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Hyperbolic Models for Two-Phase (or Two-Material) Flow

Darrell L. Hicks

Sandia National Laboratories

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HYPERBOLIC MODELS FOR TWO-PHASE (or TWO-MATERIAL) FLOW

by

D. L. Hicks

Computational Physics and Mechanics Division II - 5533
Sandia National Laboratories
Albuquerque, New Mexico 87185

Abstract

For some time it has been known that many of the two-phase flow models lead to ill-posed problems unless viscous stresses are included. The inclusion of viscous stresses changes the character of the equations from hyperbolic to parabolic. A continuing problem has been to find a well-posed hyperbolic system of equations which provide a reasonable model for two-phase flow, or to show that no such model exists. Another outstanding problem has been to understand why the derivation procedures for microstructural models produce models with the peculiar defect of being unstable. A careful investigation of the derivation procedures for the simple case of stratified flow suggests that the equal-pressures assumption is most likely the assumption leading to instability. Consideration of the alternative assumption suggests a model, namely the Unequal-Pressures Model, which is expressed by a first order system of partial differential equations with real characteristics. Thus the problem of complex characteristics (or sound speeds) which lead to the instability in the equal-pressures models is obviated. The form that the analysis takes suggests a technique for categorizing models according to the evolution equations for their internal state variables in order to aid model builders in quickly determining which models will lead to complex characteristics. Also a model with real characteristics for the two-phase flow of a bubbly liquid arises from an extension of the Unequal-Pressures model for single-layered flow to multi-layered flow. This Unequal-Pressures model has real characteristics for all physically acceptable states and has a complete set of eigenvectors except for a set of measure zero in state space and therefore is hyperbolic a.e. (almost everywhere) in state space. Also this Unequal-Pressures model is stable in the sense of von Neumann a.e. in state space.

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PREFACE

Several colleagues asked about procedures for analyzing the well-posedness of initial value problems for systems of nonlinear partial differential equations. In response to their requests I have made this report somewhat more tutorial than it otherwise would have been.

The reader should note that the equation numbering is restarted at the beginning of each section.

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SYMBOL DEFINITIONS

c_ξ	Sound speed of ξth constituent (or phase)
g	Subscript for a gas-phase variable
k	Wave number
ℓ	Subscript for a liquid-phase variable
m	Subscript for a mixture variable
$\hat{p}$	Interface pressure
p_ξ	Pressure in the ξth constitutent
ϕ_ξ	Volume fraction of the ξth constitutent
q_ξ	Viscous stress in the ξth constituent
ρ_ξ	Mass density of the ξth constituent w.r.t. the mixture
$\overline{\rho}_\xi$	Mass density of the ξth constituent w.r.t. the ξth constituent
S_ξ^ρ	The mass source term for the ξth constituent
$S_\xi^{\rho u}$	The momentum source term for the ξth constituent
$\hat{u}$	Interface x-velocity
$\hat{v}$	Interface y-velocity
x	Spatial variable
y	Spatial variable
ξ	Constituent index, ξ = g or ℓ

SUMMARY

Herein the derivation procedure for a simple case of two-phase flow, namely stratified flow, is carefully studied by enumerating and analyzing all of the approximations and assumptions made in constructing the model. In particular the approximations are studied for their effect on the stability of the model. In this way a conclusion is reached as to which of the assumptions lead to instability: the equal-pressures assumption is apparently that assumption. Then what appears to be a realistic model (at least the sound speeds are real) for two-phase flow is constructed which avoids the assumption that the pressures in the different phases are equal. This model, namely the Unequal-Pressures Model, is hyperbolic a.e. in state space. That is, the model is expressed by a system of first order partial differential equations which have real eigenvalues everywhere and a complete set of eigenvectors a.e. in state space. Also a procedure for categorizing the models according to the form of the evolution equations for their internal state variables (e.g. the volume fractions in the case of two-phase flow) is suggested here. The purpose for this categorization is to aid model builders in quickly determining which models lead to ill-posed problems. Also a hyperbolic a.e. model for the two-phase flow of a bubbly liquid is presented here. This model arises from a consideration of the extension of the hyperbolic a.e. model for single layered flow to multi-layered flow. These hyperbolic a.e. models are stable a.e. in the sense of von Neumann.

1. INTRODUCTION

1.1 PURPOSES AND PRELIMINARIES

This report has four purposes:

- present a careful analysis of the microstructural modeling procedure for a simple case in order to illuminate all of the various assumptions and approximations which might be the source of the instabilities of the two-phase (or two-material) models.

- present an a.e. hyperbolic model for stratified flow that arose when the equal-pressures assumption was removed.

- present a procedure for categorizing the models according to the evolution equations for the internal state variables.

- present an a.e. hyperbolic model for the flow of a material which has a microstructure similar to, for example, a foam, a bubbly liquid or a gas with solid particles in it.

It appears that there are basically two kinds of mixture theories: miscible and immiscible. Here we are concerned with an immiscible mixture theory. Basically that means that the interfaces between materials are always well-defined. It is beyond the scope of this report to debate the issue of whether or not the two-phase (e.g. gas and liquid) flow of some specific material (e.g. water) is well-modeled by an immiscible mixture model.

1.2 DEFINITIONS AND EQUATIONS

In DREW [1] and ISHII [2] and elsewhere the microstructural models are derived by averaging processes. These models often lead to instabilities. In order to get an idea of which of the assumptions or approximations might be leading to instabilities we follow WENDROFF [3] by looking very carefully at a very simple case of two-phase flow, namely stratified flow. See Figure 1.

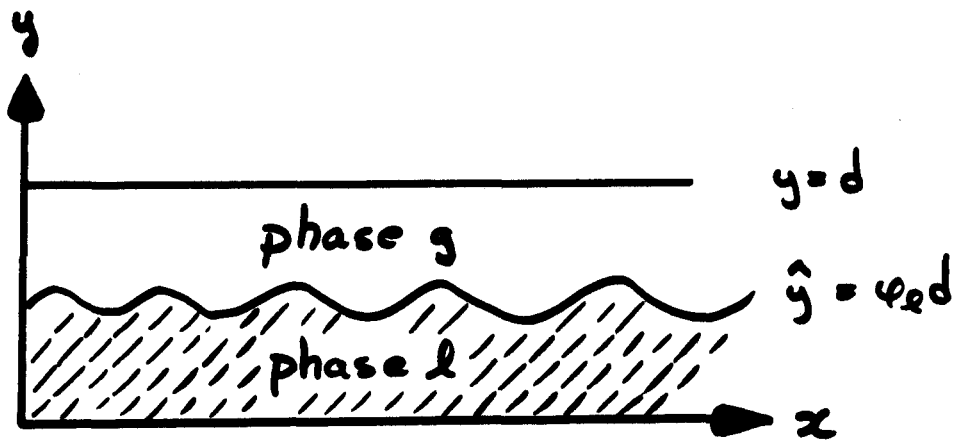

FIGURE 1. Separated, two-phase flow between two plates in zero gravity.

ASSUMPTION 1: Assume the fluid is inviscid with a mechanical equation of state.

If viscous stresses are included then stable models result [4]. However, here we are in pursuit of hyperbolic models and the introduction of viscous stresses leads to a model with parabolic character. The assumption of a mechanical equation of state simplifies the analysis and does so with apparently no loss of generality.

Let $\hat{f}$ be f evaluated at the interface. That is, for any function of x, y, t let

$$\hat{f}_\ell(x,t) = f(x,\hat{y}-,t)$$
$$\hat{f}_g(x,t) = f(x,\hat{y}+,t) \tag{1}$$

Let the volume fractions ϕ_ℓ and ϕ_g be defined by

$$\phi_\xi = (\text{vol. of } \xi)/(\text{vol. of mixture}) \quad . \tag{2}$$

Let $\hat{y}_\ell(x,t)$ be the top surface of phase ℓ (see Fig. 1). Let $\hat{y}_g(x,t)$ be the bottom surface of phase g.

ASSUMPTION 2: Assume that the phases do not detach from each other or from the top and bottom plates.

Under Assumption 2 we have $\hat{y}_\ell = \hat{y}_g$ so let $\hat{y} = \hat{y}_\ell = \hat{y}_g$. Also we have

$$\phi_\ell + \phi_g = 1 \quad . \tag{3}$$

Note that (under Assumption 2)

$$\hat{v}_\ell = \left.\frac{\partial \hat{y}_\ell}{\partial t}\right|_{X_\ell} = \left.\frac{\partial(\phi_\ell d)}{\partial t}\right|_{X_\ell} = \left(\frac{\partial \phi_\ell}{\partial t} + \hat{u}_\ell \frac{\partial \phi_\ell}{\partial x}\right) d \quad . \tag{4}$$

and

$$\hat{v}_g = \left.\frac{\partial \hat{y}_g}{\partial t}\right|_{X_g} = \left.\frac{\partial}{\partial t}(1 - \phi_g)d\right|_{X_g} = -\left(\frac{\partial \phi_g}{\partial t} + \hat{u}_g \frac{\partial \phi_g}{\partial x}\right) d \quad . \tag{5}$$

where X_ξ is the Lagrangean coordinate in the ξ-material.

From $\phi_\ell + \phi_g = 1$ it follows that

$$\partial\phi_g = -\partial\phi_\ell$$

for $\partial = \partial/\partial t$ and $\partial/\partial x$ and therefore the relation for the components of the velocities at the interface follows

$$\hat{v}_\ell - \hat{v}_g = (\hat{u}_\ell - \hat{u}_g) \frac{\partial(\phi_\ell d)}{\partial x} \quad . \tag{6}$$

Note that requiring that the phases do not detach from the plates leads to

$$v_\ell(x,0,t) = 0 = v_g(x,d,t) \quad . \tag{7}$$

<u>REMARK</u>: Assuming that the phases do not detach from each other leads to

$$\hat{v}_g = \hat{v}_\ell \tag{8}$$

except for example, for such cases as those instants when a shock wave is passing through the interface. From (8) and (6) it follows that $\hat{u}_g = \hat{u}_\ell$ unless $\partial\phi_\ell/\partial x = 0$. This suggests that $\hat{u}_\xi$ should be considered as a function of $\partial\phi_\ell/\partial x$. More about this later.

<u>Mass Density Definitions</u>:

$$\rho^\xi(x,y,t) \equiv \begin{cases} \text{mass density in phase } \xi \text{ at } (x,y,t) \\ \text{if } (x,y,t) \text{ is in phase } \xi \\ \\ \text{zero if } (x,y,t) \text{ is not in phase } \xi \end{cases} \tag{9}$$

$$\overline{\rho}_\ell(x,t) \equiv \frac{1}{\alpha_\ell} \int_0^{\alpha_\ell} \rho^\ell(x,y,t)\, dy \tag{10}$$

$$\bar{\rho}_g(x,t) \equiv \frac{1}{\alpha_g} \int_{\alpha_\ell}^{d} \rho^g(x,y,t)\, dy \tag{11}$$

where $\alpha_\xi = \phi_\xi d$ for $\xi = g$ and ℓ.

