{n', m'} C_{n, m} \frac{1}{2} \delta_{m, -m'} \left[ I_2(n, n', m) - I_2(n', n, -m) \right]. \quad (\text{IV-99})$$

Substitution of (IV-88) into (IV-99) shows that

$$\boxed{\mathcal{J}_{\hat{r} \times \bar{X} \cdot \bar{X}}(n, m, n', m') = 0} \quad (\text{IV-100})$$

This Page Intentionally Contains No Relevant Information

## Appendix V — Relevant Network Parameters

### The relationship between the S parameters and the admittance matrix

Define terminal voltages and currents at the ports of a two-port network as  $V_1, I_1$  and  $V_2, I_2$  at ports 1 and 2, respectively. The port voltages and currents can be related through the admittance matrix

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}. \quad (\text{V-1})$$

For a microwave network, the ports are typically fed with transmission lines. In this case, it is convenient to describe the response in terms of incident and scattered port voltages,  $V_i^+, V_i^-$ , respectively. These port voltages are related through the scattering parameters

$$\begin{bmatrix} V_1^- \\ V_2^- \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} V_1^+ \\ V_2^+ \end{bmatrix}. \quad (\text{V-2})$$

Associated with the incident voltage is an incident current,  $I_i^+$ , which flows into the port. Similarly, associated with the scattered voltage is a scattered current,  $I_i^-$ , flowing out of the port. The incident and scattered voltages and currents are related by the transmission-line wave admittance,  $Y_{0,i}$  at the respective ports

$$\begin{bmatrix} I_1^+ \\ I_2^+ \end{bmatrix} = \begin{bmatrix} Y_{0,1} & 0 \\ 0 & Y_{0,2} \end{bmatrix} \begin{bmatrix} V_1^+ \\ V_2^+ \end{bmatrix}. \quad (\text{V-3})$$

The total port voltage is

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} V_1^+ \\ V_2^+ \end{bmatrix} + \begin{bmatrix} V_1^- \\ V_2^- \end{bmatrix}, \quad (\text{V-4})$$

while the total port current is

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} I_1^+ \\ I_2^+ \end{bmatrix} - \begin{bmatrix} I_1^- \\ I_2^- \end{bmatrix} = \begin{bmatrix} Y_{0,1} & 0 \\ 0 & Y_{0,2} \end{bmatrix} \left( \begin{bmatrix} V_1^+ \\ V_2^+ \end{bmatrix} - \begin{bmatrix} V_1^- \\ V_2^- \end{bmatrix} \right), \quad (\text{V-5})$$

with the reference direction into the port. Substituting (V-2), (V-3), (V-4), and (V-5) into (V-1) gives the relationship between the admittance matrix and the scattering parameters

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} \frac{(1 - S_{11})(1 + S_{22}) + S_{12}S_{21}}{(1 + S_{11})(1 + S_{22}) - S_{12}S_{21}} Y_{0,1} & -\frac{2S_{12}}{(1 + S_{11})(1 + S_{22}) - S_{12}S_{21}} Y_{0,1} \\ -\frac{2S_{21}}{(1 + S_{11})(1 + S_{22}) - S_{12}S_{21}} Y_{0,2} & \frac{(1 + S_{11})(1 - S_{22}) + S_{12}S_{21}}{(1 + S_{11})(1 + S_{22}) - S_{12}S_{21}} Y_{0,2} \end{bmatrix}. \quad (\text{V-6})$$

Similarly

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \begin{bmatrix} \frac{(Y_{0,1} - Y_{11})(Y_{0,2} + Y_{22}) + Y_{12}Y_{21}}{(Y_{11} + Y_{0,1})(Y_{22} + Y_{0,2}) - Y_{12}Y_{21}} & \frac{-2Y_{12}Y_{0,2}}{(Y_{11} + Y_{0,1})(Y_{22} + Y_{0,2}) - Y_{12}Y_{21}} \\ \frac{-2Y_{21}Y_{0,2}}{(Y_{11} + Y_{0,1})(Y_{22} + Y_{0,2}) - Y_{12}Y_{21}} & \frac{(Y_{0,1} + Y_{11})(Y_{0,2} - Y_{22}) + Y_{12}Y_{21}}{(Y_{11} + Y_{0,1})(Y_{22} + Y_{0,2}) - Y_{12}Y_{21}} \end{bmatrix} \quad (\text{V-7})$$

