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A Survey of Grid-Free Methods for the Simulation of 3-D Incompressible Flows in Bounded Domains

Adrin Gharakhani, Sc.D.
InteliComp
39 Lockeland Avenue
Arlington, MA 02174

Prepared by
Sandia National Laboratories
Albuquerque, New Mexico 87185 and Livermore, California 94550

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Adrin Gharakhani, Sc.D.
InteliComp
39 Lockeland Avenue
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Sandia Contract AV-0444

ABSTRACT

The state-of-the-art in Lagrangian methods for the grid-free simulation of three-dimensional, incompressible, high Reynolds number, internal and/or external flows is surveyed. Specifically, vortex and vorticity (or impulse) element methods are introduced. The relative merits of various available techniques and the outstanding challenges in simulating the processes of convection and diffusion, as well as in satisfying the wall boundary conditions are discussed individually. Issues regarding the stretch and solenoidality of vorticity are also discussed. A potentially successful algorithm for simulating the flow around a parachute is proposed as well.

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OVERVIEW

This report presents results from the survey of available literature on grid-free methods for the simulation of high Reynolds number incompressible flow in bounded three-dimensional domains and, in particular, flow over parachutes; conducted for Sandia National Laboratories under consulting agreement AV-0444.

The discussion of grid-free methods that are of potential interest to the simulation of flow over parachutes can be classified into (1) Vortex Methods and (2) Velocity Methods. Vortex methods are techniques for approximating the vorticity transport formulation of the governing equations of fluid motion. These include the sub-classes of (1) vortex monopoles or elements or vortons, (2) vortex filaments or sticks, and (3) vortex-lattices or panels. Velocity methods, also known in the literature as magnetization or impulse or dipole, are recently developed techniques based on the Hamiltonian formulation of incompressible, inviscid fluid flows.

The review of vortex and velocity methods that have been used in the literature to solve the Navier-Stokes equations in 3-D bounded domains is simplified by focusing attention individually on (1) the solution of the unbounded Euler equations, (2) the solution of the unsteady diffusion equation, and (3) the effect of solid boundaries. These categorizations allow a hierarchical approach for the evaluation of the merits and deficiencies of available methods, and eliminate unnecessary duplication. For example, the discussion of accuracy and convergence of vortex monopoles applies equally well to filaments and panels *unless* stated otherwise. Similarly, because of the parallelism between the vorticity and velocity transport equations, most techniques that have been developed in conjunction with vortex methods are expected to be applicable to the velocity method *unless* stated otherwise.

Every attempt has been made in this survey to find and present numerical evidence on the accuracy and convergence properties of the presented methods. Particular emphasis has been put on finding parametric studies that compare the accuracy of different methods.

Only grid-free vortex and velocity methods are presented. Hybrid schemes are largely ignored, with the exception of algorithms that are based on the Delaunay tetrahedralization. Articles that discuss 2-D technologies are included in this report *only if* the ideas show promise and can be extended to 3-D. This is particularly true when dealing with the solution of the unsteady diffusion equation and the application of the no-slip boundary condition, which have mostly been attempted in the literature in 2-D space.

A deliberate attempt has been made to reduce the number of "unnecessary" references in the report. For example, if a particular subject/technology by an author appears in more than one publication, only one is cited. Also, publications that involve the implementation of a known technology to study physics, rather than numerics, have largely been ignored.

LITERATURE SEARCH STRATEGY

The literature search was conducted for the period starting from 1980 to the present. This is justified because simulation of complex three-dimensional flow phenomena using the Lagrangian vortex and/or velocity (magnetization) methods is a relatively recent enterprise, perhaps dating back 15 years. Furthermore, what might have been the state-of-the-art just a decade ago may simply be obsolete today. This allowed the use of electronic databases for carrying out the search, which enhanced the quality and speed of the survey tremendously.

The literature search was conducted using the following electronic databases:

1. Compendex Plus; a comprehensive collection of abstracts from engineering and technology related journals and conference proceedings,
2. Inspec; a database for physics, electronics and computing related journals and conferences,
3. MathSci; a reference material for mathematics and computing,
4. Aerospace; a database for journals and conference proceedings related to aeronautics and astronautics, and
5. NTIS; a database for U.S. government sponsored research and development.

A three-pronged approach was taken to conduct the literature search. Firstly, the databases were searched for selected keywords. Secondly, the references in the reviewed articles were examined for new leads. Thirdly, yet-unpublished papers were solicited directly from researchers who have been contributing significantly to the development and implementation of new, more accurate 3-D vortex and velocity methods in recent years. The keywords used to search for articles on vortex methods were vort*.AND.(meth*.OR.simul*). For the velocity method, the keywords used were (velic*.OR.magnet*.OR.dipol*.OR.impulse).AND.(meth*.OR.simul*). (* means wild card.) Articles on two-dimensional flow were included in the search; however, only relevant topics - judged from the abstracts - were selected for the review process.

A SURVEY OF GRID-FREE METHODS FOR THE SIMULATION OF 3-D INCOMPRESSIBLE FLOWS IN BOUNDED DOMAINS

1 VORTEX METHODS

1.1 Introduction

Lagrangian vortex methods are approximation techniques for the simulation of the unsteady, incompressible, Euler or high Reynolds number Navier-Stokes equations in unbounded or bounded domains. "Bounded domains" implies internal or external flow configurations with any combination of boundary conditions. In this approach, the governing equations are expressed in the vorticity transport formulation. The vorticity field is discretized using a collection of vortex elements which are identified by the strength and the direction of the local vorticity. The convection of the vorticity field is evaluated in the Lagrangian frame of reference by tracking the trajectory of the elements due to the local velocity. Additionally, the strengths and the directions of the vortex elements are updated to account for diffusion, as well as stretch due to the interaction of the vorticity with the local velocity gradients. In vortex methods, the velocity field in a bounded flow domain is expressed as the superposition of a vortical velocity component in an unbounded domain and a "corrective" potential velocity such that the normal velocity flux boundary condition is imposed at the boundaries of the domain (Lighthill, 1963; Batchelor, 1967). This velocity field produces a residual tangential velocity at the solid walls, which is different from the applied no-slip boundary condition. The observed jump in the tangential velocity is equivalent to the amount of vorticity that must be generated at the wall to satisfy the no-slip boundary condition (Lighthill, 1963; Batchelor, 1967; Chorin, 1978). The task is then to relate the velocity jump to the boundary vorticity and to devise a mechanism for transporting the generated vorticity into the domain interior.

Three-dimensional inviscid flows were first simulated using a vortex filament approach (Hama, 1962; Leonard, 1980, 1985), which is an excellent candidate for solving the Euler equations in unbounded domains. In this model, a collection of filaments, each forming a closed vortex line and segmented along the vorticity vector, is used to approximate the initial vorticity field. By construction, Kelvin's theorem for the conservation of circulation is implicitly satisfied in this scheme.

Beale & Majda (1982) proposed extending the notion of the two-dimensional vortex element to a three-dimensional element with spherical core function, and provided mathematical proof for the existence of solutions for short times. However, in this approach the velocity gradients, used to compute vorticity stretch, were obtained by a finite difference approximation, which contradicted the spirit of grid-free Lagrangian techniques. Anderson & Greengard (1985) alleviated this problem by differentiating the regularized kernel for the velocity in the Biot-Savart integral directly to obtain the regularized kernel for the velocity gradients. The convergence of the latter was subsequently proved by Beale (1986).

Vortex methods offer some significant advantages. To begin with, due to the Lagrangian nature of the computations, convection is approximated with minimal numerical diffusion, making the scheme an excellent tool for simulating high Reynolds number flows. In addition, vortex methods are generally grid-free and thus eliminate the often tedious task of meshing the volume of complex three-dimensional domains. This advantage is even more pronounced in situations where moving boundaries are encountered, and in unsteady flow topologies where the envelope of the vorticity field changes in time. Vortex methods are self-adaptive and capable of dynamically concentrating computational elements in regions with significant vorticity, such as in wakes, jets and shear layers. Furthermore, vortex methods are naturally adaptable to massively parallel computing, which can be exploited to solve large scale problems efficiently. Finally, vortex methods readily facilitate an intuitive and quantitative tool for interrogating “three-dimensionality effects” in the flow by monitoring vorticity stretch in the field. This is an immediate consequence of representing the Navier-Stokes equations in the velocity-vorticity formulation, where the three-dimensional equations are distinguished from their two-dimensional counterpart by the extra vorticity stretch term.

At present, three-dimensional vortex methods present a few computational challenges which need to be addressed with more rigor. Firstly, the most widely used and flexible class of vortex methods - vortons - which use vortex monopoles to discretize the three-dimensional vorticity field, do not implicitly guarantee the solenoidality of the discretized vorticity in space and time. Neither do they necessarily preserve flow invariants, such as linear and angular momenta, helicity or energy. Consequently, the accumulation of errors leads to the eventual blow up of the simulations at long times. (This issue will be addressed in more detail in the following sections.) Secondly, the cost for evaluating the vortex element velocities using traditional summation techniques grows quadratically with the number of elements. This imposes a severe limitation on the maximum number of elements that can be used in a simulation. Fortunately, fast multipole expansion techniques are now available, which achieve near-linear CPU dependence on the number of elements (Buttke, 1990; Greengard & Rokhlin, 1987; Winckelmans *et al.*, 1996). Furthermore, parallelization of the velocity evaluations by the direct or fast summation methods speeds up the computations considerably (Gharakhani & Ghoniem, 1996; Winckelmans *et al.*, 1996). Nevertheless, physically correct and numerically accurate vortex element merging algorithms may still be necessary to limit the proliferation of the number of elements in time (Buttke & Chorin, 1993; Chorin, 1993).

The proper choice for boundary conditions is perhaps the most challenging and controversial issue in the application of the vorticity transport formulation of the Navier-Stokes

equations to bounded domains. At the rudimentary level, the vorticity-based equations are expected to require compatible *vorticity* boundary conditions, whereas the Navier-Stokes equations admit the natural boundary conditions of no-slip and no-flux at solid walls. Consequently, one must reconcile the differences between the natural (velocity) boundary conditions and the derived boundary conditions for the vorticity by addressing the physics of vorticity generation at the walls and diffusion into the fluid. Unfortunately, the fundamental physics of the vorticity generation mechanism at solid walls are not well-understood. Morton (1984) points out that a thorough discussion of boundary conditions is absent in virtually all fluid mechanics textbooks that discuss vorticity dynamics - notable exceptions being the books by Lighthill (1963) and Batchelor (1967).

Indeed, there even seems to be disagreement (and confusion) among fluid dynamicists on whether vorticity generation is an inviscid or a viscous process. Morton (1984) and Morino (1986), using the generalized Biot-Savart formula of Wu & Thompson (1976), make the controversial assertion that vorticity generation at the wall is an inviscid process and that the no-slip boundary condition cannot provide a mechanism for the generation of vorticity! Morino (1986) further concludes that only the normal boundary conditions are necessary to establish the flow field. Wu & Wu (1993) argue that this conclusion is erroneous, and show that if the fluid is assumed to be truly inviscid, the jump in the tangential velocity from the fluid to the solid does not create any rotation of fluid particles and there is no no-slip - the fluid particles simply slide on the wall. On the other hand, if the inviscid flow is assumed to be the limiting case of the viscous flow with viscosity approaching zero, then the no-slip condition becomes meaningful and the tangential jump from the fluid to the solid creates singular vorticity. In other words, vorticity generation and viscosity are interlinked.

In addition to the debate over the inviscid versus viscous nature of vorticity generation, there are other characteristics of the vorticity boundary condition, with significant computational consequences, that remain unresolved. For example, is vorticity generation a local effect or is it linked globally to all other points on the boundary? Is the boundary vorticity a kinematic function of the boundary velocity and stream function or is it determined dynamically via the Navier-Stokes equations at the wall? Should one apply a Dirichlet boundary condition or a Neumann boundary condition for the wall vorticity?

In order to express the boundary vorticity in terms of the corresponding velocity boundary conditions, one may apply the generalized Biot-Savart law developed by Bykhovskiy & Smirnov (1983) and independently by Wu & Thompson (1973), Wu (1976, 1984), and Wu & Gulcat (1981). Alternatively, one may use the method of projection developed by Quartapelle (1981) and Quartapelle & Valz-Griz (1981). In the case of the former formulation, the kinematic definition of the vorticity (as the curl of the velocity) along with the incompressibility constraint are manipulated to obtain an integral equation that relates the velocity field to the corresponding vorticity field, inclusive of the boundary. Additionally, conservation of vorticity is imposed as a supplemental global constraint. In the discretized form, either the normal or the tangential velocity boundary conditions may be used to form a linear set of equations that solves for the vorticity distribution on the surface. Note that Wu (1976) insists that the global conservation properties of the flow must be imposed *explicitly*. Kempka *et al.* (1995) have

recently re-investigated the generalized Biot-Savart formula and have shown that the global conservation properties are satisfied *implicitly* in the derivation. Furthermore, they showed that either the tangential or the normal boundary condition would be sufficient to obtain a solution. However, while the conclusions by Kempka *et al.* (1995) may be correct in the continuous sense, numerical experiments by Uhlman & Grant (1993) suggest that both boundary conditions as well as the explicit satisfaction of the global constraints may be necessary in the discretized form.

The method of projection was developed based on the observation that the statement of the problem in terms of the Poisson equation for the vorticity-streamfunction - using boundary values for the streamfunction and its flux - will generally not yield an exact formulation for the boundary vorticity, *unless* the evaluated vorticity is projected onto a new field that satisfies a global compatibility condition. This global condition is obtained by applying Green's second identity to the product of the vorticity field and an arbitrary scalar potential field. Anderson (1989) proposed an equivalent but more economical approach to the projection method by satisfying the global compatibility condition at time zero and by maintaining a zero time rate of change of the constraint in subsequent times. This requirement on the time derivative leads to an explicit integro-differential boundary condition for vorticity, and hence to the formulation of a new scheme for vorticity generation. Anderson (1989) also showed, without a rigorous proof, that the vorticity generation scheme by Chorin (1978) - to be discussed later - is a special case of boundary conditions which impose the time-constrained evolution of vorticity that is generated at the wall due to the no-slip boundary condition.