Note that

$$\bar{\rho}_\xi = (\text{mass of } \xi)/(\text{vol. of } \xi) \tag{12}$$

or in words, $\bar{\rho}_\xi$ is the mass density of ξ with respect to phase ξ.

Further define

$$\rho_\xi(x,t) \equiv \bar{\rho}_\xi(x,t)\phi_\xi(x,t) \quad . \tag{13}$$

and note that

$$\rho_\xi = (\text{mass of } \xi)/(\text{vol. of mixture}) \tag{14}$$

or in words, ρ_ξ is the mass density of ξ with respect to the mixture.

<u>REMARK</u>: Next the equations governing continuum flow in x,y space will be written down then averaged in the y-direction to produce one-dimensional equations in x.

<u>Equations Before Averaging</u>:

$$\partial\rho^\xi/\partial t + \partial\rho^\xi u_\xi/\partial x + \partial\rho^\xi v_\xi/\partial y = S^\rho_\xi \tag{15}$$

$$\partial\rho^\xi u_\xi/\partial t + \partial\rho^\xi u_\xi^2/\partial x + \partial\rho^\xi u_\xi v_\xi/\partial y + \partial p_\xi/\partial x = S^{\rho u}_\xi \tag{16}$$

$$\partial\rho^{\xi} v_{\xi}/\partial t + \partial\rho^{\xi} u_{\xi} v_{\xi}/\partial x + \partial\rho^{\xi} v_{\xi}^{2}/\partial y + \partial p_{\xi}/\partial y = S_{\xi}^{\rho v} \tag{17}$$

$$\partial\rho^{\xi} E_{\xi}/\partial t + \partial(\rho^{\xi} E_{\xi} u_{\xi} + u_{\xi} p_{\xi})/\partial x + \partial(\rho^{\xi} E_{\xi} v_{\xi} + v_{\xi} p_{\xi})/\partial y = S_{\xi}^{\rho E} \tag{18}$$

where

$$E_{\xi} = \mathscr{E}_{\xi} + (u_{\xi}^{2} + v_{\xi}^{2})/2 \tag{19}$$

and

$$dp_{\xi} = c_{\xi}^{2} d\rho^{\xi} \tag{20}$$

with

$$c_{\xi} > 0 \tag{21}$$

<u>REMARK</u>: S_{ξ}^{ρ}, $S_{\xi}^{\rho u}$, $S_{\xi}^{\rho v}$, and $S_{\xi}^{\rho E}$ are source terms for the mass momentum, and energy equations.

<u>AVERAGING OPERATORS</u>: Let

$$\overline{f}_{\ell} \equiv \frac{1}{\alpha_{\ell}} \int_{0}^{\alpha_{\ell}} f_{\ell}\, dy \tag{22}$$

$$\overline{f}_{g} \equiv \frac{1}{\alpha_{g}} \int_{\alpha_{\ell}}^{d} f_{g}\, dy \tag{23}$$

for $f = \rho, \rho u, \rho v, \rho E$, etc.

Note that

$$\partial[\overline{f_\xi}\phi_\xi] = \overline{\partial f_\xi}\phi_\xi + \hat{f}_\xi(\partial\phi_\xi) \tag{24}$$

for $\partial = \partial t$, $\partial/\partial x$ or $\partial/\partial y$.

Also note that

$$\overline{\frac{\partial f_\ell}{\partial y}} = \frac{1}{\alpha_\ell}[\hat{f}_\ell - f_o] \tag{25}$$

where $f_o = f(x,0,t)$ and

$$\overline{\frac{\partial f_g}{\partial y}} = \frac{1}{\alpha_g}[f_d - \hat{f}_g] \tag{26}$$

where $f_d = f(x,d,t)$.

Let

$$Df \equiv \partial f/\partial t + \partial uf/\partial x + \partial vf/\partial y \tag{27}$$

Note that

$$\phi_\ell \overline{Df_\ell} = \partial[\phi_\ell f_\ell]/\partial t + \partial[\phi_\ell \overline{u_\ell f_\ell}]/\partial x +$$

$$+ [v_\ell \hat{f}_\ell - v_0 f_0]\frac{1}{d} - \hat{f}_\ell\,[\partial\phi_\ell/\partial t + \hat{u}_\ell \partial\phi_\ell/\partial x]$$

and from eqn. (4) $v_\ell = [\partial\phi_\ell/\partial t + u_\ell \partial\phi_\ell/\partial x]d$ therefore

$$\phi_\ell \overline{Df_\ell} = \partial(\phi_\ell \overline{f_\ell})/\partial t + \partial(\phi_\ell \overline{u_\ell f_\ell})/\partial x - f_0 v_0/d \tag{28}$$

and likewise

$$\phi_g \overline{Df_g} = \partial(\phi_g \overline{f_g})/\partial t + \partial(\phi_g \overline{u_g f_g})/\partial x + f_d v_d/d \tag{29}$$

and the assumption that the flow is attached to the top and bottom plates leads to (this assumption is not necessary but it simplifies the equations)

$$v_d = 0 = v_o \quad . \tag{30}$$

Consequently we have the following averaged forms for the conservation laws:

$$\partial(\phi_\xi \overline{\rho}_\xi)/\partial t + \partial(\phi_\xi \overline{\rho_\xi u_\xi})/\partial x = \overline{S}^{\rho}_{\xi} \tag{31}$$

$$\begin{aligned} &\partial(\phi_\xi \overline{\rho_\xi u_\xi})/\partial t + \partial(\phi_\xi \overline{\rho_\xi u_\xi u_\xi})/\partial x \\ &\qquad + \partial(\phi_\xi \overline{p}_\xi)/\partial x - \hat{p}_\xi \, \partial\phi_\xi/\partial x = \overline{S}^{\rho u}_{\xi} \end{aligned} \tag{32}$$

$$\partial(\phi_\xi \overline{\rho_\xi v_\xi})/\partial t + \partial(\phi_\xi \overline{\rho_\xi u_\xi v_\xi})/\partial x + \Delta_\xi p/(\phi_\xi d) = \overline{S}^{\rho v}_{\xi} \tag{33}$$

$$\begin{aligned} &\partial(\phi_\xi \overline{\rho_\xi E_\xi})/\partial t + \partial(\phi_\xi \overline{u_\xi \rho_\xi E_\xi})/\partial x + \\ &\qquad + \partial(\phi_\xi \overline{u_\xi P_\xi})/\partial x - \hat{u}_\xi \hat{p}_\xi \, \partial\phi_\xi \, \partial x \\ &\qquad + \Delta_\xi up/(\phi_\xi d) = \overline{S}^{\rho E}_{\xi} \end{aligned} \tag{34}$$

where

$$\Delta_\xi f = \begin{Bmatrix} \hat{f}_\ell - f_o \text{ if } \xi = \ell \\ \\ f_d - \hat{f}_g \text{ if } \xi = g \end{Bmatrix} \tag{35}$$

Let $\langle f_\xi \rangle \equiv \overline{f_\xi \rho_\xi} / \overline{\rho}_\xi$ (36)

and use $\rho_\xi = \overline{\rho}_\xi \phi_\xi$ to rewrite the equations as follows:

$$\partial \rho_\xi / \partial t + \partial \rho_\xi \langle u_\xi \rangle / \partial x = \overline{S}^\rho_\xi \tag{37}$$

$$\partial \rho_\xi \langle u_\xi \rangle / \partial t + \partial \rho_\xi \langle u_\xi^2 \rangle / \partial x$$

$$+ \partial (\phi_\xi \overline{p}_\xi) / \partial x - \hat{p}_\xi \, \partial \phi_\xi / \partial x = \overline{S}^{\rho u}_\xi \tag{38}$$

$$\partial \rho_\xi \langle v_\xi \rangle / \partial t + \partial \, \rho_\xi \langle u_\xi v_\xi \rangle / \partial x + \Delta_\xi p / (\phi_\xi d) = \overline{S}^{\rho v}_\xi \tag{39}$$

$$\partial \rho_\xi \langle E_\xi \rangle / \partial t + \partial (\rho_\xi \langle u_\xi E_\xi \rangle) / \partial x +$$

$$+ \partial (\phi_\xi \overline{u_\xi p_\xi}) / \partial x - \hat{u}_\xi \hat{p}_\xi \partial \phi_\xi / \partial x \tag{40}$$

$$+ \Delta_\xi up / (\phi_\xi d) = \overline{S}^{\rho E}_\xi$$

To get the Equal-Pressures Model for a mechanical equation of state, equations (39) and (40) are dropped and the following assumptions are made about variables in eqns. (1) and (2):

<u>ASSUMPTION 3:</u> $\overline{p}_\ell = \overline{p}_g = \hat{p}_\ell = \hat{p}_g \equiv p$.

ASSUMPTION 4a: $\langle u_\xi^2 \rangle = \langle u_\xi \rangle^2$.

Thus the equations for the Equal-Pressures Model in the case of a mechanical equation of state are (where $\tilde{u}_\xi \approx \langle u_\xi \rangle$ and $\tilde{u}_\xi^2 \approx \langle u_\xi^2 \rangle$) as follows

$$\partial \rho_\xi / \partial t + \partial \rho_\xi \tilde{u}_\xi / \partial x = \bar{S}_\xi^\rho \tag{41}$$

$$\partial \rho_\xi \tilde{u}_\xi / \partial t + \partial \rho_\xi \tilde{u}_\xi^2 / \partial x + \phi_\xi \partial p / \partial x = \bar{S}_\xi^{\rho u} \tag{42}$$

Let the mixture density be denoted by ρ_m then

$$\rho_m = \rho_g + \rho_\ell \quad . \tag{43}$$

Lyczskowski et al [5] define the "mixture sound speed" by

$$c_m^{-2} = \rho_m (r_g + r_\ell) \tag{44}$$

where

$$r_\xi = \phi_\xi / (\rho_\xi c_\xi^2) \tag{45}$$

for $\xi = g, \ell$.

It follows [see 4 & 6] from Assumption #3 that $c_g^2 d\bar{\rho}_g = c_\ell^2 \, d\bar{\rho}_\ell = dp$ and therefore

$$dp = \rho_m \, c_m^2 \left[d\rho_g / \bar{\rho}_g + d\rho_\ell / \bar{\rho}_\ell \right] \tag{46}$$

and

$$d\phi_\ell = \rho_m \, c_m^2 \left[r_g \frac{d\rho_\ell}{\rho_\ell} - r_\ell \frac{d\rho_g}{d\rho_\ell} \right] \phi_g \, \phi_\ell \tag{47}$$

and, of course, to get $d\phi_g$ we use $d\phi_g = - d\phi_\ell$.

