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Exact Solutions for Symmetric Deformations of Hollow Bodies of Ideal Fluids with Application to Inertial Stability

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Exact Solutions for Symmetric Deformations of Hollow Bodies of Ideal Fluids with Application to Inertial Stability

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ABSTRACT

We present exact solutions for the problems of radial motion of spherical and cylindrical bodies of an incompressible inviscid fluid, with an initial kinetic energy, and subjected to pressure on the outer (or equivalently, inner) boundary. These solutions have bearing on the inertial stability of ejecta in hypervelocity impact and of explosively compressed shells.

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1 Introduction

When a projectile traveling at high speed strikes a plate of solid material of comparable dimension, a cloud of debris consisting of both projectile and plate material is ejected from the far side. The evolution and state of this debris cloud is a question of significance for a number of applications. The steps required to model it are believed to depend on the intensity and conditions of impact, as well as the equation-of-state properties of the interacting materials, but in fact very little is known about this question.

It is observed that when the intensity of such an impact becomes sufficiently large relative to the strength of the materials involved, breakup of the ejected debris is delayed. Rather than undergoing immediate fragmentation at breakout, the ejecta will swell and balloon, exhibiting flow and plasticity to a degree which belies the limited ductility usually expected of the participating materials. This has been observed in high-speed photographs of lead projectiles impacting lead plates at speeds near one km/s [1] and is seen in the impact of higher strength materials at greater impact speeds. In the experiments on lead, the ejected debris is initially an expanding, thinning bubble of metal which achieves a diameter four to five times the initial dimension of the projectile before undergoing breakup through thinning instabilities in the bubble wall.

It is only recently that we have started to achieve a degree of understanding of the underlying physics responsible for the enhanced ductility exhibited by solids under certain impact-induced flow situations. It is not a change in material properties associated with the very high deformation rates as has been suggested in other related applications. Rather, it is a consequence of the state of motion of the body brought about by the impulsive loading conditions. [2] The state of motion is such that small perturbations or instabilities in the flow which might lead to breakup are opposed by inertial forces which counteract the growth of such instabilities. In short, *inertial stability* is responsible for the enhanced ductility and delayed breakup observed in a number of impulsive loading applications.

This is perhaps easiest to see by considering a plate of solid or liquid which is stretching biaxially in such a way that the thickness is decreasing with time, commensurate with incompressible flow. Such flow is prohibited by the motion from being stress free. Ignoring deviator stresses, it can be shown that such motion implies a compressive pressure within the body and a positive pressure gradient directed into the body from the stress-free surfaces. The magnitudes of the pressure and pressure gradient depends on the rate of deformation and the geometry of the body.

This inertial pressure stabilizes the flow in several ways: First, defects or perturbations on the body surface are prevented from growing by the pressure gradient at the same surface. Stated differently, such surfaces are Rayleigh-Taylor stable to sur-

face perturbations. Second, the inertial confining pressure prohibits breakup through internal cavitation.

The inertial pressure brought about by the dynamics of the event is a transient effect. As the body flows and thins, the magnitude of this pressure decreases. Eventually, the destabilizing forces associated with strength, capillarity or cavitation will become dominant and failure will ensue.

In this report we put aside the question of formal stability analysis and focus instead on the underlying motion. We perform this analysis to gain an understanding of the strength and behavior of the stabilizing inertial pressure. A spherical geometry is considered because it is a reasonable approximation to the initial behavior of ejecta produced in the projectile impact events discussed previously. In addition to the freely flowing body, we also consider the motion of a hollow sphere with arbitrary constant pressure applied to either the inner or outer surface. The problem in this form has broader applications for explosively loaded metal shells. We also analyze the motion of a hollow cylinder subjected to similar initial and boundary conditions.

It is important to emphasize that the solutions obtained in this report may or may not be stable solutions. They represent the symmetric solutions which would provide the baseline for a broader analysis of the stability of the flow to asymmetric perturbations. A treatment of the stability of flow has recently been performed for the cases of a plastic rod stretching along its axis. [3] Here, we consider the case which is simplest in terms of material models: an incompressible, inviscid fluid. This allows us to obtain exact, albeit somewhat complicated, solutions.

2 The Spherical Body

2.1 Kinematics

Assume a hollow spherical body as shown in Figure 1. Let the origin of a spherical polar co-ordinate system be at the center of the body, and let the body have an outside radius of b, and an inside radius of a. Let the body be composed of an incompressible material, so that it must undergo isochoric deformation only, with volume V. Then we have

$$V = \frac{4\pi}{3}\left(b^3 - a^3\right) . \tag{1}$$

The body has a reference state at which $a = 0$, $b = b_0$. Whether this reference state is realized in a particular situation of interest it not important. Thus,

$$V = \frac{4\pi b_0^3}{3} , \tag{2}$$

and thus

$$b_0^3 = b^3 - a^3 \ . \tag{3}$$

Since a and b may depend upon time, it follows that

$$a^2 \frac{da}{dt} = b^2 \frac{db}{dt} \ ,$$

and therefore

$$\frac{da}{db} = \frac{b^2}{a^2} \ . \tag{4}$$

Later is this paper, it will be convenient to work with the quantity

$$\beta = \frac{b}{a} \ . \tag{5}$$

we record the following consequences of this definition:

$$\frac{a}{b_0} = \frac{1}{(\beta^3 - 1)^{\frac{1}{3}}} \ , \quad \frac{b}{b_0} = \frac{\beta}{(\beta^3 - 1)^{\frac{1}{3}}} \ ,$$

$$\frac{d}{dt}\left(\frac{b}{b_0}\right) = \frac{-\frac{d\beta}{dt}}{(\beta^3 - 1)^{\frac{4}{3}}} \ ,$$

so that

$$\frac{1}{b}\frac{db}{d\beta} = -\frac{1}{\beta\,(\beta^3 - 1)} \ . \tag{6}$$

2.2 Dynamics – The General Case

Generally, the solution of a problem in the hydrodynamics of an ideal fluid requires that the pressure p and the velocity v be specified as functions of spatial position x and time t, such that the equations of conservation of mass and balance of momentum be satisfied. In this particular class of problems, a parameterization in which time t is replaced by either the outside radius b or the inside radius a is convenient, in that it allows for exact solutions of boundary-value problems. It transpires that $b(t)$ is governed by an ordinary differential equation which is sufficiently well-behaved so that it may be shown that $b(t)$ is piecewise monotonic, but not such that it may be solved to give the function $b(t)$ in a convenient analytic form. For the purpose of solution of the problem, we assume that $b(t)$ is invertible with inverse $t(b)$. We let

$$\dot{b}(t) \equiv \frac{db(t)}{dt} \ ,$$

and

$$\dot{b}(b) = \dot{b}(t(b)) \ .$$

Consider the velocity field*

$$v(r,b) = \frac{b^2}{r^2}\,\dot{b}(b)\ . \tag{7}$$

This field satisfies the equation of conservation of mass. For it to satisfy the equation of balance of momentum with no body force, there must be a pressure field $p(r,b)$ such that

$$\frac{\partial}{\partial r}\left(\frac{p}{\rho}+\frac{v^2}{2}\right) = -\,\frac{\partial v}{\partial t}\ , \tag{8}$$

where ρ is the density of the material. We have

$$\begin{aligned}\frac{\partial v}{\partial t} &= \frac{\partial v}{\partial b}\,\frac{db}{dt}\ ,\\ &= \frac{\partial v}{\partial b}\dot{b}\ .\end{aligned}$$

By Equation 7

$$\frac{\partial v}{\partial b} = \frac{1}{r^2}\left(b^2\frac{d\dot{b}}{db}+2b\dot{b}\right)\ ,$$

so that

$$\begin{aligned}\frac{\partial v}{\partial b}\,\dot{b} &= \frac{1}{r^2}\left(b^2\dot{b}\,\frac{d\dot{b}}{db}+2b\dot{b}^2\right)\ ,\\ &= \frac{1}{r^2}\left(\frac{b^2}{2}\frac{d}{db}(\dot{b}^2)+2b\dot{b}^2\right)\ .\end{aligned}$$

Thus, Equation 8 becomes

$$\frac{\partial}{\partial r}\left(\frac{p}{\rho}+\frac{b^4}{r^4}\,\frac{\dot{b}^2}{2}\right) = -\,\frac{1}{r^2}\left(\frac{b^2}{2}\,\frac{d}{db}(\dot{b}^2)+2b\dot{b}^2\right), \tag{9}$$

which may be integrated with respect to r, giving

$$\frac{p}{\rho}+\frac{b^4}{r^4}\,\frac{\dot{b}^2}{2} = \frac{1}{r}\left(\frac{b^2}{2}\,\frac{d}{db}(\dot{b}^2)+2b\dot{b}^2\right)+f(b)\ , \tag{10}$$

where $f(b)$ is to be determined from boundary conditions. We apply the boundary conditions

$$p(a,b) = 0\ , \tag{11}$$

$$p(b,b) = p_0\ . \tag{12}$$

By Equation 12, we have from Equation 10,

$$f(b) = \frac{p_0}{\rho}\ -\ \frac{1}{2}\left(b\frac{d}{db}(\dot{b}^2)+3\dot{b}^2\right)\ ,$$

*Rayleigh has considered the velocity-potential associated with this field. He considered only problems with infinite outer boundaries. These are considerably simpler analytically than the problems with two finite boundaries that we consider.

so that Equation 10 is

$$\frac{p}{\rho} = \frac{p_0}{\rho} + \frac{b}{2r}(b-r)\frac{d}{db}(\dot{b}^2) \; - \; \frac{(b-r)^2(3r^2+2br+b^2)}{2r^4}\dot{b}^2 \; . \tag{13}$$

Substituting Equation 11 into the above gives

$$a^3 b(b-a)\frac{d}{db}\left(\dot{b}^2\right) \; - \; (b-a)^2(b^2+2ab+3a^2)\dot{b}^2 = -2a^4\frac{p_0}{\rho} \; . \tag{14}$$

With $a(b)$ given by Equation 3, this becomes a nonlinear differential equation for $\dot{b}^2(b)$ which appears to be intractable. However, using Equation 5 reduces it to

$$(\beta-1)^2\left[\beta(\beta^2+\beta+1)\frac{d}{d\beta}(\dot{b}^2) + (\beta^2+2\beta+3)\dot{b}^2\right] = 2\frac{p_0}{\rho} \; , \tag{15}$$

which is easily solved, giving

$$\dot{b}^2 = \frac{(\beta^2+\beta+1)\left(c - \frac{2\Pi_0}{3(\beta^3-1)}\right)}{\beta^3} \; , \tag{16}$$

where $\Pi_0 = p_0/\rho$, and c is a constant of integration. Again using Equation 5 gives

$$\begin{aligned} \dot{b}^2 &= \frac{3c(b^3-a^3) - 2a^3\Pi_0}{3b^3\left(\frac{b}{a}-1\right)} \; , \\ &= \frac{3cb_0^3 - 2a^3\Pi_0}{3b^3\left(\frac{b}{a}-1\right)} \; . \end{aligned} \tag{17}$$

This completes the solution of the kinematics of this problem.