Note that the Biot-Savart formulation as well as the projection method are kinematic in nature. Furthermore, they implement global integral constraints which link the boundary vorticity at each point to all other points on the boundary. Until recently, the vorticity boundary condition was accepted to be of the global type (Wu, 1976; Hung & Kinney, 1988; Gresho, 1991; Wu *et al.*, 1994), thus making its implementation in a local algorithm difficult and expensive. However, using a finite-difference argument, E & Liu (1996) showed that many of the recently developed global methods can actually be rewritten into local formulae. They also proposed a general recipe for converting the discrete form of the time-constrained global boundary condition by Anderson (1989) into a local one. Casciola *et al.* (1996) derived a boundary integral equation for the three-dimensional vorticity source at the wall, starting from the generalized Biot-Savart formula for the velocity and the generalized integral solution of the diffusion equation. They showed that, at least for a flat plate, the wall vortex sheet is a consistent approximation of the vorticity source; thus, further verifying that the boundary conditions can be enforced via local procedures. It must be noted here that the convergence of the Navier-Stokes equations using the (kinematic) vorticity boundary condition was proved by Hou & Wetton (1992).

In contrast to the kinematic boundary conditions mentioned above, Lighthill (1963) proposed a dynamic model for the rate of vorticity production and showed that the pressure gradient along a stationary wall is equal to the normal flux of the tangential vorticity at the wall. However, he did not elaborate on the connection between the vorticity flux and the strength of the boundary sheets. This was later accomplished by Kinney & Paolino (1974) who showed

that, for the case of flow over a flat plate, the slip velocity at the wall is proportional to the time integral (over an interval) of the normal flux of vorticity at the wall - the proportionality constant being the kinematic viscosity. The development by Koumoutsakos *et al.* (1994) is based on this formulation. Wu & Wu (1993) have developed a generalized theory for the dynamics of the interaction between a viscous compressible flow field and a three-dimensional rotating and translating curved body. They showed that the fluid dynamics of such an interaction can be decomposed into a pair of Stokes-Helmholtz potentials of the Navier-Stokes equations as a normal compressing process, represented by the pressure field in case of an incompressible flow, and a tangential shearing process, represented by vorticity. They showed that the solid surface and the boundary condition of acceleration adherence cause the coupling of vorticity and pressure in the form of a boundary flux due to the tangential gradient of the normal stress and vorticity. Furthermore, they showed the existence of a vorticity-vorticity coupling, which is peculiar to 3-D flows only and plays the important role of vorticity generation near regions of flow separation. Based upon this dynamic vorticity-pressure theorem Wu *et al.* (1994) showed that the Neumann boundary condition by Kinney & Paolino (1974) and Koumoutsakos *et al.* (1994) can exist only in an *approximate sense* and as a special case of their theorem. They also provided an efficient algorithm for decoupling the vorticity and the pressure for high Reynolds number flows. Note that the effectiveness of this decoupling may suggest that the proposed vorticity-pressure theorem may have only higher order contributions to the flow structure and that the *approximate* boundary conditions may be sufficient in many practical problems.

The various approaches that have thus far been presented for accounting boundary effects manifest themselves as either a Dirichlet or a Neumann boundary condition. It is not obvious which is correct or is at least a better representative of the natural boundary condition. Wu *et al.* (1994) argue that “in the vorticity-based methods it is impossible to obtain a strict Dirichlet or Neumann condition for the vorticity equation.” In this sense, they agree with Gresho (1991) that there are no boundary conditions on the vorticity. Hung & Kinney (1988) point out that, at least in terms of numerical implementation, the results obtained by the two boundary conditions are different only by a scaling factor. That is, if the time increment used with the Neumann boundary condition is set equal to the time that it takes for the new total vorticity (obtained by the Dirichlet boundary condition) to diffuse into the nearest normal nodal point from the surface, then the two methods will produce nearly identical results.

It is clear from this brief introduction that a much deeper understanding of vorticity, its inception at the boundary and transport into the fluid interior is necessary. Nevertheless, it is interesting that despite fundamental differences between competing models that have been implemented computationally to satisfy the vorticity boundary condition, the differences in accuracy of the solutions seem to be quite minimal. Indeed, in the absence of a controlled set of numerical experiments, such differences may even be attributed to factors other than the particulars of the boundary conditions used.

Excellent introductory surveys of various three-dimensional vortex methods are available in the literature (Hoeijmakers, 1983; Leonard, 1980, 1985; Pucket, 1992). For a more complete review of vorticity-based boundary conditions refer to Gresho (1991).

1.2 Formulation

The equations of motion of an incompressible viscous fluid in a bounded three-dimensional domain, D , with boundary, ∂D , may be expressed in the vorticity transport form of the Navier-Stokes equations as:

$$\frac{\partial \omega}{\partial t} + u \cdot \nabla \omega = \alpha \omega \cdot \nabla u + (1 - \alpha) \omega \cdot (\nabla u)^T + \frac{1}{R_c} \nabla^2 \omega + u (\nabla \cdot \omega) \quad x \in D \quad (1.2.1)$$

$$\nabla \cdot u = 0 \quad x \in D \quad (1.2.2)$$

$$\omega(x, t) = \nabla^{\wedge} u \quad x \in D \quad (1.2.3)$$

$$u(x, t = 0) = u_0 \quad x \in D \quad (1.2.4)$$

$$u(x, t) = (u \cdot \tau, u \cdot \rho, u \cdot n) \quad x \in \partial D \quad (1.2.5)$$

where $x = (x, y, z)$ is the position vector in Cartesian coordinates normalized by a reference length, L ; $u(x, t) = (u, v, w)$ is the velocity vector normalized by a characteristic speed, U ; t is the time normalized by L/U ; $\omega(x, t) = (\omega_x, \omega_y, \omega_z)$ is the vorticity vector normalized by U/L ; $R_c = UL/\nu$ is the Reynolds number, and ν is the kinematic viscosity. Superscripts T and \wedge denote the transpose and the cross-product operations, respectively. At the boundary surfaces, the velocity is expressed in terms of the local orthogonal coordinate system $\tau - \rho - n$, where $n = (n_x, n_y, n_z)$ is the unit outward normal, and $\tau = (\tau_x, \tau_y, \tau_z)$ and $\rho = (\rho_x, \rho_y, \rho_z)$ are the unit tangents to the boundary. α is a variable that casts the stretch term into three possible formulations. $\alpha = 1$ corresponds to the standard form, $\alpha = 0$ yields what is called the transpose form, and $\alpha = 1/2$ leads to a mixed symmetric form of the stretch term. The effect of α on the accuracy of the predictions will be discussed in the following section.

The last term in Eq. (1.2.1) results from taking the curl of the Navier-Stokes equations in the primitive variable formulation and using vector identities to simplify further. It is left intact intentionally and will be discussed shortly. For now, suffice it to say that the more familiar form of the Navier-Stokes equations (with the term removed) is valid only when the solenoidality of vorticity is satisfied everywhere in the domain (Gharakhani, 1993).

Simulation of the Navier-Stokes equations by Lagrangian vortex element methods generally comprises the sequential solution of the Euler and diffusion equations (Chorin, 1973). This is the so-called viscous splitting algorithm, and it provides the flexibility to concentrate on the solution of the simpler problems of Euler and diffusion equations individually and to develop hybrid algorithms with arbitrarily high degrees of accuracy and stability. The convergence of solutions using viscous splitting has been proved by Beale & Majda (1981) for unbounded domains. There is ample numerical evidence that the algorithm works quite well in bounded

domains as well. The standard viscous splitting approach (Euler+Diffusion) has been proved to converge uniformly at a rate that is proportional to $\Delta t/R_c$, where Δt is the computational timestep. A Strang-type splitting algorithm (Diffusion+Euler+Diffusion) with convergence rate proportional to $\Delta t^2/R_c$ has also been proposed.

Application of viscous splitting to Eq. (1.2.1) yields the following pair of Euler and diffusion equations, respectively

$$\frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla \omega = \alpha \omega \cdot \nabla \mathbf{u} + (1 - \alpha) \omega \cdot (\nabla \mathbf{u})^T \quad \mathbf{x} \in D \quad (1.2.6)$$

$$\frac{\partial \omega}{\partial t} = \frac{1}{R_c} \nabla^2 \omega \quad \mathbf{x} \in D \quad (1.2.7)$$

where the last term in Eq. (1.2.1) is removed for the sake of simplicity.

The velocity in the above equations can be decomposed into a vortical field, \mathbf{u}_ω , and a potential component, \mathbf{u}_p , according to the Helmholtz decomposition rule (Batchelor, 1967):

$$\mathbf{u} = \mathbf{u}_\omega + \mathbf{u}_p \quad (1.2.8)$$

The vortical field is obtained using the familiar Biot-Savart formula for an unbounded domain. The potential velocity is obtained by evaluating the gradients of a potential field that defines the geometry in question. This is done by solving a Neumann problem in which the normal flux boundary condition for the potential accounts for the effect of the vortical velocity on the boundary using Eq. (1.2.8). The superposition of the vortical and potential components yields a velocity field that satisfies the normal flux boundary condition but violates the no-slip boundary condition. The latter is satisfied by positioning an infinitesimally thin vortex sheet on the boundary with its magnitude equal to the magnitude of the observed jump in the tangential velocity. Refer to Gharakhani & Ghoniem (1997a) for a detailed description of the solution methodology. An equivalent generalized Biot-Savart formulation including the effects of the no-flux and no-slip boundary conditions was derived by Wu & Thompson (1973). The following is an alternative interpretation proposed by Kempka *et al.* (1995):

$$\begin{aligned} \mathbf{u}(\mathbf{x}) = & \nabla \wedge \int_D \omega(\mathbf{x}') G(\mathbf{x}, \mathbf{x}') dV(\mathbf{x}') + \\ & \nabla \wedge \int_{\partial D} [\gamma_c(\mathbf{x}'_b) - \mathbf{n}(\mathbf{x}'_b) \cdot \mathbf{u}_b(\mathbf{x}'_b)] G(\mathbf{x}, \mathbf{x}'_b) dS(\mathbf{x}'_b) + \\ & \nabla \int_{\partial D} [\mathbf{n}(\mathbf{x}'_b) \cdot \mathbf{u}_b(\mathbf{x}'_b)] G(\mathbf{x}, \mathbf{x}'_b) dS(\mathbf{x}'_b) \end{aligned} \quad (1.2.9)$$

where dV is an elemental fluid volume, dS is an elemental boundary surface, G is the Green function in the infinite domain, and the subscript b implies a point on the boundary. γ_c represents a vortex sheet that is created at the boundary to satisfy the no-slip boundary

condition. Note that the first integral represents the vortical velocity in the infinite domain, whereas the remaining surface integrals - with the exception of the term involving γ_c - represent the potential velocity.

In vortex element simulations, in which δ -distribution functions are used to discretize the vorticity field, the singularity of the velocity kernel gives rise to numerical instability and the solution blows up when vortex elements approach each other. Chorin (1973) removed this velocity singularity at the center using a “vortex blob” regularization technique. This was later interpreted as the convolution of the singular vortices with a smoothing function, or the use of finite area elements whose vorticity remains concentrated within a finite size core radius and decays rapidly outside it.

Vortex blob discretization of the vorticity field using N_V vortex elements - each with volumetric vorticity $\Gamma_i = \omega(x_i, 0)h^3$ and centered at x_i - is represented by:

$$\omega(x, t) = \sum_{i=1}^{N_V} \Gamma_i(t) f_\sigma(x - x_i) \quad (1.2.10)$$

where h is the inter-element distance and $f_\sigma(x) = \frac{1}{\sigma^3} f(|x|/\sigma)$ is a smoothing or core function with core radius σ . Substituting (1.2.10) into the Lagrangian equivalent of (1.2.6) and (1.2.7) yields the final form of the evolution equations in 3-D vortex element methods:

$$\frac{dx_i}{dt} = u_i \quad x_i(t=0) = x_o \quad (1.2.11)$$

$$\frac{d\Gamma_i}{dt} = \alpha \Gamma_i \cdot \nabla u_i + (1 - \alpha) \Gamma_i \cdot (\nabla u_i)^T \quad \Gamma_i(t=0) = \Gamma_o \quad (1.2.12)$$

$$\frac{d\Gamma_i}{dt} = \frac{1}{R_e} \nabla^2 \Gamma_i \quad i = 1, \dots, N_V \quad (1.2.13)$$

Equations (1.2.11) through (1.2.13) are vectorial ordinary differential equations which can now be integrated in time to any order of accuracy to obtain the trajectory of the blobs and the evolution of the vorticity due to stretch and diffusion. The velocity gradients in Eq. (1.2.12) may be evaluated by any numerical method. The current trend is the method of direct differentiation of the velocity kernel by Anderson & Greengard (1985).

1.3 The Euler Equations

1.3.1 Vortex Monopoles

The convergence of vortex “blob” simulations to Euler and Navier-Stokes equations has been studied extensively for 2-D domains, and to a lesser degree for 3-D. Beale & Majda

(1982a, 1982b) were the first to prove the convergence of the 3-D vortex simulations to the Euler equations for short times. In their approach, they superimposed a set of grids on the computational domain and used finite difference methods to evaluate the velocity gradients in the field. Later, Anderson & Greengard (1985) proposed the alternative of differentiating the velocity kernel directly to evaluate the velocity gradients. The proof of convergence was subsequently provided by Beale (1986) and further improvements in the analysis were given by Cottet (1988). At present, this approach is the most prevalently implemented technique in grid-free methods. Briefly, the convergence rate of the particle paths using an m -th order, k times continuously differentiable spherical core function with radius σ is $O(\sigma^m + (h/\sigma)^k \sigma)$. This suggests that in order for the method to converge we must have a vanishingly small core radius, subject to condition $(h/\sigma) < 1$; i.e., neighboring cores must overlap everywhere in the domain and at all times. Lung-An (1995) has proved the convergence of vortex methods for 3-D Euler equations in *bounded* domains.