Thus the equations for the Equal-Pressures Model for two-phase flow may be reduced to

$$\partial\rho_\xi/\partial t + \partial\rho_\xi\tilde{u}_\xi/\partial x = \overline{S}^\rho_\xi \tag{48}$$

$$\partial\rho_\xi\tilde{u}_\xi/\partial t + \partial\rho_\xi\tilde{u}^2_\xi/\partial x + \phi_\xi\rho_m c^2_m\left[\rho_g/\overline{\rho}_g + \rho_\ell/\overline{\rho}_\ell\right] = \overline{S}^{\rho u}_\xi \tag{49}$$

<u>ASSUMPTION 5</u>: All of the source terms $\overline{S}^\rho_\xi$, $\overline{S}^{\rho u}_\xi$, and $\overline{S}^{\rho E}_\xi$ for $\xi = g, \ell$ are real valued functions of p_ξ, ρ_ξ, ϕ_ξ, and u_ξ but not their derivatives.

The fifth assumption ensures that the source terms do not affect the characteristic values of the Equal-Pressures Model. It is well-known (see [4] & [6]) that the Equal-Pressures Model as defined above has complex characteristics and consequently leads to ill-posed problems.

Now let's investigate Assumptions 3, 4, and 5.

First, there seem to be cases where the source terms are zero or at least negligibly small and therefore we would like for the model to be stable for those cases. Therefore, we feel that Assumption 5 is associated with reasonable physical processses and need not be altered, at least for the purposes of this study.

Next, let's turn to Assumption 4a. Let $\delta u_\xi(x,y,t)$ be defined by

$$u_\xi(x,y,t) = \langle u_\xi\rangle(x,t) + \delta u_\xi(x,y,t) \quad . \tag{50}$$

Then $u^2_\xi = \langle u_\xi\rangle^2 + 2\langle u_\xi\rangle\delta u_\xi + (\delta u_\xi)^2$ and since $\langle\langle u_\xi\rangle\rangle = \langle u_\xi\rangle$ we have

$$\langle u^2_\xi\rangle = \langle u_\xi\rangle^2 + \langle(\delta u_\xi)^2\rangle \quad . \tag{51}$$

Thus Assumption 4a drops the term $\langle(\delta u_\xi)^2\rangle$ and it can be argued that $\langle(\delta u_\xi)^2\rangle$ looks a great deal like a viscous stress. We know that the inclusion of vicous stresses stabilize the equations and therefore we are led to suspect Assumption 4a.

To test this assumption we consider the following model. Replace Assumption 4a with

<u>ASSUMPTION 4b</u>: Let $\langle u_\xi^2\rangle$ be determined by the following modification of eqn. (40) (the averaged equation for the energy):

$$\partial\rho_\xi(\mathscr{E}_\xi(\bar{\rho}_\xi) + \frac{1}{2}\langle u_\xi^2\rangle)/\partial t + \partial\rho_\xi(\langle u_\xi\rangle\mathscr{E}_\xi(\rho_\xi) + \frac{1}{2}\langle u_\xi\rangle^3)/\partial x$$

$$+ \phi_\xi\,\partial\langle u_\xi\rangle p/\partial x = \bar{S}_\xi^{\rho E} \tag{52}$$

Also use eqn. (38) instead of (42) to determine $\langle u_\xi\rangle$.

<u>REMARK</u>: Note that we have made the approximation $\langle u_\xi^3\rangle \doteq \langle u_\xi\rangle^3$ in eqn. (52). Thus the Equal-Pressures-Model with Assumption 4b, namely Equal-Pressures Model 4b, is expressed as follows:

$$\text{(I)}\qquad \partial\rho_\xi/\partial t + \partial\rho_\xi\langle u_\xi\rangle/\partial x = \bar{S}_\xi^{\rho} \tag{53}$$

$$\text{(II)}\qquad \partial\rho_\xi\langle u_\xi\rangle/\partial t + \partial\rho_\xi\langle u_\xi^2\rangle/\partial x + \phi_\xi\,\partial p_\xi/\partial x = \bar{S}_\xi^{\rho u} \tag{54}$$

$$\text{(IV)}\qquad \partial\rho_\xi\frac{1}{2}\langle u_\xi^2\rangle/\partial t + \partial\rho_\xi\frac{1}{2}\langle u_\xi\rangle^3/\partial x + \partial\rho_\xi\mathscr{E}_\xi(\bar{\rho}_\xi)/\partial t$$

$$+ \partial\rho_\xi\langle u_\xi\rangle\mathscr{E}_\xi(\bar{\rho}_\xi)/\partial x + \phi_\xi\,\partial\langle u_\xi\rangle p_\xi/\partial x = \bar{S}_\xi^{\rho E} \tag{55}$$

where by the equal pressures assumption

$$dp_\xi = dp = \rho_m c_m^2 \left(\frac{d\rho_g}{\rho_g} + \frac{d\rho_\ell}{\rho_\ell} \right)$$

and furthermore by elementary thermodynamic identities

$$d\mathscr{E}_\xi = \frac{p_\xi}{\bar{\rho}_\xi^2} d\bar{\rho}_\xi = \frac{p_\xi}{\bar{\rho}_\xi^2 c_\xi^2} dp_\xi \tag{56}$$

which hold when the specific entropy is held constant.

Now to simplify the notation let

$$m_\xi = \rho_\xi \langle u_\xi \rangle \tag{57}$$

$$K_\xi = \frac{1}{2} \rho_\xi \langle u_\xi^2 \rangle \tag{58}$$

$$\left. \frac{\partial f}{\partial t} \right|_x = \dot{f} \tag{59}$$

and

$$\left. \frac{\partial f}{\partial x} \right|_t = f' \tag{60}$$

for $f = m_\xi, k_\xi, \rho_\xi$, etc. Then we have the following expressions for the Equal-Pressures Model 4b:

$$\rho_\xi + m'_\xi = \overline{S}_\xi^\rho \tag{61}$$

$$m_\xi + 2K'_\xi + \phi_\xi \rho_m \; cm^2 \left[\frac{\dot{\rho}_\ell}{\rho_\ell} + \frac{\dot{\rho}_g}{\rho_g} \right] = \overline{S}_\xi^m \tag{62}$$

$$\dot{K}_\xi + \dot{\rho}_\xi \mathscr{E}_\xi + r_\xi \rho_m c_m^2 \, p \, \frac{\dot{\rho}_\ell}{\bar{\rho}_\ell} + \frac{\dot{\rho}_g}{\bar{\rho}_g}$$

$$+ \left[\frac{3}{2} u_\xi^2 + \mathscr{E}_\xi + p/\bar{\rho}_\xi \right] m'_\xi - u_\xi (u_\xi^2 + p/\rho_\xi) \rho'_\xi$$

$$+ u_\xi (r_\xi p + \phi_\xi) \rho_m \, c_m^2 \, \frac{\rho'_\ell}{\bar{\rho}_\ell} + \frac{\rho_g}{\bar{\rho}_\ell} = \bar{S}_\xi^{\rho E} \tag{63}$$

It turns out (as shall be shown in Section 2) that the above model also leads to complex characteristics. This does not, of course, rigorously establish that Assumption 4a is not the assumption that leads to instability. However, it does suggest that to us. Therefore, we turn to Assumption 3 (The Equal-Pressures Assumption) and consider its reasonableness.

In the case of stratified flow we could conceivably at some instant have pressure profiles similar to the one illustrated in Figure 2.

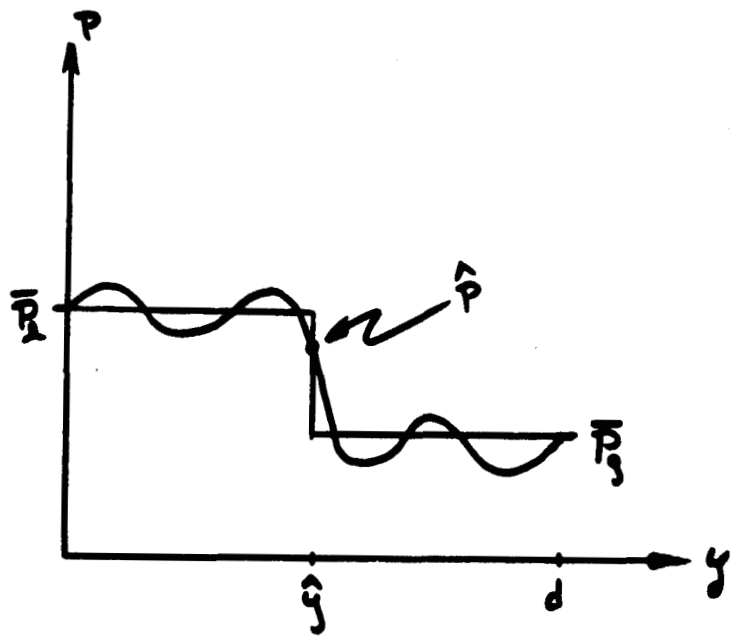

FIGURE 2
A Conceivable Pressure Profile for Stratified Flow.

From considerations of conceivable situations such as that exhibited in Figure 2, it appears that in a dynamic flow situation there is no good reason why $\bar{p}_\ell$ should equal $\bar{p}_g$. Considering Figure 2 suggests that what we should do to resolve the discontinuity at $\hat{y} = \alpha_\ell$ is solve the Riemann problem between the ℓ-state and the g-state. This would determine $\hat{p}$ and $\hat{v}$. Knowing $\hat{v}$ would then allow us to write an evolution equation for ϕ_ℓ, namely

$$\partial\phi_\ell/\partial t + \hat{u}_\ell \partial\phi_\ell/\partial x = \hat{v}/d \tag{64}$$

where we assume that $\hat{u}_\ell$ is some real valued function of the state variables.

Reasonable and physically acceptable approximations to $\hat{u}_\ell$ are, for example, u_ℓ and $(u_g + u_\ell)/2$. However, these should probably be patched up a bit from the following considerations: as $\phi_g \to 0$ we should have $\hat{u}_\ell \to u_\ell$ and as $\phi_\ell \to 0$ we should have $\hat{u}_\ell \to u_g$ if we want our equations to reduce to simple single-phase flow when the other phase vanishes. Therefore, one might take $\hat{u}_\ell = \phi_\ell u_\ell + \phi_g u_g$. The remark after equation (7) suggests also that $\hat{u}_\ell$ should be dependent on $\partial\phi_\ell/\partial x$.