In general, for a multiport network

$$\mathbf{I} = \mathbf{Y} \cdot \mathbf{V} \quad (\text{V-8})$$

$$\mathbf{V}^- = \mathbf{S} \cdot \mathbf{V}^+ \quad (\text{V-9})$$

$$\mathbf{I}^\pm = \mathbf{Y}_0 \cdot \mathbf{V}^\pm \quad (\text{V-10})$$

$$\mathbf{V} = \mathbf{V}^+ + \mathbf{V}^- \quad (\text{V-11})$$

and

$$\mathbf{I} = \mathbf{I}^+ - \mathbf{I}^- = \mathbf{Y}_0 \cdot (\mathbf{V}^+ - \mathbf{V}^-) \quad (\text{V-12})$$

so that

$$\mathbf{Y} = \mathbf{Y}_0 (\mathbf{1} - \mathbf{S})(\mathbf{1} + \mathbf{S})^{-1} \quad (\text{V-13})$$

and

$$\mathbf{S} = (\mathbf{Y}_0 + \mathbf{Y})^{-1} (\mathbf{Y}_0 - \mathbf{Y}) \quad (\text{V-14})$$

## The relationship between the S parameters and the impedance matrix

Define terminal voltages and currents at the ports of a two-port network as  $V_1, I_1$  and  $V_2, I_2$  at ports 1 and 2, respectively. The port voltages and currents can be related through the impedance matrix

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \quad (\text{V-15})$$

For a microwave network, the ports are typically fed with transmission lines. In this case, it is convenient to describe the response in terms of incident and scattered port voltages,  $V_i^+, V_i^-$ , respectively. These port voltages are related through the scattering parameters

$$\begin{bmatrix} V_1^- \\ V_2^- \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} V_1^+ \\ V_2^+ \end{bmatrix} \quad (\text{V-16})$$

Associated with the incident voltage is an incident current,  $I_i^+$ , which flows into the port. Similarly, associated with the scattered voltage is a scattered current,  $I_i^-$ , flowing out of the port. The incident and scattered voltages and currents are related by the transmission-line wave impedance,  $Y_{0,i}$  at the respective ports

$$\begin{bmatrix} I_1^\pm \\ I_2^\pm \end{bmatrix} = \begin{bmatrix} Z_{0,1} & 0 \\ 0 & Z_{0,2} \end{bmatrix}^{-1} \begin{bmatrix} V_1^\pm \\ V_2^\pm \end{bmatrix}. \quad (\text{V-17})$$

The total port voltage is

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} V_1^+ \\ V_2^+ \end{bmatrix} + \begin{bmatrix} V_1^- \\ V_2^- \end{bmatrix}, \quad (\text{V-18})$$

while the total port current is

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} I_1^+ \\ I_2^+ \end{bmatrix} - \begin{bmatrix} I_1^- \\ I_2^- \end{bmatrix} = \begin{bmatrix} Z_{0,1} & 0 \\ 0 & Z_{0,2} \end{bmatrix}^{-1} \left( \begin{bmatrix} V_1^+ \\ V_2^+ \end{bmatrix} - \begin{bmatrix} V_1^- \\ V_2^- \end{bmatrix} \right), \quad (\text{V-19})$$

with the reference direction into the port. Substituting (V-16), (V-17), (V-18), and (V-19) into (V-15) gives the relationship between the admittance matrix and the scattering parameters

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} \frac{(1+S_{11})(1-S_{22})+S_{12}S_{21}}{(1-S_{11})(1-S_{22})-S_{12}S_{21}} Z_{0,1} & \frac{2S_{12}}{(1-S_{11})(1-S_{22})-S_{12}S_{21}} Z_{0,2} \\ \frac{2S_{21}}{(1-S_{11})(1-S_{22})-S_{12}S_{21}} Z_{0,1} & \frac{(1-S_{11})(1+S_{22})+S_{12}S_{21}}{(1-S_{11})(1-S_{22})-S_{12}S_{21}} Z_{0,2} \end{bmatrix}. \quad (\text{V-20})$$