We note that the constant c appears to remain undetermined. That is not the case. The obvious specification, for time $= 0$, $\beta = \infty$, and the limiting case, time $= \infty$, $\beta = 1$ are unsuitable. However, for example, when $b = 2a$, so $\beta = 2$, we have

$$c = \frac{8\dot{b}^2 + \frac{2}{3}\Pi_0}{7} \; .$$

A specification like this simply fixes the kinetic energy of the body when it is in a particular configuration. Note that each of $\dot{b}(2a)$ and Π_0 may be positive, negative, or zero, so that c may have either sign or may vanish. For example, if $\dot{b}(2a) > 0$ and $\Pi_0 > 0$, so that when $b = 2a$ the body is expanding but with a pressure on its outer surface (or equivalently, a tension on its inner surface), then $c > 0$. Consider Equation 17. It implies that $\dot{b}$ will vanish, so that $b(t)$ will have an extremum, when

$$3c(b^3-a^3) - 2a^3\Pi_0 = 0 \; ,$$

or

$$\beta^3 = \frac{2\Pi_0}{3c} + 1 \ . \tag{18}$$

Thus, under these conditions, there will be a value of β, and thus one value of time, such that a body which is initially expanding will stop doing so, and begin to contract. A similar argument holds when the body is contracting at a particular instant, but is pressurized on the inside (or equivalently, subjected to a tension on the outside). Here, of course, there is the possibility that the inside radius a will go to zero before Equation 18 is satisfied. We note that if at any instant the outside radius of the body is moving outward, and if there is a pressure on the inside; or if the outside radius is moving inward, and there is a pressure on the outside, no such extremum occurs.

To solve for the pressure field, substitute Equation 17 into Equation 10. Use the notation

$$\Pi = \frac{p}{\rho} \ . \tag{19}$$

The result is

$$\Pi = \frac{A+B}{C} \ . \tag{20}$$

where

$$A = 2\Pi_0 a^2(r-a)(3b^3r^3 + ab^2r^3 + a^2br^3 + a^3r^3 - ab^3r^2 - a^2b^3r - a^3b^3) \ ,$$

$$B = c(b-a)(b^2+ab+a^2)(r-a)(b-r)(b^2r^2 + abr^2 + a^2r^2 + ab^2r + a^2br + a^2b^2) \ ,$$

and

$$C = 6a^2b^2(b-a)r^4 \ .$$

or, with

$$\xi = \frac{r}{b} \ , \tag{21}$$

$$\Pi = \frac{D+E}{F} \ , \tag{22}$$

where

$$D = 2\Pi_0(\beta\xi - 1)(3\beta^3\xi^3 + \beta^2\xi^3 + \beta\xi^3 - \beta^2\xi^2 - \beta\xi - 1) \ ,$$

$$E = 3c(\beta-1)(\beta^2+\beta+1)(1-\xi)(\beta\xi-1)(\beta^2\xi^2 + \beta\xi^2 + \xi^2 + \beta\xi + \xi + 1) \ ,$$

and

$$F = 6\beta^3\xi^4(\beta-1) \ .$$

In general, we have no assurance that the pressure will have an extremum inside (a, b). If it does, it is found by differentiating Equation 20 with respect to r, or equivalently by differentiating Equation 22 with respect to ξ, and setting the derivative equal to zero. The result is a cubic with one real root, having the value*

$$r_{max}^3 = \frac{8\Pi_0 a^6 b^3 - 12ca^3b^3(b^3-a^3)}{2\Pi_0 a^3(4b^3 + ab^2 + a^2b + a^3) - 3c(b^4-a^4)(b^2+ab+a^2)} \ , \tag{23}$$

*The notations r_{max} and ξ_{max} imply that an extremum, which may be a maximum, *may* exist, but not that it *does* exist.

or

$$\xi^3_{max} = \frac{8\Pi_0 - 12c(\beta^3 - 1)}{2\Pi_0(\beta^3 + \beta^2 + \beta + 1) - 3c(\beta^4 - 1)(\beta^2 + \beta + 1)} \,. \tag{24}$$

If extrema of pressure exist, they may be found by substituting Equation 23 into Equation 20, or the equivalent equations for the dimensionless forms. The results are rather too complicated to be written down here explicitly, but they have the form

$$\Pi_{max} = \frac{(A_1\Pi_0 + A_2c)(A_3\Pi_0 + A_4c)^{\frac{1}{3}} + (A_5\Pi_0 + A_6c)(A_7\Pi_0 + A_8c)^{\frac{1}{3}}}{A_9(A_{10}\Pi_0 + A_{11}c)^{\frac{1}{3}}} \,, \tag{25}$$

where the $A_i, i = 1, ..., 11$, are polynomial functions of β.

2.3 The Case of No External Pressure

The situation where $p_0 = 0$ is of special interest. Here, the appropriate results may be read off from those in the preceding section. In particular, we have

$$\dot{b}^2 = \frac{cb_0^3}{b^3\left(\frac{b}{a} - 1\right)} \,, \tag{26}$$

Thus $\dot{b}$ never changes sign; the body always either expands or contracts.* For the pressure, we have

$$\Pi = \frac{c(r-a)(b-r)(b^2r^2 + abr^2 + a^2r^2 + ab^2r + a^2br + a^2b^2)}{2a^2b^2r^4} \,, \tag{27}$$

or

$$\Pi = \frac{c(\beta^2 + \beta + 1)(1 - \xi)(\beta\xi - 1)(\beta^2\xi^2 + \beta\xi^2 + \xi^2 + \beta\xi + \xi + 1)}{2\beta^3\xi^4} \,. \tag{28}$$

Furthermore,

$$\xi^3_{max} = \frac{4}{\beta^3 + \beta^2 + \beta + 1} \,. \tag{29}$$

It is intuitively obvious that ξ^3_{max} is always within the spherical shell. To prove this, we need only note that $\beta > 1$. Then $\beta^3 + \beta^2 + \beta + 1 > 4$, so

$$\frac{4}{\beta^3 + \beta^2 + \beta + 1} < 1 \,.$$

Moreover, since $\beta > 1$, we have $\beta^3 > 1$, and $\beta^2 > \beta$, so $\beta^3 > \beta^2 > \beta$. Thus, $4\beta^3 > \beta^3 + \beta^2 + \beta + 1$, so

$$\frac{1}{\beta^3} < \frac{4}{\beta^3 + \beta^2 + \beta + 1} \,,$$

*An alternative derivation of Equation 26 is given in Appendix A.

and

$$\frac{1}{\beta} < \frac{4^{\frac{1}{3}}}{(\beta^3+\beta^2+\beta+1)^{\frac{1}{3}}} < 1, \tag{30}$$

which is the required result. The value of the nondimensional pressure $\bar{\Pi} = \Pi/c$ at the point ξ_{max} is

$$\bar{\Pi}_{max} = \frac{(\beta^2+\beta+1)\left[3(\beta+1)^{\frac{4}{3}}(\beta^2+1)^{\frac{4}{3}} - 4^{\frac{4}{3}}\beta(\beta^2+\beta+1)\right]}{2\,4^{\frac{4}{3}}\beta^3}. \tag{31}$$

It is possible to prove that $\bar{\Pi}_{max} > 0$ for all values of β.

It is illustrative to plot the results of these computations. Equation 26 is not integrable for $b(t)$ in terms of elementary functions, however, numerical integration of that equation is elementary. Here note by Equation 3 that a depends upon b and b_0, thus Equation 26 appears to depend upon an initial condition, say $b(0)$, and the two parameters b_0 and c. However, the substitutions

$$\eta = \frac{b}{b_0}, \quad \tau = \sqrt{c}\frac{t}{b_0}, \tag{32}$$

give

$$\left(\frac{d\eta}{d\tau}\right)^2 = \frac{(\eta^3-1)^{\frac{1}{3}}}{\eta^3\left[\eta - (\eta^3-1)^{\frac{1}{3}}\right]}, \tag{33}$$

a single equation with no parameters, so the only dependence is upon $\eta(0)$. Note that by Equation 32 only $\eta(0) \geq 1$ is allowed. Furthermore $\eta(0) = 1$ gives $\eta(\tau) = 1$ for all τ, so that only $\eta(0) > 1$ need be investigated. Moreover, the right-hand side of Equation 33 is strictly positive, so $\eta(t)$ is a strictly monotonic function. Graphs of that function for $\eta(0) = 1.01, 1.1, 1.2$, and 1.5 are given in Figure 2. Note that the function $\eta(\cdot)$ is almost linear, so that essentially, b may be treated as if it were proportional to time.

For the balance of the relations, it is helpful to understand the purely kinematic result implied by Equation 3. In dimensionless form, this is

$$\beta^3 = \frac{\eta^3}{\eta^3-1}. \tag{34}$$

This relation is plotted in Figure 3. Here, β is a decreasing function of η, but the decrease is highly nonlinear. Here note that b_0 is a constant, so η is a linear function of b. Furthermore, b is essentially a linear function of t, so essentially, η is a linear function of t. The quantity β may be interpreted as the "thickness ratio" of the shell, with $\beta \to \infty$ representing the thickest possible shell, while $\beta \to 1$ represents a very thin shell. Here we see that because of the incompressibility of the material, β initially decreases very quickly and almost linearly with time. In the region $1.2 \lesssim \beta \lesssim 1.5$, there is a nonlinear transition to a very slow linear decrease of β with time.