The foregoing considerations yield the following model which we shall refer to as the Unequal-Pressures Model for stratified flow:

$$\rho_\xi + m'_\xi = 0 \tag{65}$$

$$m_\xi + (m_\xi^2/\rho_\xi)' + \phi_\xi \bar{p}'_\xi + (\bar{p}_\xi - \hat{p})\phi'_\xi = 0 \tag{66}$$

for $\xi = g$ and ℓ and the system is completed with the evolution equation (64) for the internal state variable.

Let $\bar{a}_\xi = c_\xi \bar{\rho}_\xi$ be the acoustic impedance of the ξth phase. By assuming $\bar{v}_g = 0 = \bar{v}_\ell$ and by approximating the vP Hugoniots with straight lines of slopes $-\bar{a}_\ell$ and $+\bar{a}_g$, we arrive at the following approximate solution to the Riemann problem

$$\hat{v} = (\bar{p}_\ell - \bar{p}_g)/(\bar{a}_g + \bar{a}_\ell) \tag{67}$$

$$\hat{p} = (\bar{p}_g/\bar{a}_g + \bar{p}_\ell/\bar{a}_\ell)/(1/\bar{a}_g + 1/\bar{a}_\ell) \quad . \tag{68}$$

Thus we arrive at the Unequal-Pressures Model for stratified two-phase flow:

$$\dot{\rho}_\ell + m'_\ell = \bar{S}p_\ell \tag{69}$$

$$\dot{m}_\ell + (c_\ell^2 - u_\ell^2)\,\rho'_\ell + 2u_\ell m'_\ell + \phi'_\ell \mathscr{P}_\ell = \bar{S}m_\ell \tag{70}$$

$$\dot{\rho}_g + m'_g = \bar{S}p_g \tag{71}$$

$$\dot{m}_g + (c_g^2 - u_g^2)\rho'_g + 2u_g m'_g - \phi'_\ell \mathscr{P}_g = \bar{S}m_g \tag{72}$$

$$\dot{\phi}_\ell + \hat{u}_\ell \phi'_\ell = \hat{v}/d \tag{73}$$

where

$$\mathscr{P}_\xi = p_\xi - \hat{p} - c_\xi^2 \rho_\xi \quad .$$

This model has real characteristics as shall be shown in section two. Moreover this model is hyperbolic a.e. (almost everywhere) in state space. That is, except for a set of measure zero in state space (with its natural measure) this is a hyperbolic model. This shall be shown in section two.

Next let's consider what happens when instead of a single layer (one interface) of stratified flow we have a two layer (two interfaces) stratified flow as illustrated in Figure 3 (where $\alpha_{\xi i} = \phi_{\xi i} d_i$ for $\xi = g, \ell$ and $i = 1,2$ and $d_1 + d_2 = d$).

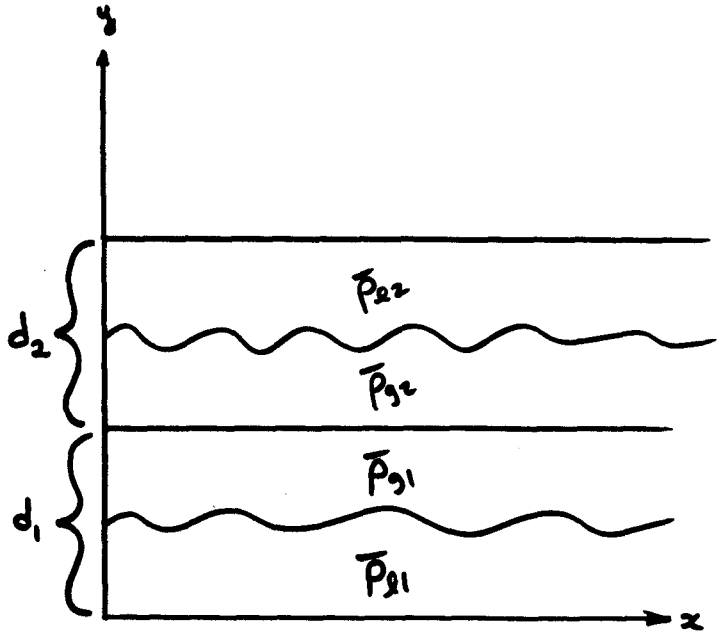

FIGURE 3: TWO-LAYERED FLOW

For two-layered flow we have the following four averaging operators:

$$\bar{f}_{\ell 1} = \frac{1}{\alpha_{\ell 1}} \int_0^{\alpha_{\ell 1}} f_{\ell 1} \, dy \tag{74}$$

$$\bar{f}_{g1} = \frac{1}{\alpha_{\ell 1}} \int_{\alpha_{\ell 1}}^{d_1} f_{g1} \, dy \tag{75}$$

$$\bar{f}_{g2} = \frac{1}{\alpha_{g2}} \int_{d_1}^{d_1 + \alpha_{g2}} f_{g2} \, dy \tag{76}$$

$$\bar{f}_{\ell 2} = \frac{1}{\alpha_{\ell 2}} \int_{d_1 + \alpha_{g2}}^{d} f_{\ell 2} \, dy \tag{77}$$

Note that for $\partial = \partial/\partial t$, $\partial/\partial x$, or $\partial/\partial y$ we have

$$\partial(\alpha_{\xi i} \bar{f}_{\xi i}) = \alpha_{\xi i} \overline{\partial f}_{\xi i} + \hat{f}_{\xi i} d_i \partial\phi_{\xi i}$$

Therefore, if we apply the averaging operators to

$$Df \equiv \partial f/\partial t + \partial uf/\partial x + \partial vf/\partial y$$

we get

$$\frac{\partial}{\partial t}(\alpha_{\ell 1} \bar{f}_{\ell 1}) + \frac{\partial}{\partial x}(\alpha_{\ell 1} \overline{u_{\ell 1} f_{\ell 1}}) - f_o v_o = 0 \tag{78}$$

$$\frac{\partial}{\partial t}(\alpha_{g1} \bar{f}_{g1}) + \frac{\partial}{\partial x}(\alpha_{g1} \overline{u_{g1} f_{g1}}) + f_{d1} v_{d1} = 0 \tag{79}$$

$$\frac{\partial}{\partial t}(\alpha_{g2} \bar{f}_{g2}) + \frac{\partial}{\partial x}(\alpha_{g2} \overline{u_{g2} f_{g2}}) - f_{d1} v_{d1} = 0 \tag{80}$$

$$\frac{\partial}{\partial t}(\alpha_{\ell 2} \bar{f}_{\ell 2}) + \frac{\partial}{\partial x}(\alpha_{\ell 2} \overline{u_{\ell 2} f_{\ell 2}}) + f_d v_d = 0 \tag{81}$$

where

$$f_0 \text{ is } f \text{ at } y = 0 \quad , \tag{82}$$

$$f_{d1} \text{ is } f \text{ at } y = d_1 \quad , \tag{83}$$

and

$$f_d \text{ is } f \text{ at } y = d_2 + d_1 = d \quad . \tag{84}$$

Using

$$\phi_\xi(d_1 + d_2) = \phi_{\xi 1} d_1 + \phi_{\xi 2} d_2 \tag{85}$$

and $v_o = 0 = v_d$ and adding eqns. (78) and (81) we get

$$\partial(\phi_\ell \overline{f}_\ell)/\partial t + \partial(\phi_\ell \overline{u_\ell f_\ell})/\partial x = 0 \tag{86}$$

and similarly adding eqns. (79) and (80) we get

$$\partial(\phi_g \overline{f}_g)/\partial t + \partial(\phi_g \overline{u_g f_g})/\partial x = 0 \tag{87}$$

where

$$\phi_\xi \overline{f}_\xi = (\alpha_{\xi 1} \overline{f}_{\xi 1} + \alpha_{\xi 2} \overline{f}_{\xi 2})/(d_1 + d_2) \tag{88}$$

for $\xi = g,\ell$.

Therefore, the conservation of mass equations are

$$\partial(\phi_\xi \overline{\rho}_\xi)/\partial t + \partial(\phi_\xi \overline{\rho_\xi u_\xi})/\partial x = 0 \tag{89}$$

for $\xi = g,\ell$. Similarly the conservation of momentum equations are

$$\partial(\phi_\xi \overline{\rho_\xi u_\xi})/\partial t + \partial(\phi_\xi \overline{\rho_\xi u_\xi u_\xi})/\partial x + \partial(\phi_\xi \overline{p}_\xi)/\partial x - \hat{p}_\xi \partial\phi_\xi/\partial x = 0 \tag{90}$$

where

$$\hat{p}_\xi \partial\phi_\xi/\partial x(d_1 + d_2) = \hat{p}_{\xi 1}\ \partial\phi_{\xi 1}/\partial x\ d_1 + \hat{p}_{\xi 2}\ \partial\phi_{\xi 2}/\partial x\ d_2 \tag{91}$$

for $\xi = g,\ell$.

Solving (approximately) the Riemann Problem at the interfaces we get (recall that $a = c\rho$ and that we are using the approximations $dp/dv = \pm \bar{a}$):

$$\hat{v}_i = (-1)^i \frac{\bar{p}_{gi} - \bar{p}_{\ell i}}{\bar{a}_{gi} + \bar{a}_{\ell i}} \tag{92}$$

$$\hat{p}_i = (\bar{p}_{gi}\bar{a}_{gi} + \bar{p}_{\ell i}/\bar{a}_{\ell i})/(1/a_{gi} + 1/a_{\ell i}) \tag{93}$$

and

$$d_i(\partial\phi_{\ell i}/\partial t + \bar{u}_{\ell i}\,\partial\phi_{\ell i}/\partial x) = (-1)^{i+1}\hat{v}_i \tag{94}$$

for $i = 1,2$. Now, mainly to simplify these equations, let us assume that

$$f_{g1} = f_{g2} \tag{95}$$

and

$$f_{\ell 1} = f_{\ell 2} \tag{96}$$

for $f = a, c, \bar{p}, \bar{\rho}$, and $\bar{u}$. Then we get

$$\left[\frac{\partial\phi_\ell}{\partial t} + \hat{u}_\ell \frac{\partial\phi_\ell}{\partial x}\right] d = 2\left(\frac{\bar{p}_\ell - \bar{p}_g}{\bar{a}_g + \bar{a}_\ell}\right) \tag{97}$$

Recall (from eqns. (66) and (67)) that for the single layer case we had

$$\left[\frac{\partial\phi_\ell}{\partial t} + \hat{u}_\ell \frac{\partial\phi_\ell}{\partial x}\right] d = 1\left(\frac{\bar{p}_\ell - \bar{p}_g}{\bar{a}_g + \bar{a}_\ell}\right) \tag{98}$$

Note that the only difference between eqn. (97) (two-layer case) and eqn. (98) (single-layer case) is the factor of two (2) appearing on the right

hand side of eqn. (97). Consideration of this result leads to a conclusion that it is reasonable because if we have two interfaces moving from phase ξ (g or ℓ) toward phase ζ (ℓ or g) at the same speed as one interface moving from ξ toward ζ then we should change the volume fractions twice as fast.