Anderson & Greengard (1985) have also studied convergence due to temporal discretization. For a class of explicit multi-step Adams-Bashforth methods with local truncation errors in the order of $r+1$, they showed the convergence rate to be $O(\Delta t^r)$. Additionally, Anderson & Greengard (1985) proved that the modified Euler method converges quadratically. For a concise review of the convergence properties of vortex methods refer to Hald (1991).

In order to improve the convergence rate, Hald (1987) and Nordmark (1996) have developed high-order core functions, while Beale & Majda (1985) have proposed a technique to construct arbitrarily high-order core functions from a given list of lower order ones. However, numerical implementation and testing of various high-order core functions have revealed that the computed convergence rates agree with theory only initially, and the errors caused by severe vorticity smearing, for example, lead to suboptimal performance quickly (Beale & Majda, 1985; Perlman, 1985). This loss of accuracy in long time simulations (of even 2-D flows) is a serious deficiency of grid-free methods and an active area of research. One idea put forth by Beale & Majda (1985) is the periodic regridding or rezoning of the vorticity field. Regridding has been shown to improve long time predictions (Bernard, 1995; Winckelmans & Leonard, 1993); however, it eliminates two of the most attractive advantages of Lagrangian vortex methods: (1) it inevitably introduces numerical diffusion, and (2) it uses grids to project the vorticity field onto uniformly distributed collocation points! Beale (1988) proposed an alternative method for improving the long time solution accuracy without having to resort to regridding. In Eq. (1.2.10), vorticity was formerly initialized using the relation $\Gamma_i(0) = \omega(\mathbf{x}_i, 0)h^3$ to assign the field vorticity to the vorticity at a collocation point i . This effectively assumes a point-wise, δ -distribution of vorticity and ignores the influence of the core functions from the neighboring elements. In the new method, Beale (1988) suggested assigning the field vorticity to the left hand side of Eq. (1.2.10) and inverting the resulting matrix to obtain $\Gamma_i(0)$ at collocation points i . This approach improves the quality of long time solutions significantly (Knio & Ghoniem, 1990, 1991; Winckelmans & Leonard, 1993). Marshall & Grant (1996) have proposed and tested an improved matrix inversion technique that effectively cuts off high-frequency oscillations from the inversion process and leads to even longer time solution accuracy. Strain

(1996) has recently devised a fast adaptive vortex method for the 2-D Euler equations which utilizes the free-Lagrangian formulation, an adaptive quadrature rule for the Biot-Savart summations, the fast multipole method, and a non-standard error analysis technique to obtain long-time arbitrarily high order accuracy at asymptotically optimal cost. The key here is to apply time-dependent quadrature weights that adapt to the flow map; traditional vortex methods use fixed weights which do not take flow disorganization into consideration. At present, the 3-D equivalent of the method is not available (Strain, 1997).

Humi (1993) analyzed the spectral dependence of vortex methods on the choice of the core function and showed that the infinite order core function by Hald (1987) and higher order functions by Beale & Majda (1985) have fixed spectral range of accuracy. Therefore, he argued, a large number of vortex elements will be necessary in order to capture the required scales of the flow accurately. He developed a core function that gives the user control over the spectral accuracy of the predictions, and used it to predict the drag over a 2-D plate. He showed that for a fixed number of vortex elements the increase in the core wave-number improves the drag prediction significantly. This appears to be one promising approach to reduce the total number of elements for a particular resolution.

Alternatives to the traditional vortex elements with spherical core functions have also appeared in the literature. Hou (1992) and Hou *et al.* (1993) reintroduced point vortex methods, without core functions, in 3-D and showed the predictions to maintain long time accuracy! The idea was to apply a novel exact desingularization technique that reduces the order of the singularity in the Biot-Savart kernels by one, and thus makes the quadratures (the summations) more stable. Cottet *et al.* (1991) proved the method to be stable and convergent to second order accuracy in 3-D space. Marshall & Grant (1995a) developed an approximate method for determining the induced velocity from highly anisotropic 3-D vorticity blobs. Anisotropic elements are highly useful in discretizing boundary layer and wake-type flows where the variation in the vorticity is more across the thin layer of the flow than along it. As a result, the computational domain can be discretized using substantially fewer number of elements. Marshall & Grant (1995a) claimed several orders of magnitude reduction in the number of elements for 3-D simulations! Grant *et al.* (1995) have embarked on the development and testing of a code that is based on the advancing front idea and the Delaunay triangulation method proposed by Russo & Strain (1994). In this approach, new tetrahedral elements are generated on solid boundaries and their vertices convected with the local velocity in the Lagrangian frame of reference. Vorticity is assumed to be constant within each control volume which allows the exact integration of the Biot-Savart formula.

It has been known for some time now that vorton simulations of the standard 3-D Euler equations - $\alpha = 1$ in Eq. (1.2.6) or (1.2.12) - are unstable (Choquin & Huberson, 1990; Winckelmans & Leonard, 1993). The source of instability has been attributed to vorticity stretch and the fact that (1) vortons do not preserve the basic invariants of the flow represented by $\alpha = 1$, including the total vorticity, and (2) Eq. (1.2.12) does not form a weak solution of Eq. (1.2.6) (Saffman & Meiron, 1986; Winckelmans & Leonard, 1993). In an attempt to improve the stability of the simulations, two alternative representations of the vorticity stretch -

$\alpha = 0$ and $\alpha = 1/2$ derived by Rehbach (1978) - have been analyzed and numerically tested in the past (Beale *et al.*, 1991; Choquin & Cottet, 1988; Choquin & Huberson, 1990; Greengard & Thomann, 1988; Knio & Ghoniem, 1991; Winckelmans & Leonard, 1988, 1993). Note that for the continuous solenoidal vorticity field all three forms of the stretch term are identical; however, since vorton simulations are only approximately divergence-free, we can expect differences in the accuracy and stability of the predictions. Simulations using $\alpha = 1/2$ were preferred by Rehbach (1978) because the symmetry of the formulation resulted in computational savings. This case does not preserve the invariants of the flow, neither does it lend itself to a weak solution of Eq. (1.2.6) (Winckelmans & Leonard, 1993). The case with $\alpha = 0$ - the transpose method - has been shown to preserve total vorticity (Choquin & Cottet, 1988) as well as form a weak solution of the Euler equations (Winckelmans & Leonard, 1988). Therefore, one may expect the transpose method to be superior to the other two. Unfortunately, unlike the standard method which has the propensity to maintain the solenoidality of an initially divergence-free vorticity field, the transpose method destroys this solenoidality (Cottet, 1996). Certainly, results from the theoretical analyses are inconclusive as to which of the three methods is superior, and one must resort to numerical experimentation and comparison with available data to select the more robust method. Using a test problem of two colliding vortex rings, Winckelmans & Leonard (1993) concluded that the transpose method is superior. Choquin & Huberson (1990) computed the Norbury (1973) steady problem and arrived at a similar conclusion. On the other hand, in a study of the evolution of an initially two-dimensional shear layer, perturbed in the streamwise and spanwise directions, Knio & Ghoniem (1991) concluded that the standard method is superior. Beale *et al.* (1991), using the test problem of a vortex ring with swirl, concurred with the latter. The fact that the computations are as inconclusive as the analyses may actually suggest that the currently prevailing hypothesis about the source of the numerical instability in 3-D; i.e., the stretch term, may be inaccurate or incomplete. Greengard & Thomann (1988) pointed out that systems of singular vortex elements cannot form weak solutions of the Euler equations because the non-linear term is not defined in the distribution sense. Thus, they argue, the conclusion made by Winckelmans & Leonard (1988) about the existence of weak solutions for the transpose method is not general and depends on the specific choice of the regularization. Furthermore, Pedrizzetti (1992) notes that vortons not representing a weak solution of the Euler equations has little bearing on the validity of the method as an operative tool. Greengard & Thomann (1988) and Pedrizzetti (1992) conclude that vortons may be a poor choice for solving Euler equations *not* because they don't preserve the invariants of the flow, but because they are not divergence-free! (The fact that divergence-free vortex filament methods can successfully simulate vortex ring instabilities, with severe stretch, may be an indirect confirmation of the latter conclusion.) It is worth recalling that all 3-D convergence studies implicitly assume the solenoidality of the vorticity field.

The problem of maintaining the solenoidality of the vorticity field is currently an active area of research. Pedrizzetti (1992) devised a linear filtering feedback procedure which reconstructs a divergence-free vorticity field from the non-solenoidal computations. The frequency of this tuning is scaled with the physical phenomenon under study. This model is implicitly dissipative; nevertheless, improvements in the stability of the solutions were observed. Winckelmans & Leonard (1993) reconstruct the solenoidal vorticity field by setting the left hand

side of Eq. (1.2.10) to the divergence-free representation of the vortons (Novikov, 1983) and by solving a system of equations for the volumetric vorticity of the elements, Γ_i . They show marked improvement in the stability of the predictions with this approach. Interestingly, the invariants of the flow are better preserved as a result of “enforcing” the solenoidality of vorticity. In unpublished work, Gharakhani (1993) argued that the continuous form of the Euler equations, in their familiar form, explicitly assume solenoidality of vorticity and thus remove the last term in Eq. (1.2.1). However, if vorticity is non-solenoidal, as in the case of the vorton approximation, the last term in (1.2.1) *must* be included to keep the correct balance between the rate of change of vorticity on the left sides of (1.2.1) and vorticity stretch on the right. The failure to correctly keep the balance leads to the growth of vorticity and eventual numerical blow up. Additionally, taking the divergence of Eq. (1.2.1), it was shown that the rate of change of the divergence of vorticity is zero everywhere in the field when the last term is included. This implies that an initially solenoidal vorticity field made up of vortons will remain solenoidal at all times. In contrast, when the last term in Eq. (1.2.1) is discarded, the rate of change of the divergence of vorticity remains zero only on the Lagrangian path of the vortons. This simple method was shown to maintain very long time stability using the test problem of a self-propelling vortex ring. The idea was recently published by Cottet (1996) subsequent to private conversations with Gharakhani. Wu *et al.* (1995) proposed a novel projection method for recovering the solenoidality of the vorticity field in bounded domains, which will be discussed in Section 1.5.

1.3.2 Vortex Filaments

The vortex filament method is a special sub-class of the generalized Lagrangian vortex element methods and is best utilized in the study of 3-D flow phenomena within *inviscid unbounded domains*. Examples of typical problems that vortex filaments perform extremely well are the study of vortex ring instability (Knio & Ghoniem, 1990), shear and mixing layers (Ashurst & Meiburg, 1988; Inoue, 1989; Knio & Ghoniem, 1991), jets (Martin & Meiburg, 1996) and wing-tip vortices (Bliss *et al.*, 1987).

The premise of the vortex filament method is the realization that the evolution of an infinitesimal line element satisfies an equation that is identical to Eq. (1.2.6) with $\alpha=1$ (Batchelor, 1967). Therefore, for an inviscid motion, vortex lines move as material lines (Helmholtz’s Theorem.) Furthermore, in inviscid flow, the circulation of a vortex tube (a bundle of vortex lines) remains constant in time (Kelvin’s Theorem.) Thus, a tube of vorticity retains its identity as it moves with the fluid.

In the vortex filament approach, the curve that defines a vortex line is discretized using a set of contiguous vector segments (or sticks) along the curve, and by assigning its circulation value to the sticks. Therefore, the tail node of one segment is connected to the head node of the next segment. Similarly, a vortex *tube* with a finite core radius is discretized by sub-dividing the cross-section of the tube into a bundle of vortex “lines” and by discretizing each individual line as described earlier. Note that the filament method preserves the solenoidality of the vorticity field by construction (Chorin, 1973, 1993; Leonard, 1980, 1985). Another advantage of this

approach is that the motion of the nodes, due to the local fluid velocity, accounts for vorticity stretch automatically (Helmholtz's Theorem.) However, in order to maintain the stick resolution, line segments that stretch beyond a pre-specified length must be split into smaller contiguous sticks (Knio & Ghoniem, 1990).

The velocity of a vortex filament is evaluated using the Biot-Savart law. Similar to its vorton counterpart, the velocity kernel is singular and is regularized using the concepts presented earlier. The velocity at a point due to a collection of N_F vortex filaments, each with circulation γ_j around its curve C_j is given by:

$$u_i(x, t) = \sum_{j=1}^{N_F} \gamma_j \int_{C_j} K(x - r_j(s')) \frac{\partial r_j}{\partial s'} g\left(\frac{|x - r_j|}{\sigma_j}\right) ds' \quad (1.3.2.1)$$

where K is the velocity kernel, r_j is the position vector parametrized along s' and g is the smoothing function with core radius σ_j (Hama, 1962; Leonard, 1980, 1985). The volume of vorticity is preserved by the relation:

$$\frac{d}{dt} (\sigma_j^2 L_j) = 0 \quad (1.3.2.2)$$

where L_j is the total length of filament j (Leonard, 1980, 1985). This relation implies that the core radius of a filament is reduced (increased) to account for the stretch (compression) of vortex lines. Furthermore, Leonard (1980, 1985) suggested that it is possible to conserve linear and angular momenta exactly by replacing σ_j with $\left(\frac{\sigma^2 + \sigma_j^2}{2}\right)^{1/2}$, where σ is the core radius of the filament at the site the velocity is being evaluated. Therefore, vortex filament methods preserve almost all the invariants of the flow as well as maintain the solenoidality of the vorticity field. The convergence of the regularized vortex filament method to the Euler equations in unbounded domains has been proved by Greengard (1986). This implies that most of the techniques discussed previously can be used successfully to obtain higher-order core functions for the filament method. Alternatively, one can improve the accuracy of the filament discretization by replacing the vortex sticks with quadratic curved elements (Bliss *et al.*, 1987) or by fitting splines through the collocation points (Leonard, 1980). For a given level of accuracy, Bliss *et al.* (1987) achieved over an order of magnitude reduction in the number of collocation points when using quadratic elements as compared to linear sticks. Another advantage of the filament method is that one can easily implement hairpin removal techniques to delete unwanted small scales of the flow, and thus to delay and in some cases to stop the explosive growth of the number of elements (Chorin, 1993).