Similarly

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \begin{bmatrix} \frac{(Z_{11}-Z_{0,1})(Z_{22}+Z_{0,2})-Z_{12}Z_{21}}{(Z_{11}+Z_{0,1})(Z_{22}+Z_{0,2})-Z_{12}Z_{21}} & \frac{2Z_{12}Z_{0,1}}{(Z_{11}+Z_{0,1})(Z_{22}+Z_{0,2})-Z_{12}Z_{21}} \\ \frac{2Z_{21}Z_{0,2}}{(Z_{11}+Z_{0,1})(Z_{22}+Z_{0,2})-Z_{12}Z_{21}} & \frac{(Z_{11}+Z_{0,1})(Z_{22}-Z_{0,2})-Z_{12}Z_{21}}{(Z_{11}+Z_{0,1})(Z_{22}+Z_{0,2})-Z_{12}Z_{21}} \end{bmatrix} \quad (\text{V-21})$$

In general, for a multiport network

$$\mathbf{V} = \mathbf{Z} \cdot \mathbf{I}, \quad (\text{V-22})$$

$$\mathbf{V}^- = \mathbf{S} \cdot \mathbf{V}^+, \quad (\text{V-23})$$

$$\mathbf{V}^\pm = \mathbf{Z}_0 \cdot \mathbf{I}^\pm, \quad (\text{V-24})$$

$$\mathbf{V} = \mathbf{V}^+ + \mathbf{V}^-, \quad (\text{V-25})$$

and

$$\mathbf{I} = \mathbf{I}^+ - \mathbf{I}^- = \mathbf{Z}_0^{-1} \cdot (\mathbf{V}^+ - \mathbf{V}^-), \quad (\text{V-26})$$

so that

$$\mathbf{Z} = (\mathbf{1} + \mathbf{S})(\mathbf{1} - \mathbf{S})^{-1} \mathbf{Z}_0, \quad (\text{V-27})$$

and

$$\mathbf{S} = (\mathbf{Z} - \mathbf{Z}_0)(\mathbf{Z} + \mathbf{Z}_0)^{-1}. \quad (\text{V-28})$$

From either (V-8) and (V-22) or (V-14) and (V-28), we see that

$$\mathbf{Y} = \mathbf{Z}^{-1}, \quad (\text{V-29})$$

just as one would expect from the definitions of admittance and impedance in the network.

## Appendix VI — Alternate Evaluation of Impedance Integrals

The vector harmonics can be written in terms of the radially independent vector spherical harmonic,  $\bar{\mathbf{X}}_{n,m}(\theta, \phi)$ , as follows

$$\bar{\mathbf{M}}_{n,m}^{(i)}(\bar{\mathbf{r}}) = z_n^{(i)}(kr) \bar{\mathbf{X}}_{n,m}(\theta, \phi),$$

and

$$\bar{\mathbf{N}}_{n,m}^{(i)}(\bar{\mathbf{r}}) = j\sqrt{n(n+1)} \frac{z_n^{(i)}(kr)}{kr} Y_n^m(\theta, \phi) \hat{\mathbf{r}} + \frac{1}{kr} \frac{\partial}{\partial r} (r z_n^{(i)}(kr)) \hat{\mathbf{r}} \times \bar{\mathbf{X}}_{n,m}(\theta, \phi).$$

In the cross products in (2.43), only the vector components transverse to  $\hat{\mathbf{r}}'$  can produce terms parallel to  $\hat{\mathbf{r}}'$ , so we have

$$\bar{\mathbf{M}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') \times \bar{\mathbf{N}}_{n,m}^{(2)}(\bar{\mathbf{r}}') \cdot \hat{\mathbf{r}}' = \frac{z_{n'}^{(i)}(kr')}{kr'} \frac{\partial}{\partial r'} [r' h_n^{(2)}(kr')] [\bar{\mathbf{X}}_{n',m'}(\theta', \phi') \times [\hat{\mathbf{r}}' \times \bar{\mathbf{X}}_{n,m}(\theta', \phi')]] \cdot \hat{\mathbf{r}}', \quad (\text{VI-1})$$

$$\bar{\mathbf{M}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') \times \bar{\mathbf{M}}_{n,m}^{(2)}(\bar{\mathbf{r}}') \cdot \hat{\mathbf{r}}' = h_n^{(2)}(kr') z_{n'}^{(i)}(kr') \bar{\mathbf{X}}_{n',m'}(\theta', \phi') \times \bar{\mathbf{X}}_{n,m}(\theta', \phi') \cdot \hat{\mathbf{r}}', \quad (\text{VI-2})$$