Therefore, we see that the basic difference between single layered (one interface), two layered (two interfaces), or L-layered (L interfaces) flow occurs only in the evolution equation for ϕ, namely

$$\left(\partial\phi_\ell/\partial t + \hat{u}_\ell \partial\phi_\ell/\partial x\right)d = L\,(\bar{p}_\ell - \bar{p}_g)/(\bar{a}_g + \bar{a}_\ell) \tag{99}$$

where L is the number of interfaces crossed in the distance d.

We can use the preceding concepts to generate a model for the flow of a bubbly liquid. Assume the bubbles are spherical of radius r with $0 \leq 2r < d$. Then given a distribution on r we can compute the expected number of interface crossings L as y goes from 0 to d. For example, suppose for simplicity that r is constant and there is on the average only one bubble in a cube of dimensions d x d x d. Then an elementary calculation gives the expected value of L to be $2\pi(\frac{r}{d})^2$. This value can be related to ϕ_g because another elementary calculation shows $\phi_g = \frac{4}{3}\pi(\frac{r}{d})^3$. Therefore since

$$L = 2\pi(\tfrac{r}{d})^2 \tag{100}$$

and

$$\phi_g = \tfrac{4}{3}\pi(\tfrac{r}{d})^3 \tag{101}$$

then

$$L = \left(\frac{9\pi\phi_g^2}{2}\right)^{\frac{1}{3}} \tag{102}$$

follows by eliminating r from eqns. (100) and (101). It will be shown in section 2 that the model for the two phase flow of a bubbly liquid which results from using the equations (99) and (102) to define the evolution equation for ϕ_ℓ with equations (89) and (90) to complete the system is hyperbolic a.e.

REMARK: An implicit assumption has been made throughout this report and that assumption shall be named

ASSUMPTION 0: The interface between the two materials (or two phases) is well-defined at all times.

This assumption is based on the fact that here we are treating the immiscible mixture (e.g. oil and water or air and sand grains) rather than a miscible mixture (e.g. oxygen and nitrogren). Whether or not this assumption is valid for a given mixture is beyond the scope of this report.

1.3 WELL-POSEDNESS AND STABILITY

Let $\underset{\sim}{U}$ be a vector of M components and consider a system of partial differential equations of the form

$$\partial \underset{\sim}{U}/\partial t = \mathscr{F}(t,x,\underset{\sim}{U},\partial \underset{\sim}{U}/\partial x, \ldots,\partial^n \underset{\sim}{U}/\partial x^n) \tag{1}$$

whose solution is desired for the finite time interval $[t^0,t^f]$. Let the equation of first variation of (1) be written

$$\partial \delta \underset{\sim}{U}/\partial t = \underset{\sim}{A}\ (\partial/\partial x)\delta \underset{\sim}{U} \tag{2}$$

where δ is the variation operator and $\underset{\sim}{A}\ (\partial/\partial x)$ is an M by M matrix polynomial in $\partial/\partial x$ with coefficients depending on $t,x,\underset{\sim}{U},\partial \underset{\sim}{U}/\partial x,\ldots,\partial^n \underset{\sim}{U}/\partial x^n$. Let $A_0\ (\partial/\partial x)$ be $\underset{\sim}{A}\ (\partial/\partial x)$ evaluated at D_0 where $D_0 = (t_0,x_0,\underset{\sim}{U}_0,\partial \underset{\sim}{U}_0/\partial x, \ldots,\partial^n \underset{\sim}{U}_0/\partial x^n)$. Note that $A_0\ (\partial/\partial x)$ is a linear operator with constant coefficients. The equation

$$\partial \delta \underset{\sim}{U}/\partial t = \underset{\sim}{A}_0(\partial/\partial x)\delta \underset{\sim}{U} \tag{3}$$

is called the local (at D_0) equation of first variation of eqn. (1). The von Neumann method of stability analysis involves the inspection of the spectral radius r_k^t of $\mathscr{P}_k^t \equiv \exp\ (A_0(ik)t)$ where k is the wave number, $-\infty < k < \infty$.

<u>Definition: Stable in the sense of von Neumann:</u>

If r_k^t is bounded uniformly in k for all t in the finite time interval $[t^0,t^f]$, then eqn. (1) is said to be stable in the sense of von Neumann at D_0.

<u>REMARKS:</u> For a motivation of this definition and its relation to well-posedness see [6].

THEOREM 1: Let $\alpha_j(k)$, $1 \leq j \leq M$, be the eigenvalues of $\underset{\sim}{A}_0(ik)$. Then eqn. (1) is stable in the sense of von Neumann at D_0 iff $\max_j \operatorname{Re} \alpha_j(k)$ is bounded above uniformly in k, $-\infty < k < \infty$.

PROOF: This follows from Theorem 1.1 and corollary 1.1B on pages 18, 19 of [6].

Consider the case when eqn. (1) has the following form

$$\partial \underset{\sim}{U}/\partial t = \underset{\sim}{\mathscr{A}}(\underset{\sim}{U})\, \partial \underset{\sim}{U}/\partial x + \underset{\sim}{B}(\underset{\sim}{U}) \tag{4}$$

where $\underset{\sim}{\mathscr{A}}(\underset{\sim}{U})$ is a real M by M array and $\underset{\sim}{B}(\underset{\sim}{U})$ is a real M vector. We follow Courant and Hilbert (see [7] p. 425) in the following definition of hyperbolic.

DEFINITION: The matrix $\underset{\sim}{\mathscr{A}}$ is said to be hyperbolic at $\underset{\sim}{U}$ iff $\underset{\sim}{\mathscr{A}}(\underset{\sim}{U})$ has all its eigenvalues real and a complete set of eigenvectors. The system (4) is said to be hyperbolic at $\underset{\sim}{U}$ provided $\underset{\sim}{\mathscr{A}}$ is hyperbolic at $\underset{\sim}{U}$.

REMARK: In [8], Lax has given a nice discussion of the notion of hyperbolicity.

THEOREM 2: The system (4) is stable in the sense of von Neumann for all $\underset{\sim}{B}$ at D_0 iff $\mathscr{A}$ is hyperbolic at $\underset{\sim}{U}_0$.

PROOF: Note that the equation of first variation of (4) is of the form

$$\partial \delta \underset{\sim}{U}/\partial t = \underset{\sim}{\mathscr{A}}(\underset{\sim}{U}) \partial \delta \underset{\sim}{U}/\partial x + \underset{\sim}{C}(\underset{\sim}{U}, \underset{\sim}{U}')\delta \underset{\sim}{U} \tag{5}$$

where $\underset{\sim}{U}' = \partial \underset{\sim}{U}/\partial x$ and where the structure of $\underset{\sim}{C}$ depends upon the structure of $\underset{\sim}{\mathscr{A}}$ and $\underset{\sim}{B}$. Note that the $A_0(\partial/\partial x)$ in the definition of stable in the sense of von Neumann is for this case given by

$$\underset{\sim}{A}_0\, (\partial/\partial x) = \underset{\sim}{\mathscr{A}}(\underset{\sim}{U}_0)\partial/\partial x + \underset{\sim}{C}(\underset{\sim}{U}_0, \underset{\sim}{U}_0)$$

and therefore

$$\underset{\sim}{A}_0(ik) = \underset{\sim}{\mathscr{A}}(\underset{\sim}{U}_0)ik + \underset{\sim}{C}(\underset{\sim}{U}_0,\underset{\sim}{U}_0') \quad . \tag{6}$$

First, we prove that if $\underset{\sim}{\mathscr{A}}(\underset{\sim}{U}_0)$ is hyperbolic then eqn. (4) is stable in the sense of von Neumann at D_0 for all $\underset{\sim}{B}$. By definition, if $\underset{\sim}{\mathscr{A}}(\underset{\sim}{U}_0)$ is hyperbolic then it is diagonalizable. Therefore, without loss of generality, assume that $\underset{\sim}{\mathscr{A}}(\underset{\sim}{U}_0)$ is diagonal with diagonal elements $d_{11}, d_{22}, \ldots, d_{MM}$. Let λ be an eigenvalue of $\underset{\sim}{\mathscr{A}}(\underset{\sim}{U}_0)$ of multiplicity m and let the d_{ii} be so arranged that $\lambda = d_{11} = \cdots = d_{mm}$. Let $\mu = ik\lambda - \alpha$ and consider the upper m by m block of $\underset{\sim}{A}_0(ik) - \alpha 1$. It has the following form

$$\underset{\sim}{D}(\mu) = \begin{bmatrix} \mu+C_{11} & C_{12} & \cdot & \cdot & \cdot & C_{1m} \\ C_{21} & \mu+C_{22} & \cdot & \cdot & \cdot & C_{2m} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ C_{m1} & C_{m2} & \cdot & \cdot & \cdot & \mu+C_{mm} \end{bmatrix} \tag{7}$$

Note that ik appears nowhere in this array except in the μ. It follows that det $\underset{\sim}{D}(\mu)$ is an mth degree polynomial in μ with coefficients that do not depend on ik and therefore the roots of the polynomial do not depend on ik. Hence, the roots are of the form $\mu_j = O((ik)^0)$ for $1 \leq j \leq m$. It follows that the roots of det $(\underset{\sim}{A}(ik) - \alpha\underset{\sim}{1})$ are of the form

$$\alpha_j(k) = ikd_{jj} + O\,(ik)^0 \text{ for } 1 \leq j \leq M$$

where the d_{jj} are all real by hypothesis. Consequently, $\max_j \operatorname{Re} \alpha_j(k)$ is bounded above uniformly in k and thus we have stability in the sense of von Neumann at D_0 for all B.

Second, we prove that if $\underset{\sim}{\mathscr{A}}(\underset{\sim}{U}_0)$ is not hyperbolic then there exists a $\underset{\sim}{B}$ such that eqn. (4) is unstable at D_0 in the sense of von Neumann. There are two ways to violate hyperbolicity: case 1 is when the eigenvalues are not real; case 2 is when there is not a complete set of eigenvectors. Consider case 1: The Jordan canonical form for $\underset{\sim}{\mathscr{A}}(\underset{\sim}{U}_0)$ has the form

$$\underset{\sim}{J}(\underset{\sim}{U}_0) = \begin{bmatrix} \lambda_1 & b_1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \lambda_2 & b_2 & 0 & \cdot & \cdot & 0 \\ 0 & 0 & \lambda_3 & b_3 & 0 & \cdot & \cdot \\ \cdot & \cdot & 0 & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & & \cdot & & 0 \\ \cdot & \cdot & \cdot & & & \lambda_{m-1} & b_{m-1} \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 & \lambda_m \end{bmatrix} \tag{8}$$

where the $b_j = 0$ or 1 and the λ_j may not be distinct and at least one of the λ_j, say λ_1, is complex in case 1. Let $\lambda_1 = a + ib$ where a and $b \neq 0$ are real. Since the matrix $\underset{\sim}{\mathscr{A}}(\underset{\sim}{U}_0)$ was real, the complex conjugate $\bar{\lambda}_1 = a - ib$ is also an eigenvalue of $\underset{\sim}{\mathscr{A}}(\underset{\sim}{U}_0)$. Consider

$$p(\alpha) = \det\,[\,-\alpha\underset{\sim}{1} + \underset{\sim}{J}(\underset{\sim}{U}_0)ik + \underset{\sim}{E}(\underset{\sim}{U}_0,\underset{\sim}{U}_0')]$$

where $\underset{\sim}{E}$ is the result of applying the canonical transformation (the one that takes $\underset{\sim}{\mathscr{A}}(\underset{\sim}{U}_0)$ into $\underset{\sim}{J}(\underset{\sim}{U}_0)$) to $\underset{\sim}{C}$.