1.3.3 Vortex-Lattices and Panels

Vortex-Lattice and panel methods are yet another set of closely-related sub-branches of vortex methods, which have widely been used in applications that compute the attached flow loading of lifting surfaces. These methods work well when the wakes can be idealized by thin

sheets, and when the variation of the field in the cross-stream direction is not so strong. See Hoeijmakers (1983) for a concise presentation of various wake modeling techniques.

In the vortex-lattice method, the lifting surface is divided into elements. Each element carries a line vortex along its $\frac{1}{4}$ -chord line, which is connected to line vortices that trail downstream to infinity along the side edges of the element. In this way a horse-shoe vortex is associated with each element. The strength of these vortices are determined by imposing the no-flux boundary condition at the midpoint of the $\frac{3}{4}$ -chord line of the elements (Butter & Hancock, 1971). A variant of this method uses quadrilateral vortex rings on the wing section of the discretization, instead of horse-shoe vortices (Maskew, 1976).

In panel methods the wing is represented more accurately by singularity distributions on the surface of the wing. Furthermore, the wake is discretized using panels with piece-wise polynomial distribution of doublets (Johnson *et al.*, 1980).

These methods are not well-suited for the parachute study for the following reasons. (1) They assume the wakes to be concentrated in thin regions similar to those behind airplane wings. The wake of a parachute is more similar to that of a bluff body than a wing. (2) In the case of a parachute, there is a strong possibility that the elements will touch or cross each other, as well as interact with the parachute. It is not immediately obvious how the intersection of an element with a solid body or another element can be modeled. For these reasons, the application of the vortex-lattice and panel methods to the parachute problem cannot be recommended; a strategy that uses the vortex filament method may actually be superior to the latter.

1.4 The Diffusion Equations

1.4.1 *The Random Walk Method*

The random walk method is a stochastic approximation of Eq. (1.2.7) and is based on the observation that the fundamental solution for the unsteady diffusion equation - the Green function - is identical to a Gaussian probability density function with zero mean and variance equal to $2t/R_\nu$. It can be shown that displacing a collection of vortex elements randomly with zero mean and Gaussian variance equal to $2\Delta t/R_\nu$ mimics the spreading of the vorticity field during Δt (Chorin & Marsden, 1979).

The random walk method was first applied in the context of Lagrangian vortex methods by Chorin (1973). It has since been used extensively in vortex simulations primarily because of its simplicity and ease of implementation in bounded domains, and also because until recently there were no other viable options for a purely grid-free simulation of diffusion. The convergence of the random vortex method in 2-D unbounded domains has been demonstrated by Marchioro & Pulvirenti (1982), Goodman (1987) and Long (1988). The proof of

convergence in 3-D unbounded domains was given by Esposito & Pulvirenti (1989). They assert that the stochastic vortex dynamics is not a solution of the Navier-Stokes equations. Nevertheless, they continue, the continuity property satisfied in the inviscid case and the law of large numbers ensure, at least in the absence of boundaries, an approximate solution.

The random walk method diffuses the vorticity field by spreading out the centers of the vortex elements while keeping their amplitudes unchanged. Therefore, total circulation is conserved with this approach. Furthermore, the computations are extremely inexpensive, involving the addition of three independently generated Gaussian random numbers to the three coordinates of the vortex centers. However, the method has a few disadvantages. It does not conserve the mean position of the vorticity in free space exactly, because the mean of the generated random numbers is only approximately equal to zero. Additionally, the computed solutions are noisy due to statistical errors. As a result, in order to predict the flow accurately, simulations are usually performed for longer times and the statistical noise is reduced by averaging the results. Milinazzo & Saffman (1977) presented a correction for the error in the mean position and tested it for the case of an initially finite region of vorticity. They found that the error in the computed mean size of the vortex system is proportional to $\sqrt{1/N}$, where N is the number of vortex elements. Later Roberts (1985) showed that the error in the angular momentum computed by the random walk method is proportional to $\sqrt{1/R_c N}$. Chang (1988) devised Runge-Kutta type 2-D random vortex methods with an efficient variance reduction technique and tested them on the model problem of a rotating flow with the initial vorticity distributed uniformly on a circular patch. The variance reduction was shown to reduce the sampling errors by one to two orders of magnitude. Fogelson & Dillon (1993) introduced the concept of a smoothed random walk method and, with the proper combination of the number of vortex elements and smoothing radius, obtained accurate predictions of the diffusion equation with relatively fewer number of vortex elements. However, despite the marked improvements obtained by the implementation of smoothing and variance reduction, the number of elements remain too high.

It is worth mentioning that the reduction in the error at higher Reynolds numbers implies that the random walk method may be adequate for simulating flow around parachutes. In addition, while the predicted vorticity field may be noisy, its integral - the velocity field - is expected to be smoother.

The slow rate of convergence of the random walk method, $\sqrt{1/N}$, is often cited as its most significant disadvantage. It turns out that the corrected core spreading vortex method by Rossi (1996) displays a similar convergence rate, although the predictions by this deterministic method are much more accurate than by the random walk method.

1.4.2 The Core Expansion Method

The core expansion method of Leonard (1980) was the first attempt at solving the diffusion equations deterministically. In this approach the core of the vortex element expands in time according to the unsteady diffusion equations. The core expansion method is deterministic

and is as inexpensive as the random walk method. Unfortunately, although the method solves the Stokes equations exactly, it does not approximate the Navier-Stokes equations correctly (Greengard, 1985).

More recently, Rossi (1996) developed a corrected core spreading vortex method and proved its convergence to the Navier-Stokes equations. This method is deterministic and Lagrangian. It also circumvents viscous splitting, which is a useful property when dealing with boundaries. The idea is to allow the core of the vortex element to expand out to a predefined limit, beyond which the element is replaced by an arbitrary number of new elements with smaller cores. In the present implementation in 2-D space, each element is substituted by four new ones which are placed at the four corners of a square that is centered on the old element. Conservation of zeroth through second moments assigns the circulation and the relative positions of the new elements, as well as the constraint on the size of the square box.

The predefined core expansion limit manifests itself as a constrained free parameter, which essentially controls the frequency of adaptation and hence the accuracy and rate of convergence of the method. Clearly, higher rates of vortex element splitting yield more accurate predictions, but at a substantially higher CPU cost. In fact, the refinement process grows the number of elements by $4^{T/\Delta t}$ per “parent” element, where T is the duration of the simulation! Rossi (1996) showed that the circulation of an original element (in 2-D) after $T/\Delta t$ refinements may be interpreted as the probability density function of two independent Bernoulli trials (with probability = 1/2), which incidentally corresponds exactly to the random walk method in the limit of large number of vortex elements. He also proved that the convergence rate is proportional to $\sqrt{1/N}$, similar to the random walk method! However, for a given N , the predictions obtained by this method were shown to be significantly more accurate than those by the random walk method (Rossi, 1996).

Clearly, the main disadvantage of the corrected core spreading method is its tendency to explode the number of vortex elements indiscriminately. That is, rather than adding new vortex elements only at the boundaries of the vorticity region to account for the volumetric expansion of the field, the method faithfully replaces a “parent” element with four “child” elements irrespective of its position in the fluid domain. As a result, a large percentage of “child” elements may/will end up practically on top of each other, adding only to the computational complexity but not accuracy. For this purpose, Rossi (1997) devised a merging algorithm with uniform estimates of the merger-induced error for vortex elements with Gaussian basis functions. He tested the accuracy of this technique in conjunction with the corrected core spreading method using standard examples. Generally, only about 25-30 percent of N was shown to be necessary for an accurate simulation, verifying the earlier assessment about the tendency of the corrected core spreading method to generate unnecessary “child” elements.

The corrected core spreading method has not been tested in 3-D. However, it is clear that a direct extension of the method from 2-D to 3-D would exacerbate the situation further, requiring $8^{T/\Delta t}$ vortex elements at the conclusion of a simulation. It is conceivable that, if the plane of the square box is chosen to be normal to the vorticity vector and diffusion along the

vector is assumed to be zero, only four “child” elements may be necessary in 3-D. However, a much more sophisticated vortex merging algorithm will be necessary for such a scenario.

1.4.3 *The Weighted Particle Method*

The weighted particle or resampling or particle strength exchange methods, initially developed by Cottet and Mas-Galic (1990), are another class of deterministic techniques for solving the Navier-Stokes equation.

In the original approach, the Navier-Stokes equations were solved using viscous splitting - similar to the logic used in the random walk method. Convection was evaluated by the standard Lagrangian vortex element method. Next, vortex element centers were used as quadrature points to evaluate the exact integral solution of the unsteady heat equation (Cottet & Mas-Galic, 1990). The weighted particle method is a technique which redistributes the strength of the vorticity field among the vortex elements, while keeping their positions fixed. This is in contrast to the random walk method, in which case vortex element positions are redistributed while keeping their circulation fixed. Note that since the integral solution of the heat equation is exact (no core functions are used with the kernel) this method is limited only by the accuracy of the viscous splitting algorithm and the accuracy of the quadrature, which is affected by the regularity of the vortex element distribution. The proof of convergence of the weighted particle method was given by Cottet & Mas-Galic (1990). Two comments are in order here. First, convergence was proved based on the stability condition that inter-element distances must be less than or equal to $C(\Delta t/R_c)^{1/2+s}$ for arbitrary positive constants C and s . Second, in the original derivation of the method total circulation was not conserved. Subsequently, Choquin & Huberson (1989) derived a formulation that conserves total circulation, and applied their method to the model problem of shear layers. A parametric study of the accuracy of the method was conducted by Choquin & Lucquin-Desreux (1988).

Degond & Mas-Galic (1989a) proposed and proved the convergence of what is now popularly referred to as the particle strength exchange method. In this approach, the exact integral form of the diffusion equation is approximated by a new integral which replaces the Gaussian kernel with an arbitrarily high order core function. The new integro-differential approximations of the Navier-Stokes equations are then solved directly, and without having to resort to viscous splitting. Degond & Mas-Galic (1989b) generalized the method further to incorporate matrix-valued anisotropic viscosity, which is an attractive feature when considering large eddy simulations. The particle strength exchange method has been used in 2-D (Koumoutsakos & Leonard, 1995) and 3-D (Winckelmans & Leonard, 1993) flow simulations.

From the perspective of the convergence properties of this method, the errors due to the approximation of the Laplacian using a generalized core function are quite similar to the results obtained from the analysis of the Lagrangian vortex method. Briefly, the convergence rate using an m -th order, k times continuously differentiable spherical core function with radius σ is $O(\sigma^m + (h/\sigma)^k \sigma^{-1})$, subject to the stability condition $\sigma^2 R_c \geq 1$. h is the inter-particle distance. This suggests that in order for the method to converge we must have a vanishingly small core

radius, subject to condition $(h/\sigma) < 1$; i.e., neighboring cores must overlap everywhere in the domain and at all times. Unfortunately, the more desirable non-negative core functions, which are uniformly stable, are only second order in σ ; i.e., $m = 2$.

The particle strength exchange method offers a few attractive advantages. It is grid free and deterministic. It maintains the conservation properties of the system and allows the implementation of relatively inexpensive kernels. Convection and diffusion can be simulated simultaneously with this method - without viscous splitting - which facilitates the simultaneous application of no-flux and no-slip boundary conditions. The method can be used successfully for the purpose of large eddy simulations, because of its ability to handle anisotropic, matrix-valued viscosity. However, the particle strength exchange method suffers from some serious disadvantages. First, it exchanges the strength of the particles without redistributing their positions. Therefore, in order to capture the effect of vorticity field expansion, one has to continuously add layers of *zero* vorticity elements at the field front to allow the exchange mechanism to “push” the vorticity field outward. This may result into unnecessary book-keeping exercises (to keep track of the vorticity front) as well as extraneous computations involving vortex elements with zero or near-zero strengths. Second, since the method uses the particle positions as quadrature points to evaluate the Laplacian integral, the accuracy of the integral evaluations deteriorates sharply as the vortex element distributions inevitably become more disordered (Marshall & Grant, 1995b). The practiced remedy is to frequently project the vorticity field onto a uniform distribution of vortex elements and discard the disordered ones (Koumoutsakos & Leonard, 1995). However, this approach suffers from two significant consequences that defeat the entire purpose of utilizing a Lagrangian method. First, it introduces a level of numerical diffusion into the computation. Second, the regridding requires special attention near the boundaries, which are bound to be difficult in 3-D computations. Another disadvantage of the particle strength exchange method is its resolution; i.e., although the particles are separated by a distance h , the resolved diffusion length scale is only $\sigma \gg h$. This implies that a much larger number of vortex elements will be necessary in order to resolve the flow field at the level of inter-particle length scale, h . Lastly, the particle strength exchange method cannot be higher than second order accurate in σ , if the desirable positivity of the kernel is to be utilized.

1.4.4 The Cutoff Differentiation Method

The method of cutoff differentiation was utilized by Fishelov (1990b). The idea is to approximate the vorticity field as the convolution of a core function with the vorticity. The Laplacian operator of the diffusion equation is then evaluated by an explicit differentiation of the cutoff function. Alternatively, one can view this as the convolution of a core function with the Laplacian operator. Therefore, the accuracy of the method is determined by the choice of the core function and the numerical approximation of the integral involved in the convolution operator. In this respect, Fishelov’s method is similar to the particle strength exchange mechanism. However, its advantage over the latter is that it can accommodate infinite-order core functions. A closely related approach was proposed by Najm (1992). He used a Taylor series expansion of the vorticity with respect to time and retained the first and second derivative

terms. He then substituted the unsteady diffusion equation and its time derivative in the latter to obtain the final equation, which contains the Laplacian and the bi-harmonic of vorticity. The spatial derivatives were then evaluated by differentiating the core functions directly. For the cases tested, this approach was shown to be three to four orders of magnitude more accurate than the particle strength exchange method.