$$\bar{\mathbf{N}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') \times \bar{\mathbf{N}}_{n,m}^{(2)}(\bar{\mathbf{r}}') \cdot \hat{\mathbf{r}}' = \frac{1}{k^2 r'^2} \frac{\partial}{\partial r'} (r' h_n^{(2)}(kr')) \frac{\partial}{\partial r'} (r' z_{n'}^{(i)}(kr')) \left[ \begin{array}{l} [\hat{\mathbf{r}}' \times \bar{\mathbf{X}}_{n',m'}(\theta', \phi')] \\ \times [\hat{\mathbf{r}}' \times \bar{\mathbf{X}}_{n,m}(\theta', \phi')] \end{array} \right] \cdot \hat{\mathbf{r}}', \quad (\text{VI-3})$$

and

$$\bar{\mathbf{N}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') \times \bar{\mathbf{M}}_{n,m}^{(2)}(\bar{\mathbf{r}}') \cdot \hat{\mathbf{r}}' = \frac{h_n^{(2)}(kr')}{kr'} \frac{\partial}{\partial r'} (r' z_{n'}^{(i)}(kr')) [[\hat{\mathbf{r}}' \times \bar{\mathbf{X}}_{n',m'}(\theta', \phi')] \times \bar{\mathbf{X}}_{n,m}(\theta', \phi')] \cdot \hat{\mathbf{r}}'. \quad (\text{VI-4})$$

Applying the identity for the vector triple product to (VI-1) and (VI-4)

$$\bar{\mathbf{M}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') \times \bar{\mathbf{N}}_{n,m}^{(2)}(\bar{\mathbf{r}}') \cdot \hat{\mathbf{r}}' = \frac{z_{n'}^{(i)}(kr')}{kr'} \frac{\partial}{\partial r'} [r' h_n^{(2)}(kr')] \bar{\mathbf{X}}_{n',m'}(\theta', \phi') \cdot \bar{\mathbf{X}}_{n,m}(\theta', \phi'), \quad (\text{VI-5})$$

and

$$\bar{\mathbf{N}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') \times \bar{\mathbf{M}}_{n,m}^{(2)}(\bar{\mathbf{r}}') \cdot \hat{\mathbf{r}}' = -\frac{h_n^{(2)}(kr')}{kr'} \frac{\partial}{\partial r'} (r' z_{n'}^{(i)}(kr')) \bar{\mathbf{X}}_{n',m'}(\theta', \phi') \cdot \bar{\mathbf{X}}_{n,m}(\theta', \phi'). \quad (\text{VI-6})$$

Applying the identity for the scalar triple product to (VI-2)

$$\bar{\mathbf{M}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') \times \bar{\mathbf{M}}_{n,m}^{(2)}(\bar{\mathbf{r}}') \cdot \hat{\mathbf{r}}' = h_n^{(2)}(kr') z_{n'}^{(i)}(kr') \hat{\mathbf{r}}' \times \bar{\mathbf{X}}_{n',m'}(\theta', \phi') \cdot \bar{\mathbf{X}}_{n,m}(\theta', \phi'). \quad (\text{VI-7})$$

Applying the identity  $(\bar{\mathbf{a}} \times \bar{\mathbf{b}}) \times (\bar{\mathbf{c}} \times \bar{\mathbf{d}}) = (\bar{\mathbf{a}} \times \bar{\mathbf{b}} \cdot \bar{\mathbf{d}}) \bar{\mathbf{c}} - (\bar{\mathbf{a}} \times \bar{\mathbf{b}} \cdot \bar{\mathbf{c}}) \bar{\mathbf{d}}$ , (VI-3) becomes

$$\bar{\mathbf{N}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') \times \bar{\mathbf{N}}_{n,m}^{(2)}(\bar{\mathbf{r}}') \cdot \hat{\mathbf{r}}' = -\frac{\frac{\partial}{\partial r'} (r' h_n^{(2)}(kr')) \frac{\partial}{\partial r'} (r' z_{n'}^{(i)}(kr'))}{k^2 r'^2} \hat{\mathbf{r}}' \times \bar{\mathbf{X}}_{n',m'}(\theta', \phi') \cdot \bar{\mathbf{X}}_{n,m}(\theta', \phi'). \quad (\text{VI-8})$$