From Puiseux's Theorem (see Eichler [9] p. 118) it follows that

$$\alpha_j = ik\lambda_j + 0\,(k)^{p_j} \tag{9}$$

where $0 < p_j < 1 - 1/m_j$ with m_j being the multiplicity of λ_j.

Consider $\lambda_1 = a + ib$ with $b \neq 0$. Then

$$\mathrm{Re}\,(\alpha_1) = -\,kb + 0\,(k)^{p_1}$$

where $p_1 < 1$. Therefore Re (α_1) is not bounded above uniformly in k for $-\infty < k < \infty$. Thus if any of the eigenvalues of $\underset{\sim}{\mathscr{A}}(U_0)$ are complex then the system (4) is unstable at D_0 in the sense of von Neumann for all $\underset{\sim}{B}$.

Consider Case 2: If there is not a complete set of eigenvectors then $\underset{\sim}{J}(\underset{\sim}{U})$ has a 2 by 2 subblock of the form

$$\underset{\sim}{J}_s(\underset{\sim}{U}_0) = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} . \tag{10}$$

Now consider

$$p(\alpha) = \det\ [\ -\alpha \underset{\sim}{1} + \underset{\sim}{J}_s(\underset{\sim}{U}_0)ik + \underset{\sim}{C}_s(\underset{\sim}{U}_0, \underset{\sim}{U}_0)]$$

where $\underset{\sim}{C}_s$ is a subblock of $\underset{\sim}{C}$. Let

$$\underset{\sim}{C}_s = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} .$$

Then

$$p(\alpha) = \alpha^2 - \alpha(2\lambda ik + c_{11} + c_{22}) - \lambda^2 k^2 + ik[\lambda(c_{11} + c_{22}) - c_{21}]$$
$$+ c_{11}\, c_{22} - c_{12}\, c_{21}$$

and

$$\alpha_\pm(k) = \lambda ik \pm \frac{c_{11} + c_{22}}{2} \pm \sqrt{ik\ c_{21} + \left(\frac{c_{11} - c_{22}}{2}\right)^2 + c_{12}\ c_{21}}$$

Note that if λ is real and $c_{21} > 0$ then max Re $(\alpha_\pm\ (k))$ is not bounded uniformly in k for $-\infty < k < \infty$.

Q.E.D.

REMARK: Note that from the proof of Theorem 2 we have:

(i) If $\underset{\sim}{\mathscr{A}}(\underset{\sim}{U}_0)$ has all real eigenvalues and a complete set of eigenvectors then (4) is stable at D_0 in the sense of von Neumann for all $\underset{\sim}{B}$.

(ii) If $\underset{\sim}{\mathscr{A}}(\underset{\sim}{U}_0)$ has some complex eigenvalues then (4) is unstable at D_0 in the sense of von Neumann for all $\underset{\sim}{B}$.

(iii) If $\underset{\sim}{\mathscr{A}}(\underset{\sim}{U}_0)$ has all real eigenvalues but an incomplete set of eigenvectors then the stability at D_0 of (4) in the sense of von Neumann depends upon the structure of $\underset{\sim}{B}$ relative to $\underset{\sim}{\mathscr{A}}$.

1.4 BACKGROUND

In [4] it was shown that the characteristic polynomial for the equal-pressures model (with Assumption 4a) is

$$P_4(\lambda) \equiv (\lambda + u_g)^2 (\lambda + u_\ell)^2 - \Xi_g(\lambda + u_g)^2 - \Xi_\ell (\lambda + u_\ell)^2 \qquad (1)$$

where

$$\Xi_\xi = \rho_m c_m^2 \phi_\xi / \rho_\xi \qquad (2)$$

for $\xi = g, \ell$. This can be simplified by the Galilean transformation $u_g \to w$, $u_\ell \to -w$, where $w = (u_g - u_\ell)/2$. Then P_4 becomes

$$P_4(\lambda) \equiv (\lambda^2 - w^2)^2 - \Xi_g (\lambda + w)^2 - \Xi_\ell (\lambda - w)^2 \quad . \qquad (3)$$

Let

$$\Sigma \equiv (\Xi_g + \Xi_\ell)/2 \qquad (4)$$

and

$$\Delta \equiv (\Xi_\ell - \Xi_g)/2 \qquad (5)$$

and then P_4 may be written

$$P_4(\lambda) = \lambda^4 + a_2\lambda^2 + a_1\lambda + a_o \qquad (6)$$

where

$$a_2 = -2(w^2 + \Sigma)$$

$$a_1 = -4w\Delta$$

and $$a_o = w^2(w^2 - 2\Sigma) \quad .$$

The following lemma (which we call the W-lemma) turns out to be useful in determining when the roots of the quartics that arise in the stability analyses of these two-phase flow models are all real.

Lemma (W): Consider the real quartic

$$P(\lambda) = \lambda^4 + b_3\lambda^3 + b_2\lambda^2 + b_1\lambda + b_o \tag{7}$$

Necessary and sufficient conditions for P to have four real roots are that $P'(\lambda)$ have three real roots $r_L < r_M < r_R$ such that the following three inequalities are satisfied

$$P(r_L) < 0 \tag{8}$$

$$P(r_M) > 0 \tag{9}$$

and

$$P(r_R) < 0 \quad . \tag{10}$$

<u>PROOF SKETCH</u>: Consider the graph of a typical quartic. Roughly it has the shape of a W (hence the name of the lemma). The relative maximum corresponds to the central, upper vertex of the W and the relative minima correspond to the two lower vertices of the W. Note that inequalities (8), (9), (10) insure that either all four legs of the W are cut (therefore four distinct roots) by the horizontal axis (strict inequalities) or (nonstrict inequalities) the axis goes through a vertex (or vertices) and we have multiple roots but still all real roots. Q.E.D.

In the case of $P_4(\lambda)$ of eqn. (6) let

$$a = -(w^2 + \Sigma) \tag{11}$$

and

$$b = -w\Delta \tag{12}$$

and

$$\Lambda_k = a\lambda_k^2 + 3b\lambda_k + a_0 \tag{13}$$

where

$$\lambda_k = 2\left|\frac{a}{3}\right|^{\frac{1}{2}} \sin^{-1}\left(\alpha + \frac{2\pi}{3}k\right) \tag{14}$$

with

$$\alpha = \frac{1}{3}\sin^{-1}\left(\frac{b}{2}\left|\frac{3}{a}\right|^{\frac{3}{2}}\right) \tag{15}$$

then

$$[\Lambda_0 > 0] \text{ iff } [P_4(r_M) > 0] \tag{16}$$

and

$$[\Lambda_1 < 0 \text{ and } \Lambda_2 < 0] \text{ iff } [P_4(r_L) < 0 \text{ and } P_4(r_R) < 0] \tag{17}$$

was proved in [4]. Thus from the W-Lemma the conditions for four real roots are $\Lambda_0 > 0$, $\Lambda_1 < 0$, and $\Lambda_2 < 0$.

Wendroff [8] has proved that the Λ_1 and Λ_2 constraints are vacuous and the Λ_0 constraint is equivalent to

$$\left(\Xi_g^{1/3} + \Xi_\ell^{1/3}\right)^3 < 4w^2 = (u_g - u_\ell)^2 \tag{18}$$

which we shall refer to as Wendroff's inequality.

In summary Wendroff [8] has proved the following theorem.

THEOREM (WENDROFF): Consider the polynomial

$$p_4(\lambda) \equiv (\lambda^2 - w^2)^2 - \Xi_g(\lambda + w)^2 - \Xi_\ell(\lambda - w)^2 \tag{19}$$

where w is real and Ξ_ξ for $\xi = g, \ell$ are nonnegative. Then we have two cases in which the roots of p_4 are all real:

(i) $\Xi_g = 0$ or $\Xi_\ell = 0$ or $w = 0$

or

(ii) $\Xi_g > 0$, $\Xi_\ell > 0$, $w \neq 0$ and w, Ξ_g, and Ξ_ℓ satisfy Wendroff's inequality (18).

Therefore, when the relative velocity (w) and the volume fractions (ϕ_g, ϕ_ℓ) are nonzero then Wendroff's inequality (18) is necessary and sufficient for the sound speeds in the Equal-Pressures model to be real.

2.1 STATEMENT AND PROOFS OF RESULTS

First consider the Equal-Pressure Model 4b where it is not assumed that $\langle u\rangle^2 = \langle u^2\rangle$ but the energy equation is used to generate an evolution equation for $\langle u^2\rangle$ and then the approximation $\langle u^3\rangle \doteq \langle u\rangle^3$ is used in the energy equation. The model is expressed by

$$\rho_g + m_g' = \bar{S}_g^{\rho} \tag{1}$$

$$\dot{m}_g + 2K_g' + \phi_g \rho_m c_m^2 \left[\frac{\dot{\rho}_g}{\bar{\rho}_g} + \frac{\dot{\rho}_\ell}{\bar{\rho}_\ell}\right] = \bar{S}_g^{m} \tag{2}$$

$$\begin{aligned}\dot{K}_g &+ \rho_g \mathscr{E}_g + r_g \rho_m c_m^2 p\left[\frac{\dot{\rho}_g}{\bar{\rho}_g} + \frac{\dot{\rho}_\ell}{\bar{\rho}_\ell}\right] \\ &+ \left[3/2\, u_g^2 + \mathscr{E}_g + p/\bar{\rho}_g\right] m_g' - u_g\,(u_g^2 + p/\bar{\rho}_g)\,\rho_g' \\ &+ u_g\,(r_g p + \phi_g)\,\rho_m c_m^2\left[\frac{\rho_g'}{\bar{\rho}_g} + \frac{\rho_\ell'}{\bar{\rho}_\ell}\right] = \bar{S}_g^{\rho E}\end{aligned} \tag{3}$$

$$\dot{\rho}_\ell + m_\ell' = \bar{S}_\ell^{\rho} \tag{4}$$

$$\dot{m}_\ell + 2K_\ell' + \phi_\ell \rho_m c_m^2 \left[\frac{\dot{\rho}_g}{\bar{\rho}_g} + \frac{\dot{\rho}_\ell}{\bar{\rho}_\ell}\right] = \bar{S}_\ell^{m} \tag{5}$$

$$\begin{aligned}\dot{K}_\ell &+ \dot{\rho}_\ell \mathscr{E}_\ell + r_\ell \rho_m c_m^2 p\left[\frac{\dot{\rho}_g}{\bar{\rho}_g} + \frac{\dot{\rho}_\ell}{\bar{\rho}_\ell}\right] + [3/2\, u_\ell^2 + \mathscr{E}_\ell + p/\bar{\rho}_\ell]\, m_\ell' \\ &- u_\ell(u_\ell^2 + p/\bar{\rho}_\ell)\rho_\ell' + u_\ell(r_\ell p + \phi_\ell)\,\rho_m c_m^2\left[\frac{\rho_g'}{\bar{\rho}_g} + \frac{\rho_\ell'}{\bar{\rho}_\ell}\right] = \bar{S}_\ell^{\rho E}\end{aligned} \tag{6}$$

where $P = \bar{P}_g = \bar{P}_\ell$.