We consider the pressure distribution given by Equation 28. Again with

$$\bar{\Pi} = \frac{\Pi}{c} , \tag{35}$$

Equation 28 becomes

$$\bar{\Pi} = \frac{(\beta^2 + \beta + 1)(1 - \xi)(\beta\xi - 1)(\beta^2\xi^2 + \beta\xi^2 + \xi^2 + \beta\xi + \xi + 1)}{2\beta^3\xi^4} . \tag{36}$$

In terms of representative phenomena, we show plots of $\bar{\Pi} \sim \xi$ for $\beta = 1.05, 1.1, 1.3, 1.5, 2.0, 3.0$ and 5.0 in Figures 4–6. We also plot $\xi_{max} \sim \beta$ (Equation 29) in Figure 7 and $\bar{\Pi}_{max} \sim \beta$ in Figure 8, where

$$\bar{\Pi}_{max} = \frac{\Pi_{max}}{c} . \tag{37}$$

Here, of course, by the boundary conditions, the pressure vanishes at $\xi = 1/\beta$ (the inside surface) and at $\xi = 1$ (the outside surface). The quantity of interest is the pressure gradient, which for small times is very steep near the inside surface, so that the pressure is a maximum closer to the inside surface than to the outside surface. After that, the pressure decreases slowly to its boundary value at the outside surface. With the evolution of time, the pressure gradient profile becomes more symmetric, with the location of the maximum pressure moving asymptotically toward the midpoint of the shell. We note from Figure 2, and as intuitively obvious, that β decreases monotonically to 1 as time increases. In Figure 7, we see that for large β (and thus small time) ξ_{max} is close to the inside edge of the shell, while as $\beta \to 1$, it approaches the center. The maximum pressure $\bar{\Pi}_{max}$ as given by Equation 37 is shown in Figure 8 for values of β between 10.0 and 1.0. Here, the high pressure at small times is seen, and the precipitous drop with time is illustrated.

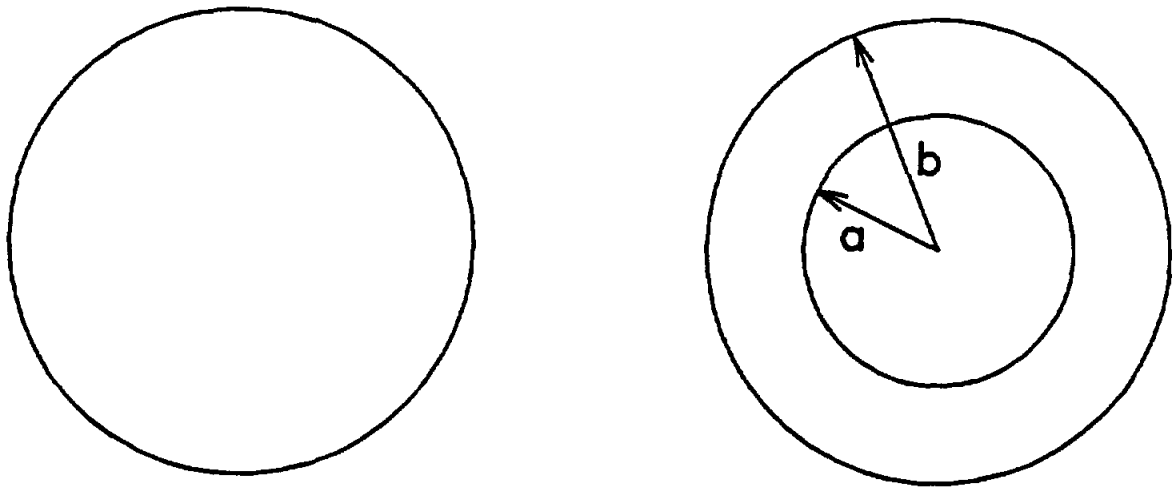

Figure 1: Geometry and Loading of the Spherical Body.

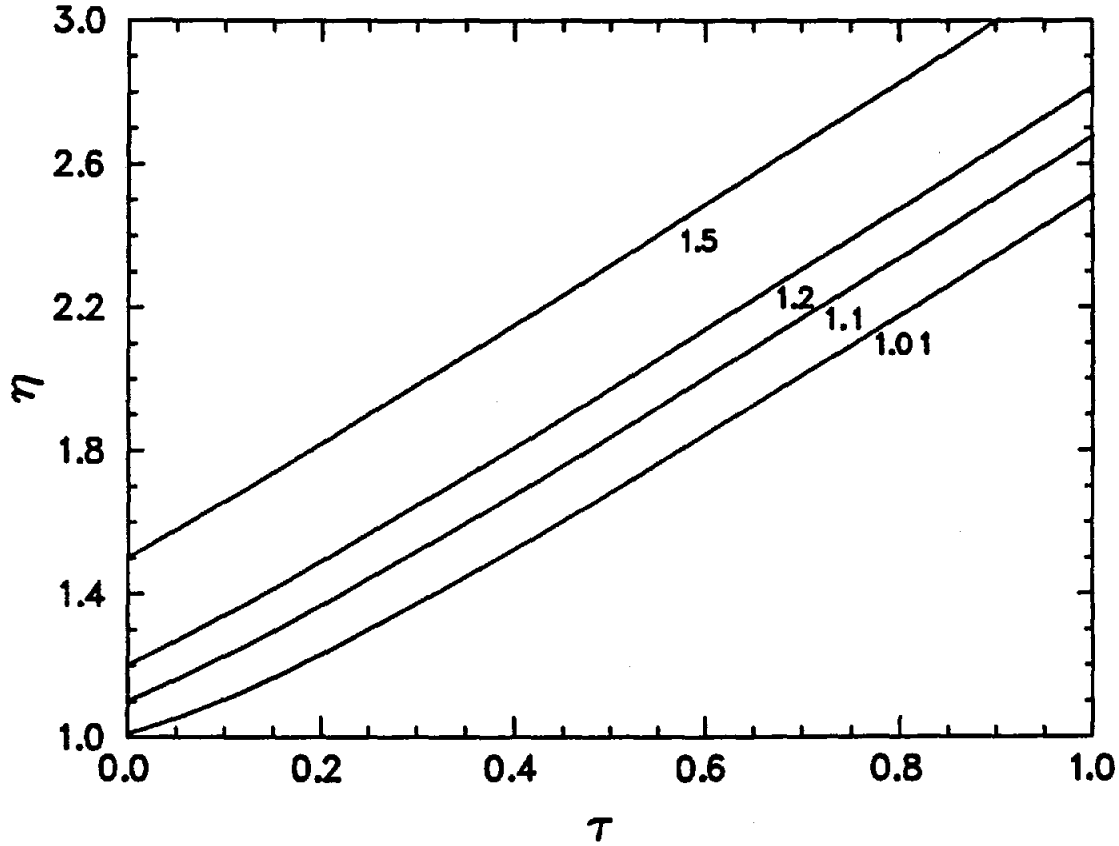

Figure 2: Growth of the Nondimensional Outside Radius η of the Sphere with Nondimensional Time τ for $\eta(0) = 1.01, 1.1, 1.2, 1.5$.

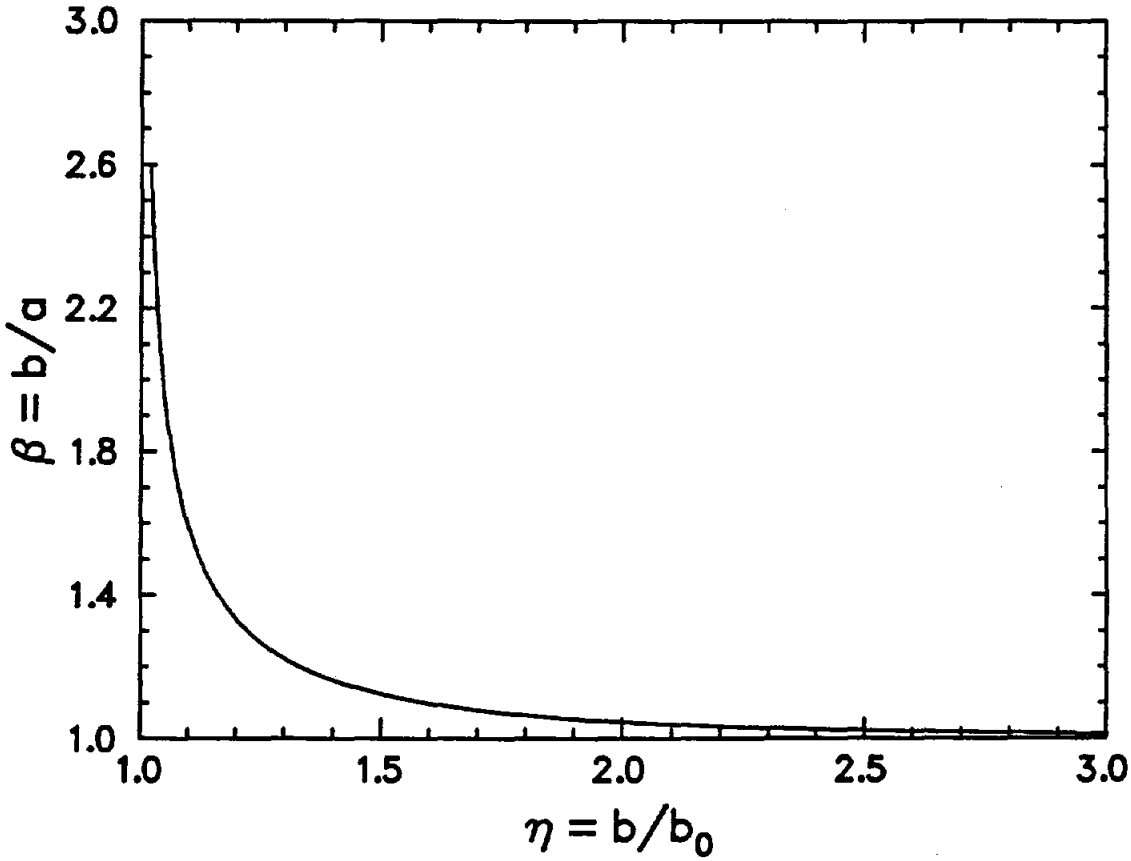

Figure 3: Change of Thickness with Growth of Radius for the Sphere.