The convergence of Fishelov's method (1990b) was proved, for the heat equation, to be in the order of $O(\sigma^m + (h/\sigma)^k \sigma^{-2})$ - equal to that of the particle strength exchange method using non-symmetric core functions. Additionally, the accuracy of the method was demonstrated using two test problems (Fishelov, 1990b). The cores in these tests were nominally set to $\sigma \approx 2\sqrt{h}$. This implies that for inter-particle distances, initially set at 0.1 to 0.2 one can at best expect resolutions of 0.64 to 0.9, respectively, for the diffusion computations! Furthermore, parametric studies by Marshall & Grant (1995b) reveal that Fishelov's method displays severe degradation in the solution accuracy when the elements are disordered. In general, the advantages and drawbacks of Fishelov's method parallel those of the particle strength exchange method and will not be reiterated here.

1.4.5 The Diffusion Velocity Method

The diffusion velocity method was developed, apparently independently, by Degond & Mustieles (1990) and Ogami & Akamatsu (1991). It was derived as an intermediate method between the random walk and the particle strength exchange or core differentiation methods. That is, the diffusion velocity method is deterministic, yet it models the volumetric expansion of the vorticity field. The idea was based on the discovery that the Laplacian of vorticity can be reformulated into sort of a convection of vorticity, if the corresponding velocity u_d is assigned a value proportional to the ratio of the vorticity gradient to the vorticity ($u_d \approx \frac{\nabla\omega}{\omega}$). As a result, a purely Lagrangian simulation of the Navier-Stokes equations becomes possible if the vortex elements travel with the local convective and diffusive velocities. In a series of numerical experiments, Degond & Mustieles (1990) discovered that their predictions were quite accurate everywhere, except toward the tail ends, where significant oscillations were present. They recognized that the motion of the particles under diffusion is expansive. Consequently, they argued, the inter-element distances grow rapidly at the tails and the elements lose core overlap, causing significant oscillations there. They showed that larger cores will diminish the oscillations and even proposed the implementation of time varying cores. It turns out that the derivation given by Degond & Mustieles (1990) and Ogami & Akamatsu (1991) is incorrect. The correct equations were later derived by Kempka & Strickland (1993) from first principles, linking the kinematics of circulation to viscous dynamics. They recognized that the diffusion velocity is non-solenoidal and that it changes the area over which vorticity diffuses. This amounts to a source term which is proportional to the divergence of the diffusion velocity or, equivalently, to the square of the diffusion velocity minus the ratio of the Laplacian of vorticity to the vorticity, $\nabla \cdot u_d \approx \left(\frac{\nabla\omega}{\omega}\right)^2 - \frac{\nabla^2\omega}{\omega}$. In terms of implementation, Kempka & Strickland

(1993) proposed a core expansion to account for the effect of divergence, in addition to convection by the diffusion velocity.

The diffusion velocity method is a relatively new field and theoretical research as well as more computational experiments are necessary to understand the range of stability and accuracy of the method. As of today, no theoretical proof exists for the convergence of the diffusion velocity method (with core expansion) to the Navier-Stokes equations. The proof by Greengard (1985) that the core spreading method of Leonard (1980) approximates the wrong equation raises legitimate concern whether the diffusion velocity method predicts the Navier-Stokes equations correctly.

Degond & Mustieles (1990) pointed out two problems that are specific to the diffusion velocity method (without core expansion.) First, the diffusion velocity is nonlinear which makes it impossible to obtain a correct solution based on the principle of superposition. That is, using a particle representation of the vorticity field, the diffusion of vorticity can be evaluated as the superposition of the diffusion of the individual particles representing that field (because of the linearity of the Laplacian operator.) However, the nonlinear diffusion velocity at a point cannot be evaluated as the linear combination of the diffusion velocities of the particles. Second, since vorticity is infinitesimally small at the boundary of the vorticity field and since its gradient is expected to be large, the exact diffusion velocity is infinite at the boundary. However, in a numerical implementation the diffusion velocity has to remain finite and bounded by a maximum velocity. Therefore, instead of emulating a diffusion at infinite velocity, the numerical method simulates a diffusion with finite velocity, which creates a shock-like diffusion front at the edges. This shock front is the source of instability at the edges which will eventually penetrate into the interior and corrupt the computation. The method developed by Kempka & Strickland (1993) may display a similar symptom. This is because the cores at the edge must be infinitely large in order to represent the physics of the flow correctly, but they can only be finite in a simulation. In a series of experiments, Shankar (1996) showed that the removal of vortical elements that carry seemingly insignificant values of circulation at the edges can lead to inaccurate solutions. For example, it was shown that accurate results were obtained when elements with circulation values smaller than 10^{-6} were eliminated, but inaccurate results were obtained when elements with circulation smaller than 10^{-5} were discarded! Therefore, since the core radius is inversely proportional to the vorticity, one may anticipate the accuracy of the diffusion velocity method to be quite sensitive to the accuracy with which vorticity fronts with very large (infinite) diffusion velocity are captured; i.e., the solution must be highly sensitive to the correctness of the core size. On the other hand, the inclusion of large cores implies that for large time simulations the computational vorticity field will be overwhelmed by the core radii of the elements at the field front, and the accuracy of the predictions may deteriorate due to the large cores. Additionally, for the case of flow in bounded domains the cores may actually cross the walls, inducing inaccurate slip velocities and subsequently introducing inaccurate vorticity into the field.

Note that, in 2-D, it is easy to verify that the diffusion velocity cancels one of the terms contained in the core expansion formula, leaving only the Laplacian operator. As a result,

intuitively, one may expect the issues raised by Degond & Mustieles (1990) to have minimal relevance to the formulation derived by Kempka & Strickland (1993). Therefore, in the absence of a rigorous proof of convergence and/or extensive numerical tests of the accuracy of the diffusion velocity method with core expansion, the above evaluation (regarding the effect of the core size of the edge elements on the accuracy and stability of the predictions) may be categorized as “healthy” skepticism rather than a rejection of the method.

The accuracy of the diffusion velocity method depends strongly on the accuracy with which the first and second degree derivatives of vorticity are evaluated as well. Degond & Mustieles (1990), Ogami & Akamatsu (1991), and Kempka & Strickland (1993) all implemented the method of core function differentiation by Fishelov (1990b). However, since the distribution of the vortex elements is nonuniform, the accuracy of the diffusion velocity evaluations is expected to be quite poor (Marshall & Grant, 1995b). Marshall & Grant (submitted) have proposed a new moving least-squares method for differentiating on irregularly spaced points. In this approach, the derivatives at a point are computed by fitting a second order polynomial to the vorticity values of the vortex elements that are in the proximity of that point, and by minimizing the local least-square error. This yields a system of two linear equations for the unknown coefficients of the polynomial, which can then be used to evaluate the derivatives. Preliminary test results with irregularly distributed points have displayed the superiority of the moving least-squares method over the particle strength exchange and core function differentiation methods.

At present, the most serious difficulty with the diffusion velocity method is the absence of a formulation in 3-D space. The equation derived by Kempka & Strickland (1993) yields no information about the diffusion velocity in the direction of the vorticity vector. Marshall & Grant (submitted) proposed an ad-hoc argument to ignore the diffusion velocity in the third dimension, which essentially implies the assumption of local two-dimensionality of the diffusion velocity. Clearly, a rigorous mathematical derivation is necessary in order to find the correct answer in 3-D.

1.4.6 The Vorticity Redistribution Method

The vorticity redistribution method was developed recently by Shankar & Van Dommelen (1996). The idea is to simulate diffusion by redistributing fractions of the circulation of each vortex element to the neighboring elements, such that the conservation properties of the system are preserved. The basic approach in the current implementation is to assign a radius of influence for each vortex element (typically in the order of the diffusion length scale) and to reformulate the conservation of zeroth to second moments for the elements inside the radius in terms of the fractions. Repeating the process for all vortex elements yields a linear system of equations for the unknown fractions, which is solved using linear programming techniques. If a solution of the linear system exists, it is guaranteed to be a stable and convergent approximation of the diffusion equations. If a solution does not exist, it is an indication of the presence of “holes” within the computational domain, in which case a judicious filling of the holes with new elements (with zero circulation) will lead to a guaranteed solution. The same mechanism enforces the addition of new elements at the vorticity field front to account for its volumetric

expansion. Note that the (lack of) existence of a solution equips the redistribution method with an automatic mechanism that detects when and where an element may be added. Therefore, this method is inherently adaptive. In contrast, in order to maintain solution accuracy and stability, the particle strength exchange and core differentiation methods require the flooding of the field front with *multiple* layers of new elements, because they lack a priori information about the extent to which vorticity may have to be diffused.

The vorticity redistribution method has some attractive features which are essential for the accurate simulation of diffusion. It guarantees - by construction - the conservation of circulation for each element, the conservation of the center of vorticity, and the correct expansion of the mean diameter. Furthermore, only positive fractions are allowed, implying that reverse vorticity cannot form spontaneously in the middle of a flow field. Unlike the particle strength exchange and core differentiation methods, core functions are not used in the redistribution method. As a result, the resolution of the latter is in the order of inter-element spacing and not the nominally larger core radius. More importantly, since the redistribution formulation does not involve quadratures, frequent remeshing of the field onto a uniform distribution of vortices is not required for obtaining accurate predictions. (The vorticity redistribution method is equivalent to a finite difference scheme for a uniform distribution of points.)

The high accuracy of the method has been demonstrated by Shankar (1996) using a number of test problems. The most notable example, relevant to this report, is the direct simulation of impulsively started flow over a circular cylinder at $R_e = 9500$. More specifically, the converged solution used 60,000 vortex elements at non-dimensional $t = 3$. In contrast, the direct simulation by Koumoutsakos & Leonard (1995) using the particle strength exchange method required 350,000 elements at the same non-dimensional time!

The vorticity redistribution method suffers from the following disadvantages. First, it must employ viscous splitting to simulate the Navier-Stokes equations. Therefore, its rate of convergence is limited to that of viscous splitting. Furthermore, it cannot satisfy no-flux and no-slip boundary conditions simultaneously. Second, the method is algorithmically complicated and will require substantial initial time investment to implement. Third, the computations are expensive - approximately equal to the Biot-Savart evaluations. However, for a given accuracy (rather than a given number of elements) the redistribution method may still be competitive. Finally, the extension of the method to large eddy simulations; i.e., matrix valued viscosity, may not be a trivial task.

1.4.7 The Free-Lagrange Method

The free-Lagrange method developed by Russo (1993) and Strain (1994) is based on the idea of constructing a finite difference scheme to approximate the Laplacian using the Voronoi diagram. In this approach, the vortex elements are convected with their local velocity, and the diffusion is solved by constructing a Delaunay triangulation on the particles and evaluating the

derivatives. The evaluation of the derivatives at a point amounts to finding quadrature weights, which happen to be a function only of the position of the neighboring points.

The discrete Laplacian is weakly consistent to first order with the continuous Laplacian, but it is not pointwise consistent (Russo, 1993). The total vorticity and angular momentum were shown to be conserved, but no information was given about the first moment. Test results using a set of parameters showed that the diffusion errors are smaller than the errors in the velocity evaluations. This is because the distribution of vortex elements eventually becomes too irregular for the Biot-Savart quadratures to produce accurate results, whereas diffusion is computed optimally using the Voronoi diagram. Therefore, as far as diffusion is concerned, the method is adaptive and has been shown to maintain long-time accuracy. However, a truly adaptive scheme will have to consider the accuracy of the velocity evaluations as well.

The disadvantage of free-Lagrange methods, in general, is that if the topology of the flow changes rapidly, as would be expected in the case of the flow over parachutes, the Voronoi diagram would have to be reconstructed at each timestep. This is an unnecessarily extra overhead that vortex methods can do without. Furthermore, the accuracy of such a construction is linear in vorticity at best, and it is not obvious how the accuracy of the method can be improved.

1.5 Wall Boundary Conditions

The first successful implementation of the no-slip boundary condition, in the context of 2-D grid-free vortex methods, was provided by Chorin (1978). In this approach, in order to satisfy the no-slip boundary condition, the solid wall was modeled by an infinitesimally thin sheet of vorticity. The sheet strength is equal to the difference between the velocity that is prescribed as the boundary condition and the tangential velocity at the wall that is induced by the combined effects of the vortical velocity of the vortex elements and a potential flow that imposes the no-flux boundary condition. At the implementation level, the continuous sheet is discretized using a set of contiguous segments and a prescribed variation of the vorticity strength across each segment. The next step is to account for the diffusion (and convection) of the sheets from the wall into the flow interior. It is well-known that the use of isotropic vortex particles very near the wall leads to a backwash effect and the eventual degradation of the simulation. Chorin (1978) proposed a domain decomposition method, in which the vorticity field is represented by anisotropic sheets within a user-specified thin region near the boundary, and by isotropic vortex elements outside this region. Based on an order of magnitude analysis, Chorin (1978) argued that the flow near the boundary is dominated and characterized by the gradients normal to the wall. This led to the Prandtl boundary layer approximation of the vorticity dynamics near the wall, as well as a significant simplification of the kinematic relationship between the vorticity and velocity fields. Therefore, the sheets convect and diffuse away from the wall and into the interior according to the Prandtl approximation. The random walk method is used to simulate diffusion. Once the sheets traverse the numerical boundary layer, they are converted to isotropic vortex elements while conserving circulation. Similarly, if a vortex element jumps into the numerical boundary layer it is converted into an appropriate sheet.

A rigorous proof of convergence for the above method is not available. However, an extensive numerical convergence study has been conducted by Puckett (1989), using the example problem of flow over a flat plate. Additionally, numerous examples of the successful application of this technique to complex flow problems can be found in the literature (Ghoniem & Gagnon, 1987; Cheer, 1989). The accuracy and convergence rate of the vortex sheet method, as implemented above, are not very high. Obviously, one source of error is the well-documented poor performance of the random walk method. Another reason was given by Sod (1991) who pointed out that the computation of the velocity field involves blob-blob, sheet-sheet, and blob-sheet interactions. He argued that the method by Chorin (1978) ignores blob-sheet interactions and, thus, does not allow for a consistent transition between the two domains when a blob is near the edge of the numerical boundary layer. In other words, the blob-sheet interaction provides a coupling mechanism between the boundary and exterior sub-domains in such a way that not only circulation is conserved, but also the continuity of the tangential and the normal velocities across the edge of the numerical boundary layer is preserved. Unfortunately, Sod (1991) did not substantiate his observation by any numerical experiments.