Substitution of (VI-5) through (VI-8) into (2.43) gives

$$z_{12} = z_{21} = -\frac{1}{i_1 i_2} j \frac{k}{\omega \mu} \sum_{n'=1}^{\infty} \sum_{m'=-n'}^{n'} \sum_{n=1}^{\infty} \sum_{m=-n}^n \left( \begin{array}{l} \left( \begin{array}{l} {}_1 b_{n',m'}^{TE} \quad {}_2 b_{n,m}^{TE} \\ + {}_2 b_{n,m}^{TM} \quad {}_1 b_{n',m'}^{TM} \end{array} \right) \frac{r'}{k} \left( \begin{array}{l} z_n^{(i)}(kr') \frac{\partial}{\partial r'} [r' h_n^{(2)}(kr')] \\ - h_n^{(2)}(kr') \frac{\partial}{\partial r'} [r' z_n^{(i)}(kr')] \end{array} \right) \mathcal{I}_{\bar{\mathbf{x}} \cdot \bar{\mathbf{x}}}(n, m, n', m') \\ + \left( \begin{array}{l} {}_1 b_{n',m'}^{TE} \quad {}_2 b_{n,m}^{TM} \\ + {}_2 b_{n,m}^{TE} \quad {}_1 b_{n',m'}^{TM} \end{array} \right) \left( \begin{array}{l} r'^2 h_n^{(2)}(kr') z_n^{(i)}(kr') \\ - \frac{1}{k^2} \frac{\partial}{\partial r'} (r' h_n^{(2)}(kr')) \frac{\partial}{\partial r'} (r' z_n^{(i)}(kr')) \end{array} \right) \mathcal{I}_{\hat{\mathbf{r}} \times \bar{\mathbf{x}} \cdot \bar{\mathbf{x}}}(n, m, n', m') \end{array} \right) \quad (\text{VI-9})$$

where (see Appendix IV)

$$\mathcal{I}_{\bar{\mathbf{x}} \cdot \bar{\mathbf{x}}}(n, m, n', m') = \int_0^{2\pi} d\phi' \int_0^{\pi} \bar{\mathbf{X}}_{n',m'}(\theta', \phi') \cdot \bar{\mathbf{X}}_{n,m}(\theta', \phi') \sin \theta' d\theta' = (-1)^{m+1} \delta_{n,n'} \delta_{m,-m'}, \quad (\text{VI-10})$$

and

$$\mathcal{I}_{\hat{\mathbf{r}} \times \bar{\mathbf{x}} \cdot \bar{\mathbf{x}}}(n, m, n', m') = \int_0^{2\pi} d\phi' \int_0^{\pi} \hat{\mathbf{r}}' \times \bar{\mathbf{X}}_{n',m'}(\theta', \phi') \cdot \bar{\mathbf{X}}_{n,m}(\theta', \phi') \sin \theta' d\theta' = 0. \quad (\text{VI-11})$$

Thus, (VI-9) simplifies to

$$z_{12} = z_{21} = \frac{1}{i_1 i_2} j \frac{k}{\omega \mu} \sum_{n=1}^{\infty} \sum_{m=-n}^n (-1)^m \left( {}_1 b_{n,-m}^{TE} {}_2 b_{n,m}^{TE} + {}_2 b_{n,m}^{TM} {}_1 b_{n,-m}^{TM} \right) \frac{r'}{k} \left( \begin{array}{l} z_n^{(i)}(kr') \frac{\partial}{\partial r'} [r' h_n^{(2)}(kr')] \\ - h_n^{(2)}(kr') \frac{\partial}{\partial r'} [r' z_n^{(i)}(kr')] \end{array} \right). \quad (\text{VI-12})$$