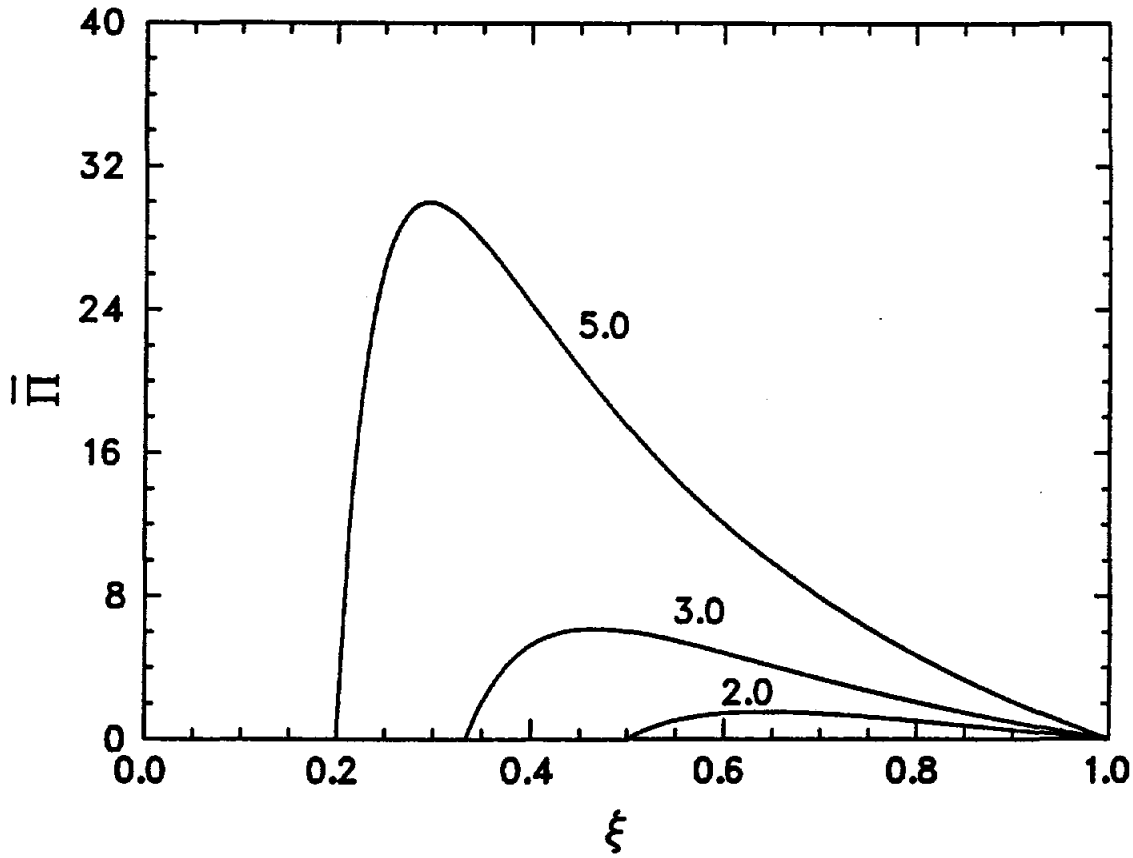

Figure 4: Pressure Distribution in the Spherical Shell for $\beta = 5.00, 3.00, 2.00$.

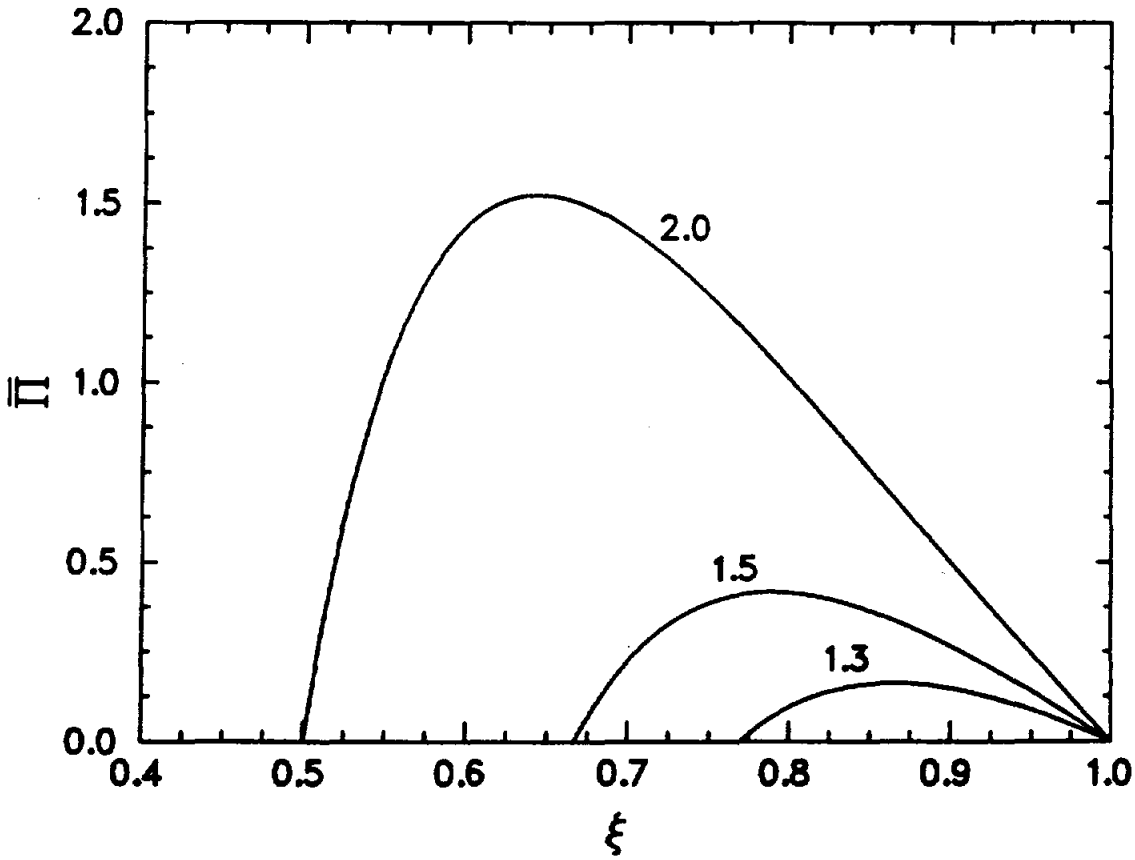

Figure 5: Pressure Distribution in the Spherical Shell for $\beta = 2.00, 1.50, 1.30$.

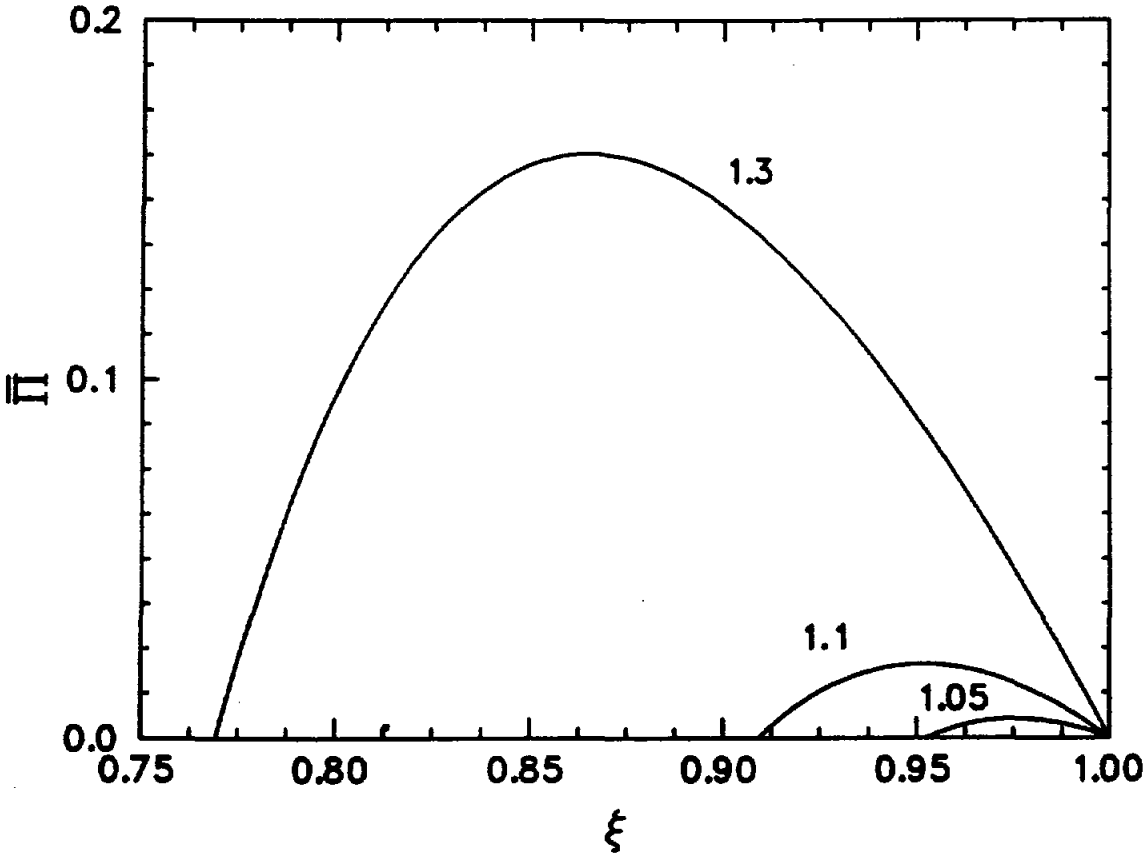

Figure 6: Pressure Distribution in the Spherical Shell for $\beta = 1.30, 1.10, 1.05$.