The validity of the Prandtl approximation, especially in separated flow simulations, has long been the source of controversy and has been suspected to be the dominant source of the observed inaccuracy in the vortex sheet method. In order to put this question to rest, Anderson & Reider (1994) reported results from a series of high resolution finite-difference simulations of flow over a 2-D cylinder at Reynolds number 1000. At this flow regime, much of the complicated boundary layer behavior, such as separation of the back flow and multiple sign changes of vorticity along the cylinder surface, are present. The high resolution finite-difference method was used to eliminate the errors caused by the random walk method and/or the sheet discretization. The flow simulations were conducted using the Navier-Stokes equations on the entire domain, as well as the finite-difference version of the Prandtl/Navier-Stokes method by Chorin (1978). Results indicated that the vorticity fields obtained by the two methods are indeed quite different and that the long-time solutions obtained by the domain decomposition method blow-up. Anderson & Reider (1994) attributed the problem to the incompatibility between the two solutions at the interface of the two sub-domains. They then devised a new scheme in which the tangential as well as the normal velocities are consistent across the interface, and showed the vorticity field to be in excellent agreement with that of the full Navier-Stokes simulation. This supports the assertion by Sod (1991) about the importance of including the sheet-blob interactions in a computation. It also suggests that the Prandtl approximation is an accurate and efficient method for resolving even complex separated flows near the boundary - given, of course, the correct boundary conditions are implemented. Cottet (1991) proposed an *overlapping* domain decomposition method that may be of value in the context of Prandtl/Navier-Stokes approximations, although he used it in conjunction with a hybrid particle-grid scheme. The idea is to allow for a limited overlap between the inner and outer domains, instead of a sharp interface between them, which may lead to a stronger coupling between the two sub-domains.

The extension of the vortex sheet method from 2-D to 3-D was first presented by Chorin (1980). In this case, the velocity jump at the boundary was discretized over a collection

of *rectangular vortex tiles*. The evolution of the tiles within a thin region near the boundary was evaluated using the Prandtl approximation, with the added simplification that vorticity stretch in the boundary region is negligible. The no-flux boundary condition was satisfied by the method of images. Once a vortex tile jumped outside the boundary domain, it was replaced by a vortex stick. Conservation of vorticity was ensured by setting the volumetric vorticity of the stick equal to the product of the magnitude of the surface vorticity of the tile and its area, and by aligning the stick direction parallel to the vectorial direction of the tile surface vorticity. Fishelov (1990a) replaced the vortex sticks with spherical vortex elements, and conducted a parametric study of the convergence of the Prandtl approximation using the simple case of flow over a flat plate, with periodic boundary conditions in the transverse direction. This method was later incorporated into a hybrid vortex-boundary element algorithm for the simulation of viscous flow inside 3-D geometries with moving boundaries of the type found in engines (Gharakhani & Ghoniem, 1997a). The boundary element method was used to evaluate the potential velocity and its gradients in the field, and to impose the no-flux boundary condition (Gharakhani & Ghoniem, 1997b). A parametric study of the accuracy and convergence of the method was also provided for the case of flow in a duct with square cross-section and, where possible, results were compared with exact solutions. Fishelov (1994) has also developed a version of the method for the spherical coordinate system and has simulated the flow over a sphere for short times. Unfortunately, the proposed approach is not general since the tiles are topologically rectangular and the potential flow is evaluated by the method of images. More recently, Gharakhani & Ghoniem (1997c) expanded the versatility of their vortex-boundary element method by generalizing the rectangular vortex tiles to polygons and used it to simulate the induction process in an engine configuration with and without a valve.

Before proceeding with the presentation of other 3-D algorithms, it seems appropriate here to introduce 2-D schemes that are potential candidates for a successful extension to 3-D. Koumoutsakos *et al.* (1994) developed a boundary vorticity creation method based on the Neumann vorticity boundary condition of Kinney & Paolino (1974). In this approach, the vortex sheet, which is established on the boundary to satisfy the no-flux velocity boundary condition, is reformulated in terms of the flux of vorticity at the wall. The latter is then used as the boundary condition for an unsteady diffusion equation with a homogeneous initial condition to diffuse the boundary vorticity into the interior. As a result, all the strength of the vortex sheet is distributed onto the existing vortex elements in the interior and the no-slip boundary condition is satisfied at the end of the time-step. Koumoutsakos *et al.* (1994) obtained the solution of the diffusion equation using an indirect boundary integral formula, which expresses the field vorticity in terms of time-dependent surface potentials that are distributed over a set of panels defining the surface geometry. A δ -function dependence of the surface potentials on time was assumed so that a closed-form solution for the boundary integrals could be obtained. The resulting algorithm is grid-free, meshes well with the particle strength exchange mechanism for diffusion, and precludes the use of the Prandtl approximation. The robustness of this method was demonstrated using the example of impulsively started flow over a cylinder at various Reynolds numbers (Koumoutsakos *et al.*, 1994; Koumoutsakos & Leonard, 1995).

Strickland *et al.* (1996) outlined a two-dimensional, unsteady, boundary-layer model that utilizes the diffusion velocity concept, and can be incorporated into vortex element methods via the domain decomposition technique. The primary objective is to minimize the number of vortex elements injected into the flow interior - endemic of the vortex sheet method - while maintaining accuracy. In this approach, the area of the thin numerical boundary layer is discretized using a set of grid points. A re-gridding scheme is implemented to control the distortion of the grids caused by their Lagrangian motion. The proposed algorithm is to diffuse the wall vorticity into the layer, allow diffusion of the existing vorticity to take place via the diffusion velocity method, convect the flow in the Lagrangian frame, and re-map the vorticity field back onto the original grid distribution. When sufficient vorticity traverses the edge of the boundary domain, it is replaced by new vortex elements such that up to the second moments are conserved (Wolfe, 1997). Strickland *et al.* (1996) used the one-dimensional version of the model proposed by Koumoutsakos *et al.* (1994) to diffuse the wall vorticity into the layer. They pointed out that the assumption of a δ -function time-dependence for the potential leads to the kernel of an impulsively started flat plate and argued that, since the diffusion equation is linear, one may obtain the vorticity distribution associated with the flux from the wall at the end of the time step as the superposition of impulsive solutions using a convolution integral. Strickland *et al.* (1996) showed that the application of the convolution integral leads to a superior representation of the vorticity distribution, especially near the boundary. The accuracy of the proposed boundary layer model has been verified using a number of standard test cases. The main disadvantage of this method is its use of the diffusion velocity method at the boundary, which does not have a rigorous solution when vorticity is zero there. An additional complication, which is actually shared by all methods that use domain decomposition, is how corners and edges (in 3-D) are handled so that two neighboring boundary domains do not overlap or parts of a boundary domain do not fall outside the physical boundaries of the geometry.

The triangulation technique by Huyer & Grant (1997) is the 2-D demonstration of the algorithm that the 3-D counterpart by Grant *et al.* (1995), which is at the development stage, is expected to adopt. In the 2-D version, the boundary surface is defined by a set of flat panels. The region near the boundary surface is triangulated using four layers of nodes that are alternately staggered above the panel centroids and the panel ends. The layers are separated by half the diffusion length scale. Additional triangular elements are generated in time using a moving front technique. Huyer & Grant (1997) use surface vorticity and potential source panels based on the recommendation by Uhlman & Grant (1993). The velocity boundary conditions are satisfied by the following iterative procedure. An initial (arbitrary) distribution of vorticity in the fluid region will generally induce a non-zero velocity at the boundary surface. The no-flux and no-slip boundary conditions are imposed at the panel nodes by solving a linear system of equations for the unknown amplitudes of the sources and vortices. Subsequently, the new boundary sources and vortices generate a new velocity distribution in the fluid domain, and thus a new vorticity distribution there. This is the beginning of the next iteration step and the process continues until the maximum vortex sheet strength is equal to a user-specified small value. In practice, this approach can be extended to grid-free methods as well.

We now return to other methods that have been implemented in 3-D.

Bernard (1995) developed a deterministic vortex sheet method in 2-D. The 3-D version was later introduced by Bernard & Thomas (1995). In this approach, the vorticity field in the boundary layer was discretized using a set of sheets (tiles in 3-D) with finite thickness along the wall and stacked on top of each other in the normal direction to the wall. The layer immediately contiguous to the wall was discretized using special sheets with half thickness, which remain stationary during the simulation. The sheet vorticity was assumed to be discontinuous; having a uniform value within the sheet and zero elsewhere. The diffusion of vorticity was evaluated using the deterministic method by Fishelov (1990b). Fictitious tiles were added below the wall and a special interpolation function used to account for the loss of vorticity due to the portion of the core radius that extends beyond the wall. The effect of vorticity stretch was included in the 3-D formulation. The velocity field was obtained by both the Prandtl approximation method (Chorin, 1980) and the Biot-Savart formula. Only the latter was used in the 3-D implementation. For the case using the Prandtl method, the no-slip boundary condition was satisfied locally by updating the vorticity of the wall sheets via a first-order discretization of the normal gradient of the tangential velocity at the wall. For the case using the Biot-Savart formula, the wall sheet vorticity was updated by solving a linear system of equations that relates the wall vorticity to its velocity - with the latter set equal to zero. The method was demonstrated to offer relatively high accuracy using a few standard boundary layer example problems. However, special care is necessary to account for the incoming and outgoing vortex element. Furthermore, as would be expected from the diffusion scheme of Fishelov (1990b), and because the gaps and overlaps caused by discontinuous vortex tiles degrade the velocity evaluations, it becomes necessary to regrid the sheets frequently to maintain an acceptable level of solution accuracy.

In the methods described thus far, no provision was made - either implicitly or explicitly - to enforce the solenoidality of the 3-D vorticity field, which is essential for obtaining accurate solutions for long times. In what follows, advanced methods will be presented, which persevere the solenoidality of the vorticity in the computational domain and, in particular, within the numerical boundary layer.

The first such attempt was made by Summers (1991). Only a brief introduction of the method will be given here, because the algorithm contains a level of hand-waving and is too complicated and costly to consider for implementation. The idea is an extension of the method of tiles by Chorin (1980) and is based on partitioning the boundary into a collection of polygonal loops which, *in aggregate*, satisfy the no-slip and solenoidality conditions and account for vorticity stretch. Each loop consists of a cluster of rectangular tiles which, individually, neither satisfy the solenoidality of the vorticity field nor account for vorticity stretch. However, the cluster is solenoidal because the tiles form a loop, and it simulates vorticity stretch as a result of the independent motion of the tiles.

Summers & Chorin (1996) have developed a new class of vorticity generation models based on the creation of vorticity at the boundary, which will be discussed in Section 2.5.

Casciola *et al.* (1996) have proposed a generalized integral formulation for a 3-D vorticity generation mechanism that will be useful in extending the applicability of the 2-D algorithms (for near-wall vorticity dynamics) to 3-D. The derivation is based on the boundary integral solution of the diffusion equation with a given initial condition, together with the Dirichlet boundary condition for the normal component of the vorticity at the wall and the Neumann boundary condition for the tangential components. The boundary vorticity vector is then meshed with the boundary velocity vector via the definition of vorticity and the generalized Biot-Savart formula (Wu & Thompson, 1973). This yields a boundary integral formulation that relates the no-slip boundary condition to the Neumann boundary condition for the diffusion equation. A significant consequence of this derivation is that for an initially divergence-free vorticity field and a solenoidal vorticity flux at the wall, the vorticity field at subsequent times will remain solenoidal. However, during a computation, the diffusion time-step is preceded by the Euler time-step which, when using the vortex blob method, may supply a non-solenoidal vorticity field as the initial condition for the diffusion step.

To this end, Wu *et al.* (1995) devised a novel projection method that maps a general vorticity field in a bounded domain onto the divergence-free space. Differential as well as integral formulations were derived. Wu *et al.* (1995) showed that the curl of the vortical velocity is in general equal to the vorticity, *plus* a volume integral term containing the divergence of vorticity and a surface integral term containing the normal component of vorticity at the boundary. They then proposed an *optimum* correction by decomposing the velocity into the familiar vortical and potential components, and an additional corrective potential term. The latter was obtained by solving a Laplace equation, with a Neumann boundary condition that reflects the existence of non-solenoidal vorticity in the field and on the boundary. Note that if the vorticity field is divergence-free everywhere, the Neumann boundary condition and subsequently the corrective potential term reduce to zero, and vorticity is trivially recovered as the curl of the vortical velocity. In this sense, the method is optimum, as it solves one additional boundary integral equation, only when necessary. Wu *et al.* (1995) demonstrated the effectiveness of the projection method in its differential derivation using the 3-D lid-driven cavity problem as the example.

2 VELICITY METHODS

2.1 Introduction

The velocity (magnetization or impulse) method is a new derivation based on the canonical Hamiltonian for the incompressible Euler equations, and is valid in all space dimensions (Oseledets, 1989; Buttké, 1993). It has the important property of preserving the invariants of the flow: impulse, angular momentum, helicity density and kinetic energy. Furthermore, by construction, velocity satisfies the solenoidality of the velocity and the vorticity everywhere in the field. Therefore, the velocity method does offer some fundamentally significant advantages over the traditional vortex methods. However, it is still in its infancy; consequently, theoretical proofs of convergence and error analyses, and rigorous numerical verification of the accuracy and overall robustness of the solutions for long times - especially in three dimensions - are just beginning to emerge in the literature. Furthermore, to date, full velocity simulations that include viscous and wall effects are not available.