Let $U^T = (\rho_g, m_g, K_g, \rho_\ell, m_\ell, K_\ell)$ and then we can write the equations (1) - (6) as

$$A_1 \dot{\underset{\sim}{U}} + \underset{\sim}{A}_2 \underset{\sim}{U}' = \underset{\sim}{A}_3 \quad .$$

which can be reduced to the form $\dot{\underset{\sim}{U}} = \underset{\sim}{A}\underset{\sim}{U}' + \underset{\sim}{B}$. Then the eigenvalues for $\underset{\sim}{A}$ satisfy the polynomial

$$P_6(\lambda) = P_{3g}(\lambda)P_{3\ell}(\lambda) - P_{1g}(\lambda)P_{1\ell}(\lambda) \tag{7}$$

where

$$P_{3\xi}(\lambda) = \frac{\lambda^3}{2} + \lambda \frac{r_\xi pb}{\rho_\xi} - \frac{\phi_\xi b}{2\rho_\xi} - F_\xi - u_\xi^2/2$$

$$- u_\xi F_\xi + u_\xi (r_\xi p + \phi_\xi) \frac{b}{\bar{\rho}_\xi} \tag{8}$$

with

$$b = \rho_m c_m^2 \tag{9}$$

and

$$F_\xi = u_\xi^2 + p/\bar{\rho}_\xi \tag{10}$$

and where

$$P_{1\xi}(\lambda) = \left[\lambda\left(\frac{\phi_\xi}{2} - r_\xi p\right) - u_\xi (r_\xi p + \phi_\xi)\right] \frac{b}{\bar{\rho}_\xi} \tag{11}$$

Note that the coefficients of P_6 depend on c_ξ, p_ξ, u_ξ, ρ_ξ, and ϕ_ξ for $\xi = g, \ell$. A state of the material in our two-phase flow model is determined by these variables.

<u>DEFINITION</u>: We say that a state $(\bar{p}_g, \bar{p}_\ell, u_g, u_\ell, \rho_g, \rho_\ell, \phi_g, \phi_\ell, c_g, c_\ell)$ <u>is physically acceptable</u> provided p_ξ, u_ξ, ρ_ξ, ϕ_ξ, c_ξ are real numbers such that $c_\xi > 0$, $\rho_\xi > 0$, $0 < \phi_\xi < 1$, and $\phi_g + \phi_\ell = 1$ for $\xi = g, \ell$.

<u>RESULT 1</u>: The roots of $P_6(\lambda)$ given by eqn. (7) are not all real for all physically acceptable states.

<u>PROOF</u>: To simplify P_6 first make the following Galilean transformation $u_g \to w$, $u_\ell \to -w$ where $w = (u_g - u_\ell)/2$. Next consider the special case of equal phases: $f_g = f_\ell$ for $f = \rho$, c, and ϕ. Then note that the λ^3 and λ^1 terms in P_6 vanish. That is,

$$P_6(\lambda) \to \lambda^6 + \lambda^4 A_4 + \lambda^2 A_2 + A_0$$

in other words P_6 becomes symmetric about $\lambda = 0$. In order for P_6 to have 6 real roots it must cross (counting contacts as two crossings) the axis 6 times and it follows from symmetry that $P_6(0) < 0$ is required. Therefore the constant term in $P_6(\lambda)$, namely A_0, must satisfy $A_0 < 0$. By an elementary calculation we find

$$A_0 = -4w^2(w^2 - c^2)(w^2 + p/\rho).$$

We have $A_0 < 0$ satisfied if $w^2 = 0$. But if $w^2 \neq 0$ then the constraint becomes

$$c^2 < w^2 .$$

Therefore in the case $w^2 < c^2$, $P_6(\lambda)$ has at least two complex roots. Q.E.D.

It follows from RESULT 1 that the Equal-Pressures Model 4b has complex characteristics at certain physically acceptable states and therefore is not stable in the sense of von Neumann (by Theorem 2) for those states.

Next consider the Unequal-Pressures Model. The model is expressed by

$$\dot{\rho}_g + m'_g = \bar{S}^{\rho}_g \tag{12}$$

$$\dot{m}_g + (c_g^2 - u_g^2)\,\rho'_g + 2u_g m'_g - \phi'_\ell \mathscr{P}_g = \bar{S}^m_g \tag{13}$$

$$\dot{p}_\ell + m'_\ell = \bar{S}^{\rho}_\ell \tag{14}$$

$$\dot{m}_\ell + (c_\ell^2 - u_\ell^2)\,\rho'_\ell + 2u_\ell m'_\ell + \phi'_\ell \mathscr{P}_\ell = \bar{S}^m_\ell \tag{15}$$

and

$$\dot{\phi}_\ell + \hat{u}_\ell \phi'_\ell = v/d \tag{16}$$

where

$$\mathscr{P}_\xi = \bar{p}_\xi - \hat{p} - c_\xi^2 \rho_\xi$$

for $\xi = g, \ell$.

Let

$$\underset{\sim}{U}^T = (\rho_g, m_g, \rho_\ell, m_\ell, \phi_\ell) \tag{17}$$

and the Unequal-Pressures Model may be expressed by

$$\dot{\underset{\sim}{U}} = \underset{\sim}{A}\,\underset{\sim}{U}' + \underset{\sim}{B} \tag{18}$$

where $\underset{\sim}{A}$ is given by:

$$\underset{\sim}{A} = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ u_g^2 - c_g^2 & -2\mu_g & 0 & 0 & \mathscr{P}_g \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & u_\ell^2 - c_\ell^2 & -2\mu_\ell & -\mathscr{P}_\ell \\ 0 & 0 & 0 & 0 & -u_\ell \end{bmatrix} \tag{19}$$

and therefore

$$\lambda \underset{\sim}{1} - \underset{\sim}{A} = \begin{bmatrix} \lambda & 1 & 0 & 0 & 0 \\ c_g^2 - u_g^2 & -\lambda + 2\mu_g & 0 & 0 & -\mathscr{P}_g \\ 0 & 0 & \lambda & 1 & 1 \\ 0 & 0 & c_\ell^2 - u_\ell^2 & \lambda + 2u_\ell & +\mathscr{P}_\ell \\ 0 & 0 & 0 & 0 & \lambda + \hat{u}_\ell \end{bmatrix} \tag{20}$$

Note that by multiplying the second row by λ and subtracting $c_g^2 - u_g^2$ times the first row therefrom and doing a similar operation on the fourth row the array above reduces to the triangular array $\underset{\sim}{D}(\lambda)$:

$$\underset{\sim}{D}(\lambda) = \begin{bmatrix} \lambda & 1 & 0 & 0 & 0 \\ 0 & (\lambda + u_g)^2 - c_g^2 & 0 & 0 & -\lambda\mathscr{P}_g \\ 0 & 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & (\lambda + u_\ell)^2 - c_\ell^2 & \lambda\mathscr{P}_\ell \\ 0 & 0 & 0 & 0 & \lambda + \hat{u}_\ell \end{bmatrix} \quad (21)$$

It follows that

$$\det(\lambda \underset{\sim}{1} - \underset{\sim}{A}) = P_5(\lambda)$$

where

$$P_5(\lambda) = [(\lambda + u_g)^2 - c_g^2]\,[(\lambda + u_\ell)^2 - c_\ell^2](\lambda + \hat{u}_\ell). \quad (22)$$

<u>RESULT 2</u>: If $\hat{u}_\ell$ is a real-valued function of the state variables then the matrix $\underset{\sim}{A}$ given by eqn. (19) has real eigenvalues for all physically acceptable states. The five eigenvalues of $\underset{\sim}{A}$ are $-\hat{u}_\ell$, $-u_g \pm c_g$, and $-u_\ell \pm c_\ell$.

<u>PROOF</u>: The eigenvalues of $\underset{\sim}{A}$ are the roots of $P_5(\lambda)$ of eqn. (22) and it is clearly seen that the roots of $P_5(\lambda)$ are the values stated above. By the definition of physically acceptable states it follows that $-u_\xi \pm c_\xi$ are real for $\xi = g, \ell$. Q.E.D.

RESULT 3: For all physically acceptable states the matrix $\underset{\sim}{A}$ has the following eigenvectors:

(i) Associated with the eigenvalues $(-u_g \pm c_g)$ are the right eigenvectors $(-1, -u_g \pm c_g, 0, 0, 0)^T$.

(ii) Associated with the eigenvalues $(-u_\ell \pm c_\ell)$ are the right eigenvectors $(0, 0, -1, -u_\ell \pm c_\ell, 0)^T$.

(iii) If $\mathscr{C}_g$ and $\mathscr{C}_\ell$ are both nonzero then associated with the eigenvalue $-\hat{u}_\ell$ is the right eigenvector

$$(\mathscr{P}_g\mathscr{C}_\ell,\ \hat{u}_\ell\mathscr{P}_g\mathscr{C}_\ell,\ -\mathscr{P}_\ell\mathscr{C}_g,\ -u_\ell\mathscr{P}_\ell\mathscr{C}_g,\ \mathscr{C}_g\mathscr{C}_\ell)^T$$

where

$$\mathscr{P}_\xi = \overline{p}_\xi - \hat{\rho} - c_\xi^2\,\overline{\rho}_\xi$$

and

$$\mathscr{C}_\xi = c_\xi^2 - (u_\xi - \hat{u}_\ell)^2$$

for $\xi = g$ and ℓ.