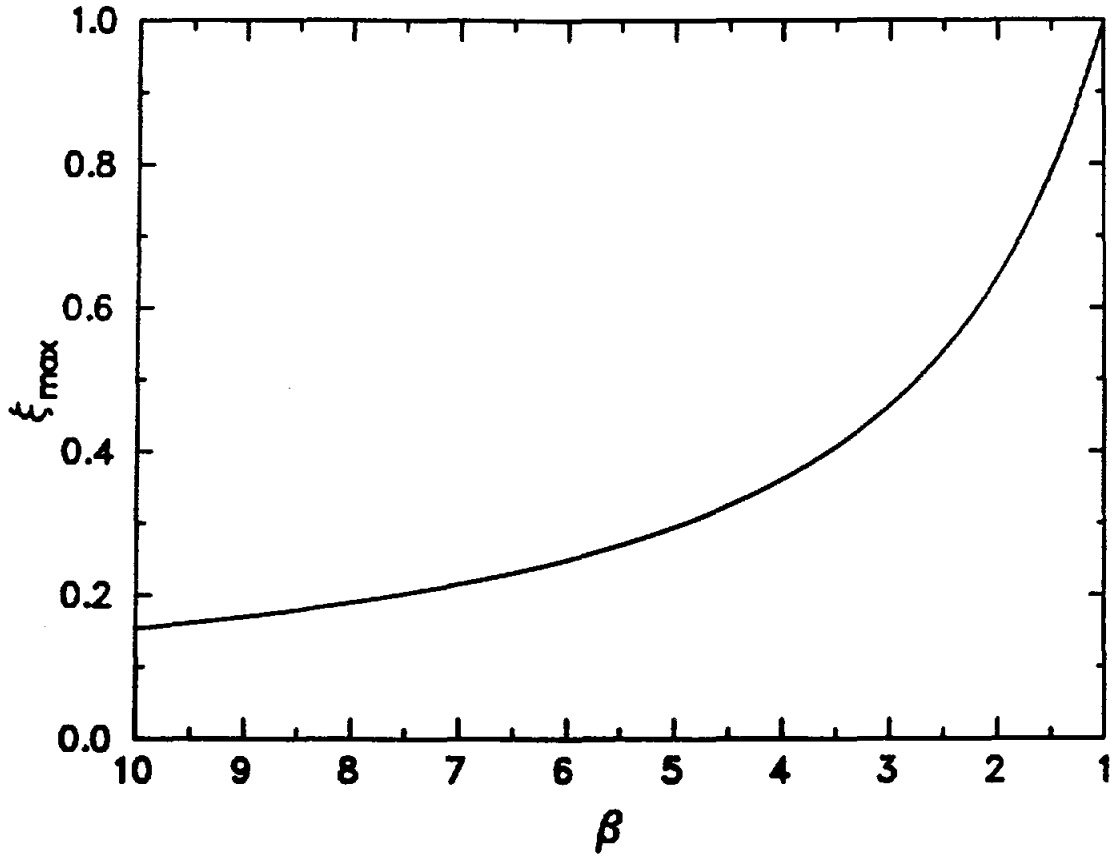

Figure 7: Location of the Maximum Pressure in the Spherical Shell as a Function of β.

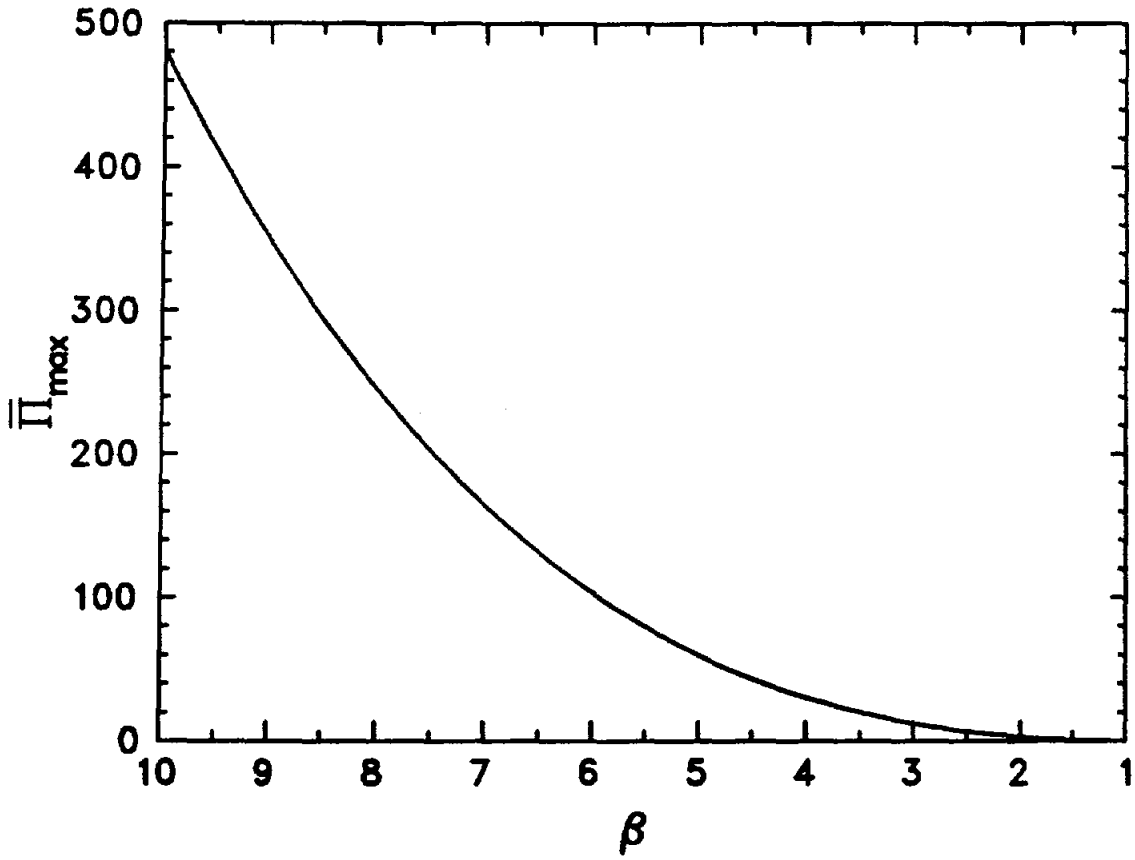

Figure 8: Evolution of the Maximum Pressure in the Spherical Shell as a Function of β.

3 The Cylindrical Body

The arguments for the cylindrical body are much like that for the spherical body, thus there is no particular purpose in repeating either the computations or the comments. Here, we simply present the equations corresponding to Section 1 for the cylindrical solution. We work in plane polar co-ordinates and consider the body as infinite in extent, but consider only a section of unit length.

3.1 Kinematics

Here, the volume of the body is

$$V = \pi(b^2 - a^2) \ , \tag{38}$$

so

$$b_0^2 = b^2 - a^2 \ , \tag{39}$$

and thus

$$\frac{da}{db} = \frac{b}{a} \ . \tag{40}$$

Again, letting

$$\beta = \frac{b}{a} \ , \quad \eta = \frac{b}{b_0} \ , \tag{41}$$

we have

$$\frac{\eta}{\beta} = \frac{a}{b_0} = \frac{1}{(\beta^2 - 1)^{\frac{1}{2}}} \ , \quad \eta = \frac{b}{b_0} = \frac{\beta}{(\beta^2 - 1)^{\frac{1}{2}}} \ , \tag{42}$$

and

$$\frac{1}{b}\frac{db}{d\beta} = -\frac{1}{\beta(\beta^2 - 1)} \ . \tag{43}$$

3.2 Dynamics – The General Case

Here, we let r be a plane polar co-ordinate, and consider

$$v(r, b) = \frac{b}{r}\,\dot{b}(b) \ . \tag{44}$$

The dynamical equation is

$$\frac{\partial}{\partial r}\left(\frac{p}{\rho} + \frac{v^2}{2}\right) = -\frac{\partial v}{\partial t} \ , \tag{45}$$

$$= -\frac{\dot{b}}{r}\frac{d}{db}(b\dot{b}) \ . \tag{46}$$

We integrate this equation and apply the boundary conditions

$$p(a,b) = 0 \ , \tag{47}$$

$$p(b,b) = p_0 \ . \tag{48}$$

The appropriate pressure field is

$$\Pi = \frac{c\left[\beta^2\xi^2 \ln\left(\beta\xi^2\right)\right] + \Pi_0 \ln\beta}{2\beta^2\xi^2(\ln\beta)^2} \ , \tag{49}$$

or

$$\Pi = \frac{cr^2 \ln\left(\frac{r^2}{ab}\right) + \Pi_0 a^2 \ln\left(\frac{b}{a}\right)}{2r^2 \ln\left(\frac{b}{a}\right)} \ , \tag{50}$$

while

$$\dot{b}^2 = \frac{(\beta^2-1)\left(c - \frac{\Pi_0}{\beta^2-1}\right)}{\beta^2 \ln\beta} \ , \tag{51}$$

or

$$\dot{b}^2 = \frac{cb_0^2 - a^2\Pi_0}{b^2 \ln\left(\frac{b}{a}\right)} \ , \tag{52}$$

with, as before, $\Pi = p/\rho$, $\Pi_0 = p_0/\rho$. Also as before, there exists the possibility that a body which is initially expanding, reaches an extremum and then contracts due to the action of an external pressure. Likewise, a body which is initially contracting may stop and then expand due to an internal pressure.

Extrema in the pressure, if they exist, are found in the usual way. The resulting equation is a quadratic with one negative root, and a positive root of

$$\xi^2_{max} = \frac{2\ln\beta\left[\Pi_0 + c(1-\beta^2)\right]}{\Pi_0\left(2\beta^2\ln\beta + \beta^2 - 1\right) - c\left(\beta^2-1\right)^2} \ . \tag{53}$$

Again, the expression for the maximum pressure is complicated.