Oseledets (1989) introduced the formalism for the velocity method for the case of constant-density Navier-Stokes equations. Later, Buttké (1993) extended the idea to the case with variable density and demonstrated the potential advantages of the method using flow induced by a generic vortex torus. Most recently, Cortez (1995, 1996) provided an analysis showing the equivalence between the velocity variable and a vortex dipole. Interestingly, the idea of discretizing the vorticity in the Euler equations by vortex dipoles, instead of monopoles or vortons, was proposed prior to the introduction of the velocity method (Saffman & Meiron, 1986). However, Saffman & Meiron (1986) dismissed the idea because, among other reasons, they argued that the self-interaction term leads to infinite speed. Later, Chefranov (1987) proposed that the self-interaction term is bounded using physical reasoning, and showed the vortex dipole method to be Hamiltonian and to preserve all the relevant flow invariants. This equivalence between the velocity and vortex dipoles suggests that technologies that have thus far been developed for vortex methods may be extended to the velocity method directly (Buttké, 1993; Cortez, 1995, 1996; Recchioni & Russo, 1995).

2.2 Formulation

The equations of motion of an incompressible viscous fluid in a bounded 3-D domain, D , with boundary, ∂D , may be expressed in the velocity transport formulation as:

$$\frac{\partial M}{\partial t} + \mathbf{u} \cdot \nabla M = -M \cdot (\nabla \mathbf{u})^T + \frac{1}{R_c} \nabla^2 M \quad \mathbf{x} \in D \quad (2.2.1)$$

$$\nabla \cdot \mathbf{u} = 0 \quad \mathbf{x} \in D \quad (2.2.2)$$

$$\mathbf{M} = \mathbf{u} + \nabla\Phi \quad \mathbf{x} \in D \quad (2.2.3)$$

$$\mathbf{u}(\mathbf{x}, t = 0) = \mathbf{u}_0 \quad \mathbf{x} \in D \quad (2.2.4)$$

$$\mathbf{u}(\mathbf{x}, t) = (\mathbf{u} \cdot \boldsymbol{\tau}, \mathbf{u} \cdot \boldsymbol{\rho}, \mathbf{u} \cdot \mathbf{n}) \quad \mathbf{x} \in \partial D \quad (2.2.5)$$

where \mathbf{M} is the velocity and Φ is an arbitrary scalar function. All other parameters were defined in Section 1.2. Equation (2.2.3) states that the velocity field is not defined uniquely, which may be useful in constructing more accurate and/or more versatile algorithms. It also implies that vorticity is simply the curl of the velocity:

$$\boldsymbol{\omega} = \nabla^{\wedge} \mathbf{u} = \nabla^{\wedge} \mathbf{M} \quad (2.2.6)$$

Application of (2.2.2) to (2.2.3) yields a Poisson equation for Φ , the solution of which can be back-substituted in (2.2.3) to obtain the velocity distribution for a given velocity field:

$$\mathbf{u} = \mathbf{M} - \nabla[\Delta^{-1}(\nabla \cdot \mathbf{M})] \quad (2.2.7)$$

Since velocity, like vorticity, is concentrated in compact regions of the flow, Buttke (1993) extended the idea of the Lagrangian vortex element method to equations (2.2.1 - 2.2.3) and constructed the Lagrangian velocity ‘‘blob’’ method for unbounded domains. He discretized the velocity field using N_V blobs, each located at \mathbf{x}_i and with volumetric velocity $\mathbf{m}_i = \mathbf{M}(\mathbf{x}_i, 0)h^3$:

$$\mathbf{M}(\mathbf{x}, t) = \sum_{i=1}^{N_V} \mathbf{m}_i(t) f_{\sigma}(\mathbf{x} - \mathbf{x}_i) \quad (2.2.8)$$

where h is the inter-element distance and $f_{\sigma}(\mathbf{x}) = \frac{1}{\sigma^3} f(|\mathbf{x}|/\sigma)$ is a smoothing or core function with core radius σ . The velocity field, Eq. (2.2.7), corresponding to the velocity distribution (2.2.8) is:

$$\mathbf{u}(\mathbf{x}, t) = \sum_{i=1}^{N_V} \left\{ \mathbf{m}_i(t) f_{\sigma}(\mathbf{x} - \mathbf{x}_i) - \mathbf{m}_i(t) \cdot \nabla[\nabla G_{\sigma}(\mathbf{x} - \mathbf{x}_i)] \right\} \quad (2.2.9)$$

where $G_{\sigma} \equiv G * f_{\sigma}$, G is the Green function in the infinite domain and $*$ represents the convolution operation. Substituting (2.2.8) into the Lagrangian equivalent of (2.2.1) and applying viscous splitting yields the discrete form of the evolution equations for the velocity field:

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{u}_i \quad \mathbf{x}_i(t = 0) = \mathbf{x}_0 \quad (2.2.10)$$

$$\frac{d\mathbf{m}_i}{dt} = -\mathbf{m}_i \cdot (\nabla \mathbf{u}_i)^T \quad \mathbf{m}_i(t = 0) = \mathbf{m}_0 \quad (2.2.11)$$

$$\frac{dm_i}{dt} = \frac{1}{R_c} \nabla^2 m_i \quad i = 1, \dots, N_V \quad (2.2.12)$$

Equations (2.2.10) through (2.2.12) are vectorial ordinary differential equations which can now be integrated in time to any order of accuracy to obtain the trajectory of the velocity blobs and the evolution of the velocity.

2.3 The Euler Equations

The accuracy and convergence rate of the Lagrangian velocity method, like its vortex element counterpart, is expected to be a function of grid resolution, the choice of the core function and its radius, the order of space and time integrations, and how the velocity distribution is initially projected onto the blobs. Some of these issues have been studied numerically by Cortez (1995, in review) and Recchioni & Russo (1995). Convergence of the method is studied theoretically by Benedetto (1996).

One severe drawback of the method is that the velocity has the same units as the velocity; subsequently, the velocity field is evaluated using an integral formulation that is one order more singular than the Biot-Savart integral used in vortex methods. In other words, the velocity in the velocity method is in the same order as the velocity gradient in the vortex method, which is one source of instability in 3-D vortex methods. Undoubtedly, the higher order singularity of the velocity gradients in the velocity method will contribute to its instability further. As a result, the solution is expected to be highly sensitive to the correct choice of the core function, the core radius, and the spatial resolution and integration scheme. Benedetto (1996) showed that in order to obtain a bounded approximation of the velocity field, the Fourier transform of the core function must be positive. Recchioni & Russo (1995) demonstrated, with a simple example, how the solution blows up suddenly if a non-positive core function is used in the calculations. Benedetto (1996) also obtained error estimates for the velocity field in the infinity norm and showed its linear dependence on the core radius, as well as a term which is

proportional to $\frac{1}{\sigma^6 N_V^{1/4}} \approx \frac{(h/\sigma)^{3/4}}{\sigma^8}$. This suggests that a large number of velocity elements,

subject to core overlap, may be necessary to obtain accurate solutions for short times. Cortez (1995, in review) conducted a series of experiments in 2- and 3-D to assess the accuracy of the velocity element method. He concluded that the initial grid refinement only delays the inevitable numerical blow up. (The Fourier transform of the core function in his computations was positive.) This is because as the flow evolves, the contour (or surface in 3-D) of the velocity field experiences strong stretching and the velocity elements become sparsely distributed along the contour (or the surface.) This leads to the loss of accuracy of the mid-point rule for spatial integrations and the eventual blow up. Note that vortex simulations of shear flow experience similar problems with sheet stretch. Curiously, even as the predictions begin to diverge from the correct solution, the velocity method faithfully preserves the invariants of the flow until the moment the spontaneous blow up occurs (Cortez, 1995, in review). Cortez (1995) proposed two techniques to remedy the problem of blow up. The first is a grid refinement approach,

similar to the ideas used in the vorticity insertion techniques. In this case, when the velocity magnitude grows beyond a preset maximum value the element is replaced by two equally-sized elements, each having a velocity vector in the same direction as and with half the magnitude of the original vector. This approach preserves linear impulse and has been shown to maintain stability for longer times. However, the proliferation of the number of elements, especially in 3-D, makes this method impractical. The second remedy is a re-gridding approach that exploits the non-uniqueness of the velocity field representing the flow. In this case, rather than allowing the field to stretch along its original surface, a new surface-minimizing distribution of velocity is constructed over the flow every few time-steps and the old elements are discarded. This technique has the advantage of bounding the number of velocity elements. However, the development of a generalized algorithm to identify a minimal surface in 3-D will require complex mathematics (Chopp, 1993), and Cortez (1995) postponed it to future work.

2.4 The Diffusion Equations

The diffusion of the velocity variable can, in theory, be evaluated using any of the methods discussed in Section 1.4. However, as discussed earlier, since the velocity field obtained by the velocity method is one order more singular than that obtained by the vortex method, it is essential to utilize methods that yield very high solution accuracy and rate of convergence. To date, parametric investigations of the relative accuracy of various available grid-free techniques for solving the velocity diffusion equation have not appeared in the literature.

2.5 Wall Boundary Conditions

The grid-free velocity method has thus far been used mainly to simulate flow in unbounded domains. However, recently, Summers & Chorin (1996) proposed a hybrid vortex-velocity model in which they used the velocity variable to satisfy the no-slip boundary condition and to simulate the flow within a thin region near the boundary, and used vorticity to simulate the flow elsewhere.

Recall that the velocity variable has the attractive property that it conserves the invariants of the flow and maintains the solenoidality of velocity and vorticity. Velocity or impulse is not a unique variable and contains a gauge freedom, which provides a degree of flexibility in designing optimum algorithms. Summers & Chorin (1996) proposed two choices for the gauge and proceeded with the formulation of the algorithm for each choice.

In the first method the impulse vector was set *normal* to the wall. This is a natural generalization of vortex methods and leads to an algorithm that prescribes the generation and subsequent conservation of circulation. In this approach, the wall surface may be discretized using a set of polygonal tiles. Then the magnitude of the tile-centered impulse vector may be approximated as the line integral of the slip velocity around the tile divided by the height of a very thin numerical boundary layer near the wall. An order of magnitude analysis reveals that the dynamics of only the normal impulse are dominant (Summers & Chorin, 1996). Therefore, once the normal impulse vectors are generated at the wall to satisfy the no-slip condition, they

are diffused using the random walk method, convected, and stretched away from the boundary. When a tile traverses the thin layer, it is converted to a vortex loop parallel to the wall with an arbitrary combination of circulation and radius such that the volumetric impulse is conserved (Summers & Chorin, 1996). Note that these loops are not unique. Summers & Chorin (1996) opted to convert the impulse tiles into vortex filaments; however, one may design a method based on spherical vortex elements instead of filaments. The feasibility of the first method was demonstrated by Summers & Chorin (1996) for short times using the example of flow over a sphere.

In the second method the impulse vector was set *parallel* to the wall. This algorithm enforces the conservation of momentum, and it implies the creation of an impulse density at the wall equal in magnitude and opposite in direction to the slip-flow there. An order of magnitude analysis shows that the normal component of the velocity vector is negligible. Therefore, in this case, the tile-centered velocity vectors parallel to the wall are set equal to the negative of the slip velocity at the wall, and their evolution toward the flow interior is obtained from the equations of motion in the direction parallel to the wall. Once a tile jumps out of the numerical boundary layer it is converted into a vortex loop, such that its plane is oriented vertically to the wall and normal to the direction of slip flow at the point of creation (Summers & Chorin, 1996). Note that no line integrations are necessary in the second method and the creation principle is localized. Furthermore, the vorticity created in this approach produces a more detailed picture of the boundary layer than the first method (Summers & Chorin, 1996). The velocity method appears to be a simple and efficient model for generating divergence-free vorticity at the walls.

SUMMARY

Lagrangian vortex element methods are techniques for the simulation of incompressible, high Reynolds number, unbounded or bounded flows. The idea is to discretize the governing equations of the fluid motion in its vorticity transport formulation, which is computationally advantageous for the following reasons. Most vortex methods are grid-free and eliminate the need for volumetric meshing of the fluid domain, which can be quite complicated and time-consuming - especially for the complex deforming boundaries of parachutes. Vortex methods are almost free of numerical diffusion, which makes them particularly suitable for the simulation of high Reynolds number flows. Vortex methods satisfy the far-field boundary conditions exactly which, combined with the fact that vorticity is generally concentrated in relatively small regions of the flow field, leads to substantial savings in computational resources as compared to grid-based methods. Vortex methods are also naturally adaptable to massively parallel computing.

The approach taken for discretizing the vorticity field defines the various categories of vortex element methods, each with its advantages and drawbacks. The vortex filament method discretizes the vorticity field using a collection of contiguous line segments or sticks. It preserves the total vorticity, as well as linear and angular momenta. It also satisfies the solenoidality of vorticity by construction, but only globally and not pointwise. The filament method is an excellent tool for simulating *inviscid unbounded* 3-D flows. The vortex-lattice or panel method is a closely related approach which uses a collection of contiguous, infinitesimally thin panel segments to discretize the vorticity field. It is regularly used to model the wake behind airfoils. Unfortunately, the filament and panel methods may not be used accurately for *viscous* flow simulations.

Vortex monopoles or vortons, commonly referred to as vortex elements, use smoothed monopoles to discretize the vorticity field. They are highly versatile and can, in principle, be used to simulate unbounded or bounded flow at any Reynolds numbers. The accuracy of vortex methods and their convergence to inviscid or viscous unbounded flows have been proved. Additionally, smoothing functions with arbitrarily high order of accuracy have been provided in the literature (Beale & Majda, 1985). A method for constructing smoothing functions that allow the user to control the spectral accuracy of the discretization has also been proposed (Humi, 1993).

The most commonly implemented vortex methods use isotropic spherical elements. However, other variants have appeared in the literature as well. Marshall & Grant (1995a) have developed an anisotropic method, which can be useful for simulating wake and boundary layer flows. Hou (1992) has derived a formulation that uses point vortices, and he has demonstrated that the predictions remain accurate for long times. Grant *et al.* (1995) are developing an advancing front technique that uses Delaunay tetrahedralization to discretize the vorticity field.