## References

- [1] R. E. Collin, *Field Theory of Guided Waves, Second Edition*, IEEE Press, Piscataway, NJ, 1991.
- [2] D. T. Paris, F. K. Hurd, *Basic Electromagnetic Theory*, McGraw-Hill Book Company, New York, 1969.
- [3] C. A. Balanis, *Advanced Engineering Electromagnetics*, John Wiley & Sons, New York, 1989.
- [4] J. H. Richmond, "A Reaction Theorem and Its Application to Antenna Impedance Calculations", *IRE Transactions on Antennas and Propagation*, vol. AP-9, no. 6, pp 515-520, November 1961
- [5] V. H. Rumsey, "Reaction Concept in Electromagnetic Theory", *Physical Review*, vol. 94, no. 6, pp 1483-1491, June 15, 1954.
- [6] W. L. Weeks, *Electromagnetic Theory for Engineering Applications*, John Wiley & Sons, Inc., New York, 1964.
- [7] Robert E. Collin and Francis J. Zucker, *Antenna Theory Part I*, McGraw-Hill Book Company, New York, 1969.
- [8] John David Jackson, *Classical Electrodynamics*, John Wiley & Sons, Inc., New York, 1962.
- [9] M. Abramowitz and I. A. Stegun, ed., *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Dover Publications, Inc., New York, 1965, (ninth Dover printing, 1972).
- [10] Weng Cho Chew, *Waves and Fields in Inhomogeneous Media*, IEEE Press, The Institute of Electrical and Electronics Engineers, Inc., New York, 1995. (Originally published by Van Nostrand Reinhold, New York, 1990.)
- [11] N. N. Lebedev (translated and edited by R. A. Silverman), *Special Functions & Their Applications*, Dover Publications, Inc., New York, 1972.
- [12] Y. T. Lo and S. W. Lee ed., *Antenna Handbook, Theory, Applications, and Design*, Van Nostrand Reinhold Company, New York, 1988.
- [13] J. E. Hanson, ed., *Spherical Near-Field Antenna Measurements*, Peter Peregrinus, Ltd., London, 1988.
- [14] John H. Bruning and Yuen T. Lo, "Multiple Scattering of EM Waves by Spheres, Part II — Numerical and Experimental Results", *IEEE Transactions on Antennas and Propagation*, Vol. AP-19, No. 3, May 1971, pp 391-400.
- [15] Julius Adams Stratton, *Electromagnetic Theory*, McGraw-Hill Book Company, New York, 1941.
- [16] George Arfken, *Mathematical Methods for Physicists*, Academic Press, New York, 1968.

- [17] I. S. Gradshteyn, I. M Ryzhik, *Table of Integrals, Series, and Products*, Academic Press, New York, 1965 (originally published as *Tablitsy Integralov, Summ, Ryadov I Proievedeniy*, Gosudarstvennoe Izdatel'stvo Fiziko-Matematicheskoy Literatury, Moscow, 1963.).
- [18] Jon Mathews, R. L. Walker, *Mathematical Methods of Physics*, 2<sup>nd</sup> edition, The Benjamin/Cummings Publishing Company, Menlo Park, 1970.
- [19] L. Tsang and J. A. Kong, "Effective propagation constants for coherent electromagnetic wave propagation in media embedded with dielectric scatters", *Journal of Applied Physics*, Vol. 53, No. 11, November 1982, pp7162-7172.
- [20] Albert Messiah, *Quantum Mechanics, Volume II*, North-Holland Publishing Company, Amsterdam, 1962.
- [21] A. R. Edmonds, *Angular Momentum in Quantum Mechanics*, Princeton University Press, Princeton, New Jersey, 1957.
- [22] A. J. Stone, C. P. Wood, "Root-Rational-Fraction Package for Exact Calculation of Vector-Coupling Coefficients", *Computer Physics Communications*, Vol. 21, 1980, pp 195-205.
- [23] James H. Luscombe, Marshall Luban, "Simplified recursive algorithm for Wigner 3j and 6j symbols", *Physical Review E*, Vol. 57, No. 6, June 1998, pp7274-7277.

This Page Intentionally Contains No Relevant Information

## Distribution

1	MS 0303	M. W. Callahan	2300
1	MS 0509	B. C. Walker	2308
1	MS 0519	B. L. Remund	2348
1	MS 0529	G. A. Kallenbach	2345
1	MS 0529	A. W. Doerry	2345
1	MS 0529	G. R. Sloan	2345
1	MS 0529	M. B. Murphy	2346
1	MS 0533	W. H. Schaedla	2343
1	MS 0533	S. E. Allen	2343
10	MS 0533	B. C. Brock	2343
1	MS 0533	K. W. Sorensen	2343
1	MS 0537	R. M. Axline	2344
1	MS 0537	J. T. Cordaro	2344
1	MS 0612	Review & Approval Desk for DOE/OSTI	9612
2	MS 0899	Technical Library	9616
1	MS 9018	Central Technical Files	8945-1