3.3 The Case of No External Pressure

Here we have

$$\dot{b}^2 = \frac{cb_0^2}{b^2 \ln\left(\frac{b}{a}\right)} \ , \tag{54}$$

or

$$\dot{b}^2 = \frac{c\left(\beta^2-1\right)}{\beta^2 \ln\beta} \ , \tag{55}$$

so that $\dot{b}$ never changes sign. The pressure is

$$\Pi = \frac{c \ln (\beta \xi^2)}{2 (\ln \beta)^2} , \tag{56}$$

or

$$\Pi = \frac{c \ln \left(\frac{r^2}{ab}\right)}{2 \ln \left(\frac{b}{a}\right)} , \tag{57}$$

with

$$\xi^2_{max} = \frac{2 \ln \beta}{\beta^2 - 1} , \tag{58}$$

or

$$\xi^2_{max} = 2 \left(\frac{a}{b_0}\right)^2 \ln \left(\frac{b}{a}\right) . \tag{59}$$

Here also, it is possible to prove that $1/\beta < \xi_{max} < 1$. The general expression for the pressure is complicated, so we do not write it out.

Again, for the case of no external pressure, it is possible to solve the equation for the outer radius of the body as a function of time numerically. With Equations 32, Equation 54 becomes

$$\left(\frac{d\eta}{d\tau}\right)^2 = \frac{1}{\eta^2 \ln \left(\frac{\eta}{\sqrt{\eta^2 - 1}}\right)} . \tag{60}$$

Graphs of the solution for $\eta(0) = 1.01$, 1.1, 1.2, and 1.5 are given in Figure 9. Again, they are monotone, and nearly linear.

The kinematical relation for a cylinder which corresponds to Equation 34 for a sphere, is

$$\beta^2 = \frac{\eta^2}{\eta^2 - 1} , \tag{61}$$

and is plotted in Figure 10. For the pressure distribution, we have

$$\bar{\Pi} = \frac{\ln (\beta \xi^2)}{2(\ln \beta)^2} , \tag{62}$$

which is plotted for $\beta = 1.05$, 1.1, 1.3, 1.5, 2.0, 3.0 and 5.0 in Figures 11 – 13. There are two points to note here. First, the vertical scales are different than for the corresponding figures for the spherical shell; that is because the nondimensional pressure is less in this case than in the former. Second, there is still the significant pressure asymmetry at early times, with a large pressure gradient near the inside boundary, a maximum in pressure nearer the inside than the outside, and a relatively gentle gradient to zero pressure at the inside boundary. This asymmetry decreases asymptotically with time. The magnitude of the maximum pressure is shown in Figure 14, and the location ξ_{max}

of that maximum pressure is shown in Figure 15. It is interesting to note that, although the analytical expressions for these last two quantities are radically different for the sphere and the cylinder, their values are quite similar.

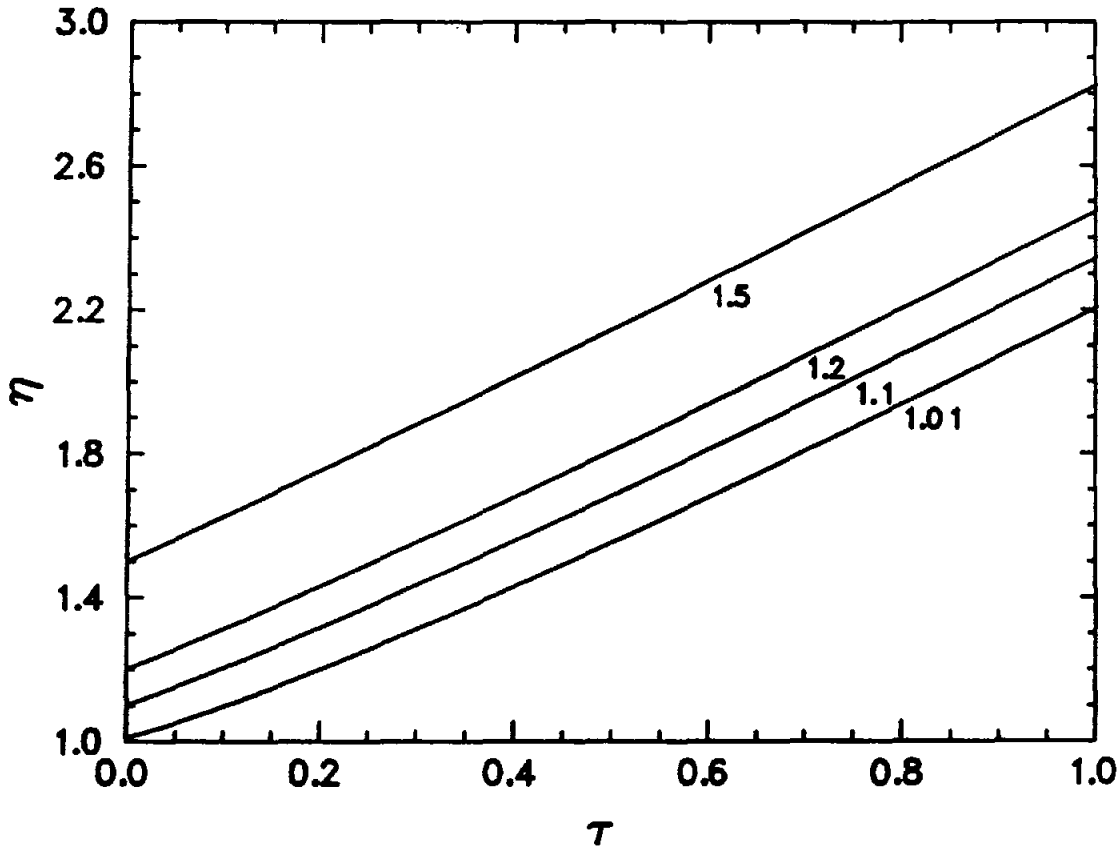

Figure 9: Growth of the Nondimensional Outside Radius η of the Cylinder with Nondimensional Time τ for $\eta(0) = 1.01, 1.1, 1.2, 1.5$.

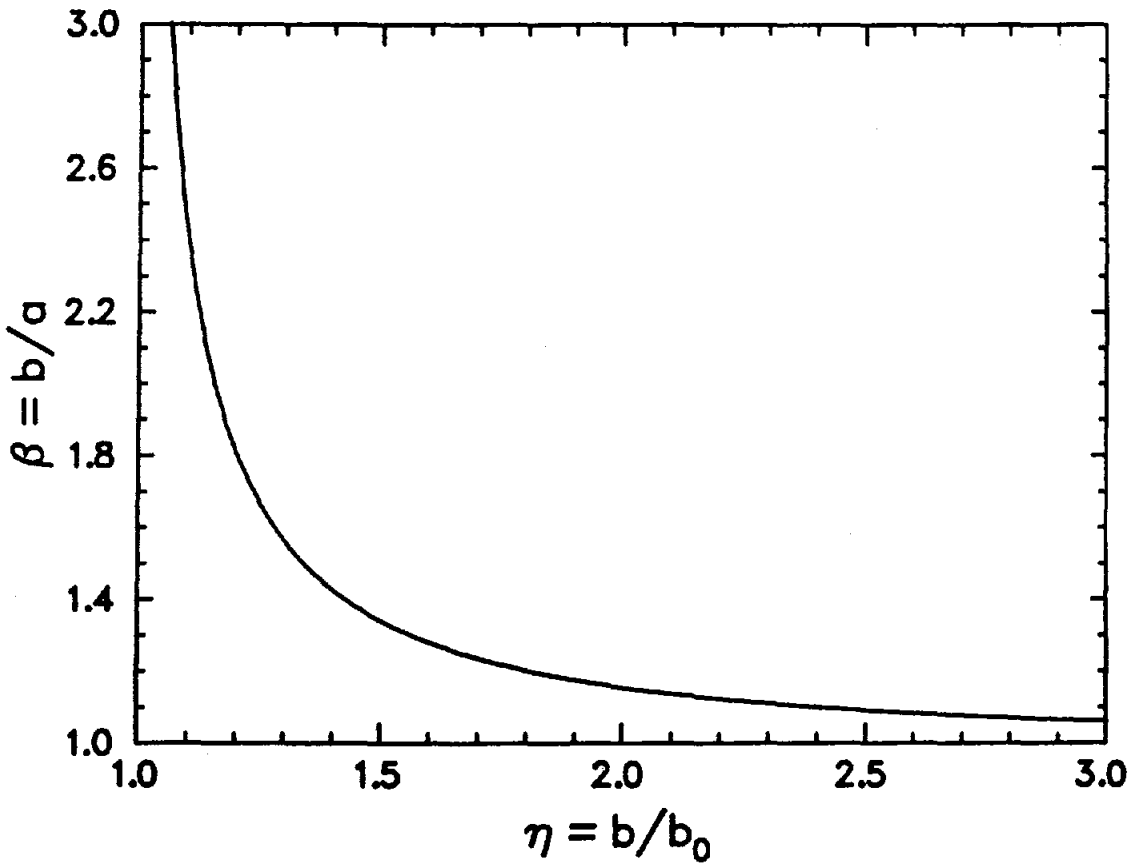

Figure 10: Change of Thickness with Growth of Radius for the Cylinder.