There are a few challenging issues that must be resolved before the full potential of vortex element methods can be realized. For example, vortex methods generally do not preserve the basic invariants of the flow. Furthermore, they satisfy the solenoidality of vorticity only approximately. Some variants of the method have been shown not to form a weak solution of the governing equations. In this regard, Pedrizzetti (1992) argues that vortons not representing a weak solution of the Euler equations has little bearing on the validity of the method. Moreover, Greengard & Thomann (1988) and Pedrizzetti (1992) suggest that vortons may predict the flow field poorly *not* because they do not preserve the invariants of the flow, but because they are non-solenoidal! Numerical experiments by Winckelmans & Leonard (1993) appear to support this conclusion. Additionally, results indicate that vortons approximating a divergence-free vorticity field tend to preserve the invariants of the flow automatically. Therefore, maintaining the solenoidality of the vorticity field is key to the successful simulation of the flow using vortons. Pedrizzetti (1992) and Winckelmans & Leonard (1993) propose techniques for periodically reconstructing a divergence-free vorticity field during the simulation. Wu *et al.* (1995) have formulated a projection technique to enforce the solenoidality of the vorticity in bounded domains.

The velocity or magnetization or impulse method is a new formulation based on the canonical Hamiltonian for the incompressible Euler equations and is valid in all space dimensions. The Lagrangian velocity method shares the advantages of the Lagrangian vortex element methods. It also preserves the invariance of the impulse, angular momentum, helicity density and energy in the discrete form, identically. Furthermore, it satisfies continuity and maintains the solenoidality of vorticity, pointwise.

The velocity formulation lends itself naturally to the 3-D simulation and modeling of turbulent flow. Buttké & Chorin (1993) argue that the vortex monopole representation of the flow is not the optimal framework for applying statistical tools to the simulation of 3-D turbulent flows. This is because vortons are only approximately solenoidal and allow numerical errors to grow and change the asymptotic equilibria. Vortex filaments are solenoidal only globally and not pointwise. This leads to difficulties with vortex merging and renormalization theories (Chorin, 1993). In contrast, since impulse is conserved in the velocity formulation, merging becomes an “effortless” process of combining velocity elements (Buttké & Chorin, 1993). This is useful in removing small scale structures and reducing computational cost.

There are applications where the velocity method has been shown to be an excellent tool. Problems of immersed boundaries with surface forces, such as elastic membranes, can be modeled with the velocity variable since these represent dipoles and the velocity equations update the dipole strengths appropriately (Cortez, 1995; Recchioni & Russo, 1995). However, the velocity element method has been shown to be accurate for fixed, short times only. This is because for certain problems, the velocity stretches without bound and its support base expands out, leading to a sparse collection of velocity elements and eventual blow up of the predictions. Certain remedies have been proposed (Cortez, 1995); however, they do not appear to be practical for 3-D computations. The velocity method is a new technique and certainly time is needed to accumulate more experience about its potential advantages and disadvantages with

respect to the more established vortex method; and, more importantly, to propose remedies for improving some of the issues that have been touched upon in the previous sections.

The following is a brief review of available methods for the grid-free simulation of diffusion.

The random walk method is used extensively in Lagrangian vortex methods and there is substantial evidence in the literature about its advantages and handicaps. It conserves circulation exactly, but conserves the linear and angular moments only approximately. The random walk method is noisy implying that longer runs and larger samples may be necessary to average out the parasitic noise from the solution. The smoothed random walk method (Fogelson & Dillon, 1993) reduces the level of noise significantly; however, it still requires a large number of elements to achieve convergence. Therefore, although the random walk method is inexpensive per vortex element, the total simulation time (controlled by the Biot-Savart evaluations) may not be as competitive, considering the large number of elements that are required to obtain convergence.

The core spreading method was introduced in this report for the sake of completeness only. Clearly, it approximates the wrong equation and cannot be recommended. The corrected core spreading method (Rossi, 1996) is a novel extension of the latter and is designed to conserve the integral properties of the system. It offers the additional advantage of circumventing viscous splitting. Selected tests have indicated the accuracy of the method compared to known solutions and the random walk method. However, since it is fundamentally a core spreading method, its accuracy is retained only by frequent splitting of the vortex elements which causes the exponential growth of the number of elements.

The diffusion velocity method (Kempka & Strickland, 1993) is a relatively new concept with no theoretical proof that the solutions approximate the Navier-Stokes equations correctly. More importantly, the method is presently limited to 2-D flows only and the formulation has not been extended to 3-D domains.

The particle strength exchange (Degond & Mas-Gallic, 1989a) and core function differentiation (Fishelov, 1990b) methods are at present the most widely implemented deterministic methods in vortex element simulations. The advantages of these methods are that viscous splitting need not be used, and that the integral properties of the flow are conserved. Furthermore, the convergence and accuracy of both methods have been established theoretically as well as numerically, using a uniform distribution of elements. However, the accuracy of the predictions diminishes severely as vortex elements become disordered due to local fluid effects. The popular remedy is to frequently project the vortex elements onto a uniform grid, which introduces numerical diffusion into the computation. Furthermore, remeshing requires the development of sophisticated grid generation techniques in order to handle the complexities of the domain boundary. This defeats the purpose of using grid-free methods entirely! Lastly, both methods are incapable of accounting for the expansion of the vorticity field automatically. At present, the approach is to fill the boundaries of the expanding computational domain with layers of new elements (with zero circulation) and to compute the exchange of vorticity among

all elements including the newly added ones. Since there is no a priori information about the location and frequency with which the elements must be added, the computations usually carry along extra elements with zero strength, which adds unnecessary overhead to an already expensive computation.

The free-Lagrange method (Russo & Strain, 1994) uses the location of the vortex elements to construct Delaunay tetrahedra, which are then used to approximate the continuous Laplacian of the diffusion with a discrete Laplacian. The method is deterministic and has been shown to maintain longtime accuracy. It conserves circulation and the second moment, but no information is available on the conservation of the linear moment. The disadvantage of the method is that it has linear order accuracy at best, and the discrete Laplacian is only weakly consistent to linear order and not pointwise consistent. Additionally, the generation of the Delaunay tetrahedra for a large number of vortex elements may prove to be an expensive overhead.

The vorticity redistribution method (Shankar & Van Dommelen, 1996) may be thought of as a generalized grid-free finite difference technique. By construction, all integral properties of the flow are conserved. The convergence and accuracy of the method have been demonstrated analytically as well as numerically. The solution is pointwise consistent and accepts arbitrarily high order accuracy (although at a commensurate increase in cost.) The algorithm is adaptive in the sense that it adds elements when and where new elements are required, including the expanding front of the vorticity field. This approach does not use the element locations as quadrature points; therefore, it is not plagued by the degradation of accuracy caused by element disorder. Additionally, it does not use core functions; therefore, the method is capable of resolving scales as low as the inter-element spacing! This results into a respectable savings in the number of vortex elements that are necessary to resolve a particular length scale. Unfortunately, the cost of evaluating the diffusion equation is equal to the cost of evaluating the Biot-Savart integrals. However, if the total time of simulation is considered, the vorticity redistribution method may be quite competitive. Consider, for example, the test problem of impulsively started flow over a cylinder. For the same level of accuracy, the vorticity redistribution method used 60,000 vortex element, whereas the particle strength exchange mechanism used roughly 350,000 elements!

The following is a brief review of issues concerning the no-slip boundary condition.

The vortex sheet and Prandtl approximation models proposed by Chorin (1980) are the most widely used algorithms for imposing the no-slip boundary condition and for computing the vorticity dynamics near the boundary, respectively. This is primarily because of the simplicity by which the algorithm can be implemented, but also because up until recently there were no reliable grid-free alternatives. The Prandtl approximation is a cost-effective, yet accurate method for evaluating the evolution of vorticity near the boundary, even in the presence of separation, provided the solution from the boundary domain is matched properly with that of the outer domain (Anderson & Reider, 1994). Another advantage of this model is that it is grid-free. Its main disadvantages are (1) the use of infinitesimally thin sheets to discretize the

vorticity field in the numerical boundary layer, which is a poor approximation, and (2) the use of the random walk method to simulate diffusion, which is noisy and converges slowly.

The deterministic sheet method by Bernard & Thomas (1995) was developed to address the disadvantages of the previous method. It discretizes the vorticity field near the boundary region using bricks or sheets with finite thickness and a piece-wise constant vorticity variation, and it simulates diffusion deterministically using Fishelov's (1990b) method. As expected, very accurate results are obtained with this method. Unfortunately, because of the choice of the diffusion scheme and discontinuous vortex sheets, the boundary domain must be re-sheeted frequently to maintain solution accuracy and stability. Therefore, this approach is not a grid-free method in a strict sense. If one has to resort to some *indirect* form of gridding to maintain accuracy, then one may as well develop a grid-based algorithm near the boundary to fully utilize some of its potential advantages, such as computational efficiency.

Strickland *et al.* (1996) did just that, and developed a grid-based algorithm to minimize the number of free vortex elements that are injected into the fluid domain. Convection of vorticity is obtained by moving the grid points using the local velocity, and the diffusion is simulated using the diffusion velocity method. A high-order remapping scheme is used to reduce numerical diffusion. Preliminary tests on commonly experienced boundary layer problems verified the robustness of the scheme. It is worth noting here that the algorithm used to reduce the number of vortex elements injected to the field is not unique to grid-based methods, and a similar strategy can be implemented in grid-free methods (Gharakhani, 1993).

It must be emphasized that a straightforward extension of a 2-D algorithm to 3-D cannot be expected to produce accurate and/or stable results for long times, because of the fundamentally different (and more complicated) kinematics and dynamics of the 3-D flow. Special care must be exercised to ensure that the vorticity vector remains divergence-free. This implies that, at a minimum, the normal component of vorticity and the divergence of the vorticity flux must be zero at the wall (Casciola *et al.*, 1996; Wu *et al.*, 1995). Since the 3-D vortex blob is non-solenoidal, it is necessary to project the incorrect vorticity onto a new solenoidal vorticity to maintain the accuracy of the simulation (Wu *et al.*, 1995).

To this end, the velocity variable method was formulated by Summers & Chorin (1996) to ensure that the generated vorticity remains solenoidal near the boundary before it is injected into the fluid domain. The algorithm is logically identical to the vortex sheet method and is, therefore, easy to implement. Another attractive feature of the velocity variable is that it is not defined uniquely, which allows one to design algorithms that are optimized for particular types of problems. Summers & Chorin (1996) proposed two alternatives for satisfying the no-slip boundary condition; one which assigns the velocity vector at the wall *parallel* to the wall, and another *normal* to it. The proposed methods enforce the conservation of momentum and circulation, respectively, to generate vorticity at the wall. The capability of each method to simulate the boundary layer flow accurately needs to be determined by a parametric study.

The algorithms discussed thus far are well-suited for simulating topologically simple boundary layer problems, such as flow over a flat plate or a cylinder; however, their

implementation on the time-dependent deformable boundary of a collapsing parachute will be a challenging task, to say the least. The vorticity generation and propagation scheme implemented by Koumoutsakos *et al.* (1994) can in principle accommodate such a complex geometry. The algorithm applies the boundary element or panel method to solve the unsteady diffusion equation, and to diffuse the wall vorticity into a layer of vortex elements that are placed adjacent to the boundary in the fluid domain. Therefore, the method is grid-free and it does not rely on the Prandtl approximation near the boundary. The version implemented by Koumoutsakos *et al.* (1994) is based on the particle strength exchange mechanism which requires frequent regridding. Nevertheless, the idea can in principle be used with other methods which do not require regridding.

RECOMMENDATIONS

The simulation of the flow around the dynamically deforming geometry of parachutes at high Reynolds numbers is very challenging, irrespective of the particulars of the computational method used to perform the task. The selected approach has to be computationally flexible and efficient, yet sufficiently accurate to capture the essential characteristics of the high Reynolds number flow around a parachute, which involves a wide range of length scales attributed to the thickness, the span, and the wake behind the parachute.

The vortex element or vorton method appears to be the most flexible alternative for simulating such a flow. However, accurate and stable vorton simulations will be possible only if the solenoidality of the vorticity field is enforced throughout the computation. The projection method by Wu *et al.* (1995) for flow in bounded domains appears to be the proper tool for this. The methods by Winckelmans & Leonard (1993), Gharakhani (1993), and Cottet (1996) are potentially powerful schemes that must at least be experimented with for the special problem at hand. Additionally, in order to optimize the accuracy of the computations for long times, it may be useful to incorporate spectral core functions of the type proposed by Humi (1993), and/or develop a 3-D adaptive Biot-Savart integration technique similar to that proposed by Strain (1996) for 2-D.

The solution of the diffusion equation can most accurately be obtained by the vorticity redistribution method by Shankar & Van Dommelen (1996). Furthermore, effective sub-grid scale diffusion models will be necessary to dissipate hairpin vortices, which inevitably develop during the simulation of nearly inviscid flows, and thus to curtail the explosive growth of the number of vortex elements.

The selection of a scheme for the generation of vorticity, and its evolution toward the fluid domain, is more difficult because of the lack of conclusive evidence about the accuracy of the existing solution techniques in 3-D. Nevertheless, a velocity generation scheme similar to that proposed by Summers & Chorin (1996) may be the most appropriate approach for this problem. This is because the velocity preserves the invariants of the flow and the solenoidality of the vorticity field. Additionally, element merging is extremely simple with the velocity variable. This allows the use of a large number of elements to resolve the flow structure near the boundary.

A potentially successful algorithm for simulating the flow around the parachute is proposed here, which is based on a dynamic domain decomposition technique and hybrid velocity and vorticity methods. In this approach, the computational domain is decomposed into an interior domain, which is a box that contains the parachute, and a remaining exterior domain. The position and the size of the box changes according to those of the parachute. The flow is simulated using the velocity element method for the interior and the vortex element method for the exterior. Velocity elements that jump into the exterior are converted into vortex elements.

However, vortex elements that jump into the interior remain unchanged. The details of the algorithm, such as the optimum choice for the box or the best method to convert the velocity elements to vortex elements, will have to be developed in the future.

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