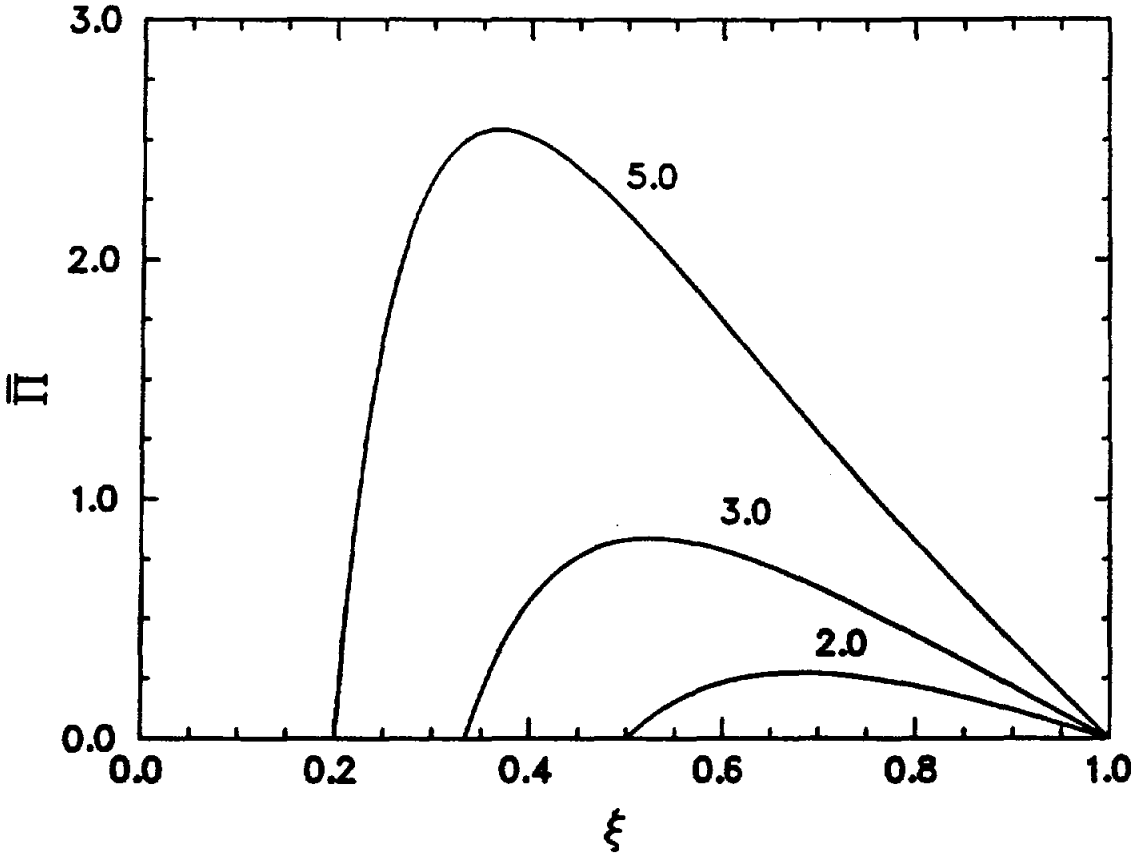

Figure 11: Pressure Distribution in the Cylindrical Shell for $\beta = 5.00, 3.00, 2.00$.

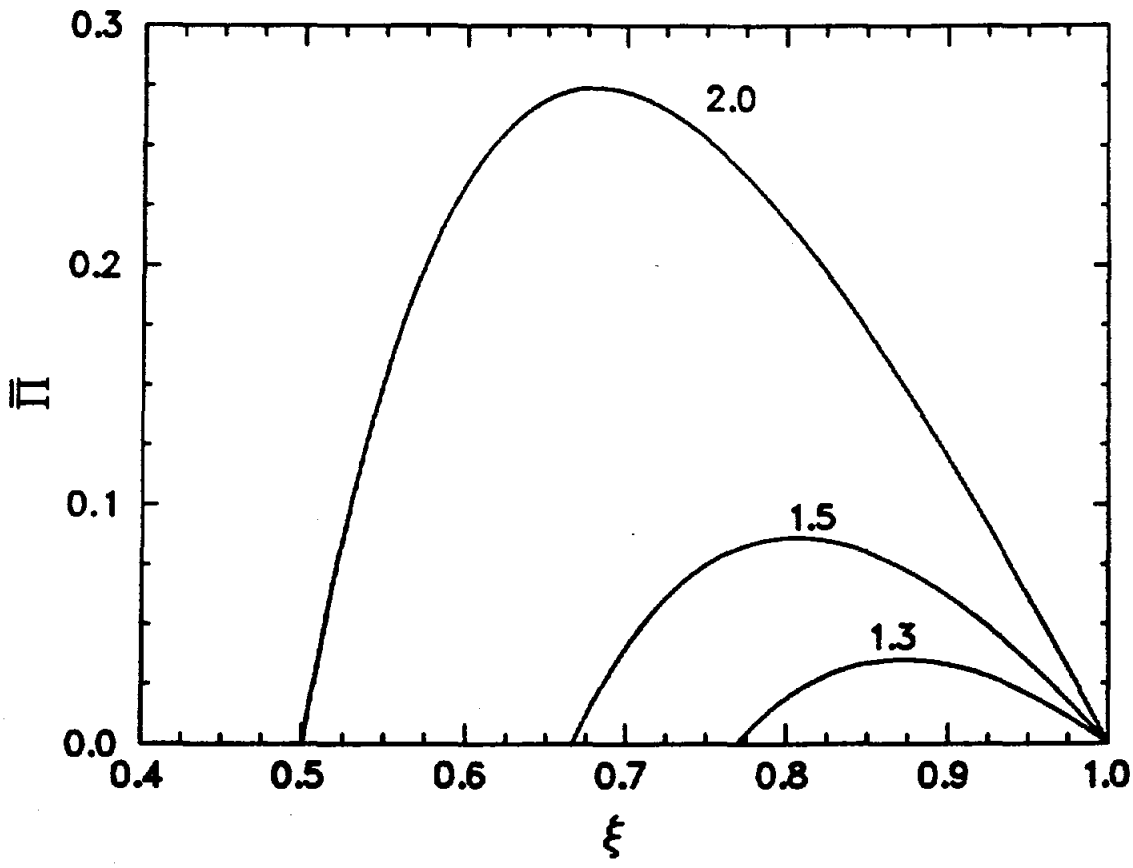

Figure 12: Pressure Distribution in the Cylindrical Shell for β = 2.00, 1.50, 1.30.

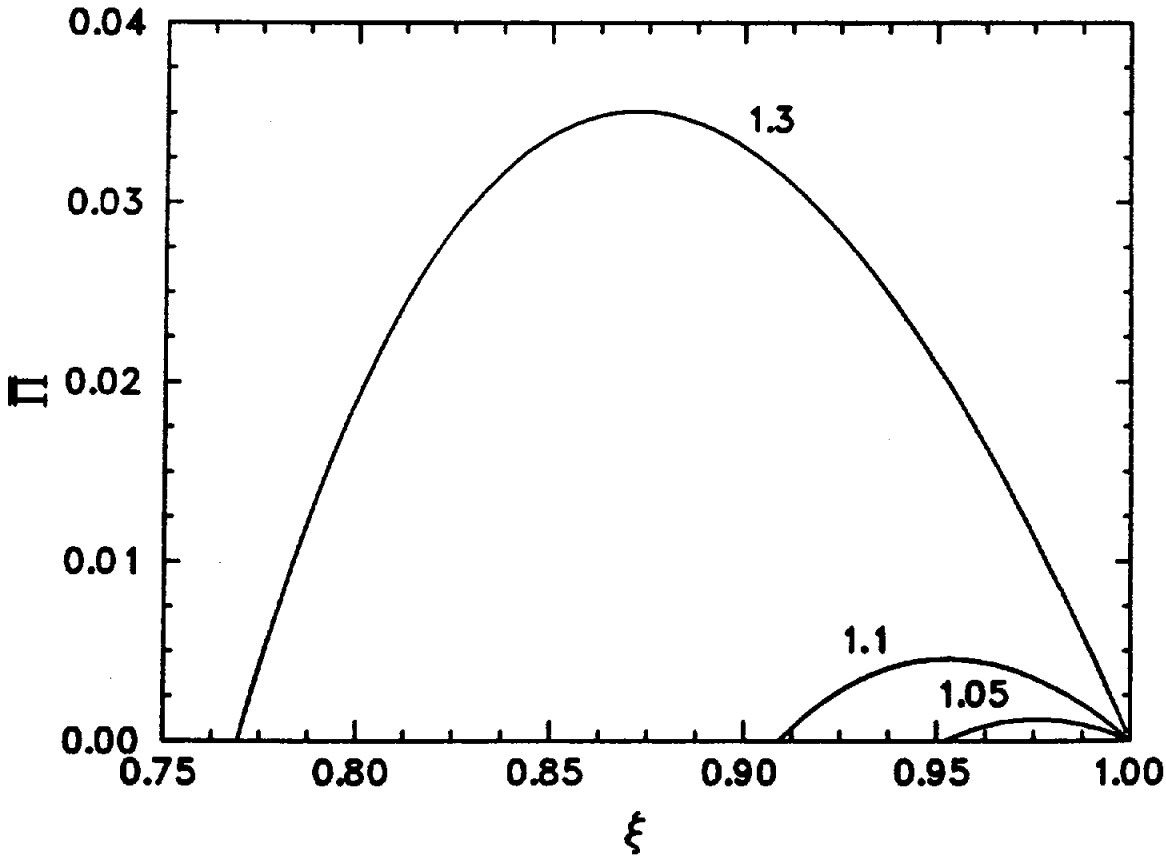

Figure 13: Pressure Distribution in the Cylindrical Shell for $\beta = 1.30, 1.10, 1.05$.

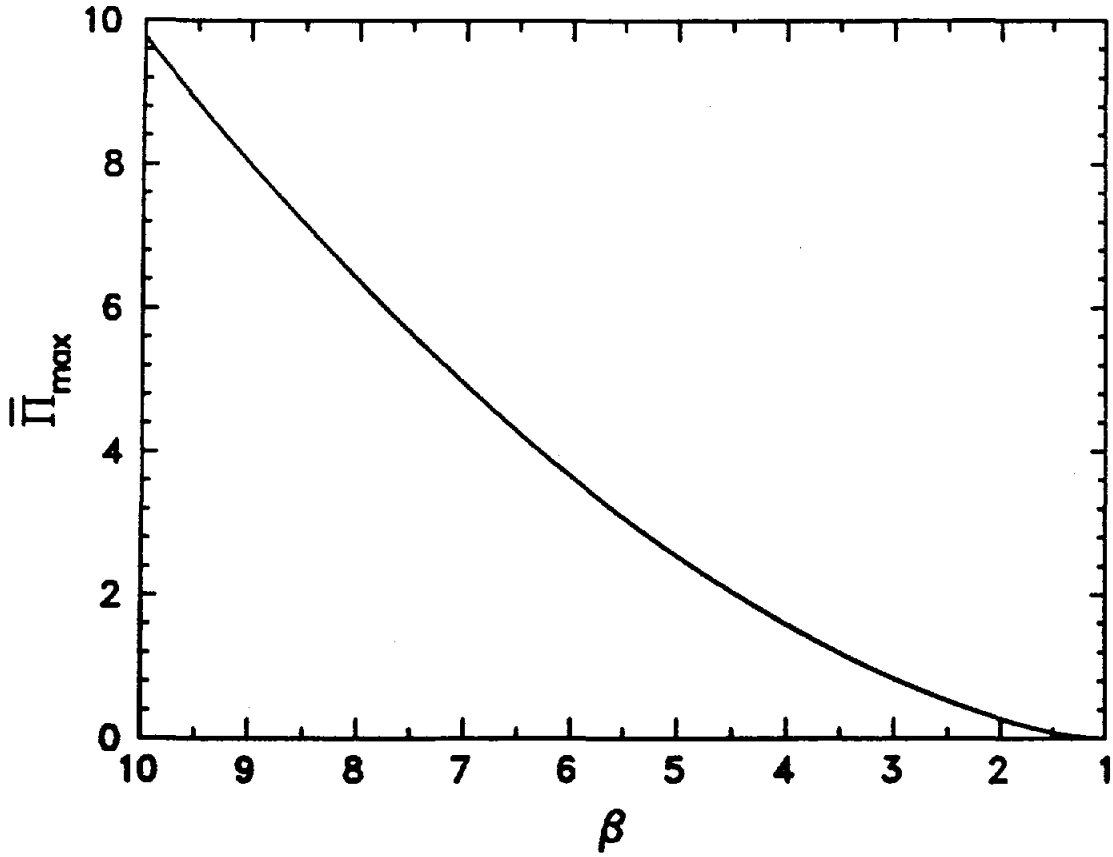

Figure 14: Evolution of the Maximum Pressure in the Cylindrical Shell as a Function of β.

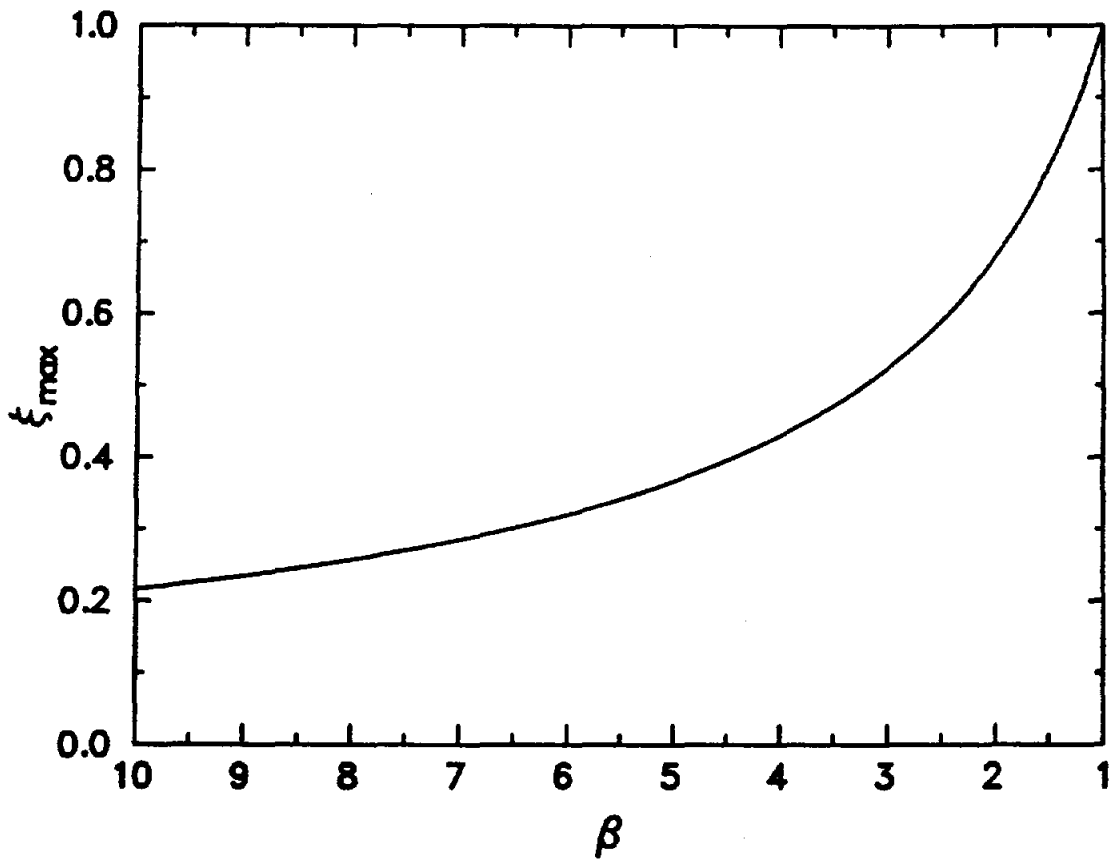

Figure 15: Location of the Maximum Pressure in the Spherical Shell as a Function of β.

4 Incompressible Elastic Cylinders and Spheres

Though the results in the previous sections hold only for incompressible fluids with no viscosity, they may be generalized to incompressible finitely elastic materials*. Both of the motions considered are circulation-preserving in the sense that, given the acceleration vector $\dot{\mathbf{v}}$, that vector satisfies

$$\operatorname{curl} \dot{\mathbf{v}} = 0 \ . \tag{63}$$

The deformation field of a body is given by $\mathbf{x} = \hat{\mathbf{x}}(\mathbf{X}, t)$, where $\mathbf{X}$ are reference ("Lagrangian") co-ordinates, and $\mathbf{x}$ are spatial ("Eulerian") co-ordinates. Then the deformation gradient is given by $\mathbf{F} = \partial \mathbf{x}/\partial \mathbf{X}$. A material is said to be incompressible and elastic if the Cauchy stress $\mathbf{T}$ is given by

$$\mathbf{T} = -\operatorname{grad} p\mathbf{1} + \mathcal{G}(\mathbf{F}) \ . \tag{64}$$

Assuming no body force, then, the appropriate dynamical equation is

$$-\operatorname{grad} p + \operatorname{div} \mathcal{G}(\mathbf{F}) = \rho \dot{\mathbf{v}} \ . \tag{65}$$

Temporarily consider t not to be the time, but rather a parameter of the motion $\hat{\mathbf{x}}(\mathbf{X}, t)$. The motion $\hat{\mathbf{x}}$ is quasi-equilibrated if each of the deformations $\hat{\mathbf{x}}(\mathbf{X}, t)$ satisfies the equations of equilibrium for the body. Then there must be a pressure field $p_0 = p_0(\mathbf{x}, t)$ such that for each value of t

$$-\operatorname{grad} p_0 + \operatorname{div} \mathcal{G}(\mathbf{F}) = 0 \ .$$

Thus it follows that

$$\operatorname{grad}(p_0 - p) = \rho \dot{\mathbf{v}} \ ,$$

or

$$\dot{\mathbf{v}} = -\operatorname{grad} \varsigma, \quad \rho\varsigma = p - p_0 \ . \tag{66}$$

Equation 66 is equivalent to Equation 63.

The following result is true under the conditions of zero body force*: Given a quasi-equilibrated motion of a particular body, this motion is dynamically possible, if and only if the motion is circulation-preserving with a single-valued acceleration potential ς. Then the stress is given by the formula

$$\mathbf{T} = -\rho\varsigma\mathbf{1} + \mathbf{T}_0 \ , \tag{67}$$

where $\mathbf{T}_0$ is the most general stress corresponding to equilibrium at time t.

*The arguments given here are those of Truesdell and Noll [4].

*Truesdell and Noll [4], page 208.

The essence of the above result in that a quasi-equilibrated motion is dynamically possible for an incompressible elastic material if and only if it is a possible motion for an incompressible, inviscid fluid. In the preceding two sections of this paper, we have found the pressure and velocity fields in two boundary-value problems for such fluids. For each, the streamlines are rectilinear, and thus the motion is circulation-preserving. In each case, the potential ς may be found by an elementary integration of Equation 66_1. In our solutions, we have found the pressure fields p in Equation 66_2. That suffices for the material of interest to us. If, in addition, one wanted to consider elastic materials undergoing the same motions, one would have to find the static pressure fields p_0 in these motions, and that would require the solutions of static problems in finite elasticity. It so happens that these exact problems have been solved.* Analysis of the resulting equations has been concentrated on consideration of the case where either the bodies are thin [5], or on finding sufficient conditions that the motion of the body be periodic [6], [7], [8], [9], [10].

5 Recommendations and Closure

In this report we have provided analytic solutions which allow a quantitative evaluation of the stabilizing inertial forces brought about in the motion of symmetric hollow bodies. Although not explicity done here, it is straightforward to demonstrate that inertial pressures can significantly exceed characteristic destabilizing pressures, such as a plastic flow stress for example, for motions commensurate with applications of interest.

The geometries considered in the present study are good approximations to the motions in two applications which should provide the basis for continued analysis of this problem. First is the symmetric implosion and rebound of an explosively-loaded metal shell. Of interest here is stability of both the inward and outward motion and evaluation of the degree of distention corresponding to stability and breakup. Second is the ejecta resulting from a hypervelocity impact event – the regime of stable motion and the point of instability. The present analysis provides the framework for useful calculations concerning the physics of stability and fragmentation in both applications.

*Truesdell and Noll [4], pages 212–214.

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Appendix A. Kinetic Energy for the Spherical Body

Consider a hollow spherical body, as described in 1, and assume zero pressure on the inside and outside boundaries. By Equation 7 the velocity field is

$$v(r,b) = \frac{b^2}{r^2}\dot{b}(b) \ . \tag{68}$$

Here, we will find $\dot{b}(b)$ by an energy argument. Since the material model we use does not allow for dissipation, and there is no pressure on the outer boundaries, the total kinetic energy, $\mathcal{T}$ of the body, $\mathcal{B}$, is a constant. This energy is calculated by:

$$\mathcal{T} = \int_{\mathcal{B}} \frac{\rho v^2}{2} dV \ .$$

Since the body is symmetric

$$\int_{\mathcal{B}} \cdots dV = 4\pi \int_a^b \cdots dr \ ,$$

so by Equation 68

$$\mathcal{T} = 2\pi\rho b^4 \dot{b}^2 \int_a^b \frac{dr}{r^2} \ .$$

This leads to

$$\dot{b}^2 = \frac{\mathcal{T}}{2\pi\rho b^3 \left(\frac{b}{a} - 1\right)} \ ,$$

which is identical to Equation 26, with the identification

$$c = \frac{\mathcal{T}}{2\pi\rho b_0^3} \ .$$

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