(iv) If $\mathscr{P}_g = 0$ and $\mathscr{P}_\ell = 0$ then associated with the eigenvalue $-\hat{u}_\ell$ is the right eigenvector $(0, 0, 0, 0, 1)^T$.

(v) If $\mathscr{P}_g = 0$ and $\mathscr{P}_\ell \neq 0$ then associated with the eigenvalue $-\hat{u}_\ell$ is the right eigenvector $(0, 0, -\mathscr{P}_\ell, -\hat{u}_\ell\mathscr{P}_\ell, \mathscr{C}_\ell)^T$.

(vi) If $\mathscr{P}_\ell = 0$ and $\mathscr{P}_g \neq 0$ then associated with the eigenvalue $-\hat{u}_\ell$ is the right eigenvector $(\mathscr{P}_g, \hat{u}_\ell\mathscr{P}_g, 0, 0, \mathscr{C}_g)^T$.

(vii) The eigenvectors of (i) and (ii) are linearly independent.

(viii) The eigenvectors of (i), (ii) and (iii) are linearly independent provided

$$-\hat{u}_\ell \neq -u_\xi \pm c_\xi \tag{23}$$

for $\xi = g$ and ℓ.

(ix) The eigenvectors of (i), (ii), and (iv) are linearly independent.

(x) The eigenvectors of (i), (ii), and (v) are linearly indpendent provided $\mathscr{C}_\ell \neq 0$.

(xi) The eigenvectors of (i), (ii), and (vi) are linearly independent provided $\mathscr{C}_g \neq 0$.

(xii) The matrix A has a complete set of eigenvectors iff [$\mathscr{C}_g$ and $\mathscr{C}_\ell$ are both nonzero; or $\mathscr{P}_g$ and $\mathscr{P}_\ell$ are both zero; or $\mathscr{P}_g$ is zero and $\mathscr{C}_\ell$ is not; or $\mathscr{P}_\ell$ is zero and $\mathscr{C}_g$ is not].

PROOF: Elementary computations of the form $\lambda\vec{a} = \underset{\sim}{A}\vec{a}$ establish that the given vectors are the eigenvectors associated with the stated eigenvalues. To see the significance of (23) note that

$$\mathscr{C}_\xi = 0 \text{ iff } -\hat{u}_\ell = -u_\xi \pm c_\xi \tag{24}$$

for $\xi = g,\ell$.

The eigenvectors of (i) and (ii) are seen to be linearly independent because $\mathscr{C}_\xi > 0$ for $\xi = g,\ell$.

The eigenvectors of (iii) is independent of those of (i) and (iii) when (23) holds because of (24).

The eigenvector of (iv) is easily seen to be independent of those of (i) and (ii).

The eigenvector of (v) is independent of those of (i) and (ii) provided $\mathscr{C}_\xi \neq 0$.

The eigenvector of (vi) is independent of those of (i) and (ii) provided $\mathscr{C}_g \neq 0$. Q.E.D.

REMARK: RESULT 3 shows that $\underset{\sim}{A}$ has a complete set of eigenvectors if (23) holds. Since RESULT 2 says that the eigenvalues of $\underset{\sim}{A}$ are all real it follows that $\underset{\sim}{A}$ is hyperbolic if (23) holds. Then by Theorem 2 it follows that for

any definition of the source terms $\bar{S}^{\rho}_{\xi}$, $\bar{S}^{m}_{\xi}$ for $\xi = g$ and ℓ that satisfies Assumption 5 (source terms don't depend on derivatives), the Unequal-Pressures Model is stable in the sense of von Neumann if (23) holds. If (23) is violated then the stability depends upon how the source terms are defined. If we put the usual Lebesgue measure on each of the state variables and the usual product measure on the state space then the set of points in state space for which (23) is violated is a set of measure zero for any physically acceptable definition of $\hat{u}_{\ell}$ (e.g. $\hat{u}_{\ell} = u_{\ell}$ or $\hat{u}_{\ell} = \left(\frac{u_g + u_{\ell}}{2}\right)$ are physically acceptable definitions for $\hat{u}_{\ell}$). It then follows from RESULTS 2 and 3 that the matrix $\underset{\sim}{A}$ is hyperbolic a.e. (almost everywhere) in state space. From Theorem 2 it then follows that the Unequal-Pressures Model is stable in the sense of von Neumann a.e. for any physically acceptable definitions of $\hat{u}_{\ell}$, $\bar{S}^{\rho}_{\xi}$, and $\bar{S}^{m}_{\xi}$. One way to remove that set of measure zero where (23) is violated is simply to define $\hat{u}_{\ell}$ in such a way that (23) is never violated. However, this seems somewhat artificial. The alternative is to define $\hat{u}_{\ell}$ in some physically acceptable way and then do a full von Neumann stability analysis to see which definitions of the source terms lead to stability and which do not when (23) is violated. This latter alternative is beyond the scope of this report.

It is interesting to note that the Unequal-Pressures Model encounters this "nonhyperbolicity problem" only in the transonic regions. To clarify this remark, let $\hat{u}_{\ell} = (u_g + u_{\ell})/2$ and then make the Galilean transformation $u_g \to w$, $u_{\ell} \to -w$ where $w = (u_g - u_{\ell})/2$ and note that $\hat{u}_{\ell} \to 0$, $-u_g \pm c_g \to -w \pm c_g$ and $-u_{\ell} \pm c_{\ell} \to w \pm c_{\ell}$. By RESULT 3 the only time that the set of linearly independent eigenvectors of $\underset{\sim}{A}$ reduces from five to four is when $-\hat{u}_{\ell} = -u_{\xi} \pm c_{\xi}$ for $\xi = g$ or ℓ which for the case at hand reduces to $w = \pm c_{\xi}$. Thus if $w < \min_{\xi} c_{\xi}$ then we are in the subsonic region and $\underset{\sim}{A}$ is hyperbolic. If

$w > \max_{\xi} c_{\xi}$ then we are in the supersonic region and $\underset{\sim}{A}$ is hyperbolic. Only in the transonic region, i.e. only as $|w|$ passes through c_g or c_ℓ does the matrix $\underset{\sim}{A}$ lose its hyperbolicity.

Next consider the Unequal-Pressures Model for L-layered, stratified, two-phase flow or for a flow of a bubbly liquid:

$$\dot{\rho}_g + m'_g = \overline{S}^{\rho}_g \tag{25}$$

$$\dot{m}_g + (c_g^2 - u_g^2)\,\rho'_g + 2u_g m'_g - \phi'_\ell\, p_g = \overline{S}^{m}_g \tag{26}$$

$$\dot{\rho}_\ell + m'_\ell = S^{\rho}_\ell \tag{27}$$

$$\dot{m}_\ell + (c_\ell^2 - u_\ell^2)\,\rho'_\ell + 2u_\ell m'_\ell + \phi_\ell p_\ell = S^{m}_\ell \tag{28}$$

$$\dot{\phi}_\ell + \hat{u}_\ell \phi'_\ell = L\hat{v}/d$$

where L may be just a constant integer L = 1,2,3,... for L-layered, stratified, two-phase flow or L may be a function of ϕ_ℓ for the flow of a bubbly liquid. Note that if L does not depend on the derivatives of the primary variables, i.e. ρ_g, m_g, ρ_ℓ, m_ℓ, ϕ_ℓ, then the characteristic values (eigenvalues) are not affected and the characteristic polynomial is $P_5(\lambda)$ as defined in eqn. (22). Let's augment Assumption 5 to include this Assumption:

<u>ASSUMPTION 6</u>: S^{ρ}_{ξ}, S^{m}_{ξ}, $\hat{p}$, $\hat{v}$, $\hat{u}_\ell$, and L are assumed to be real-valued functions of the state variables p_ξ, ρ_ξ, m_ξ, and ϕ_ξ for $\xi = g, \ell$ but are assumed not to depend on their derivatives.

For concreteness let us take

$$\hat{u}_\ell = \phi_g u_g + \phi_\ell u_\ell \quad . \tag{30}$$

RESULT 4: Under Assumption 6, the Unequal-Pressures Model for L-layered, stratified flow ($L = 1,2,3,\ldots$) or for a bubbly liquid ($L = L(\phi_\ell)$) given by eqns. (25) - (30) has real characteristic values for all physically acceptable states and has a complete set of eigenvectors a.e. in state space.

PROOF: As observed in the proof of RESULT 2 the characteristic values are $-u_g \pm c_g$, $-u_\ell \pm c_\ell$, and $-u_\ell$. The set of points for which

$$-\hat{u}_\ell = -u_\xi \pm c_\xi \tag{31}$$

for $\xi = g$ or ℓ is a set of measure zero because it is a finite union of sets of lower dimension than the dimension of the state space. Q.E.D.

REMARK: Recall that in any wave propagation model the characteristic values are associated with the sound speeds or wave speeds. One of the most physically unacceptable things about the Equal-Pressures Model is the fact that it has "complex sound speeds." This problem is corrected in the Unequal-Pressures Model.

2.2 DISCUSSION OF RESULTS

In the case of the Unequal-Pressures Model recall the triangulated form (eqn. 21 of section 2.1) of $\lambda \underset{\sim}{1} - \underset{\sim}{A}$ when the evolution equation for the internal state variable is given by

$$\phi_\ell + \hat{u}_\ell \phi'_\ell = L\hat{v}/d \tag{1}$$

Recall that the triangulated form was named $\underset{\sim}{D}(\lambda)$ and that

$$\underset{\sim}{D}(\lambda) = \begin{bmatrix} \lambda & 1 & 0 & 0 & 0 \\ 0 & (\lambda + u_g)^2 - c_g^2 & 0 & 0 & \mathscr{P}_g \\ 0 & 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & (\lambda + u_\ell)^2 - c_\ell^2 & \mathscr{P}_\ell \\ 0 & 0 & 0 & 0 & \lambda + \hat{u}_\ell \end{bmatrix} \tag{2}$$

Recall that this matrix comes from the following system of five equations

$$\rho_g + m'_g = \overline{S}\rho_g \tag{3}$$

$$m_g + (c_g^2 - u_g^2)\,\rho'_g + 2u_g m'_g - \phi'_\ell \mathscr{P}_g = \overline{S}_g^m \tag{4}$$

$$\rho_\ell + m'_\ell = \overline{S}\rho_\ell \tag{5}$$

$$m_\ell + (c_\ell^2 - u_\ell^2)\,\rho'_\ell + 2u_\ell m'_\ell + \phi'_\ell \mathscr{P}_\ell = \bar{S}_\ell^m \tag{6}$$

and

$$\dot{\phi}_\ell + u_\ell \phi'_\ell = Lv/d \quad . \tag{7}$$

It is quite convenient to categorize the equations in this form; that is, in the form of eqns. (3), (4), (5), (6) plus an evolution equation for the internal state variable. Because as we change models only eqn. (7) and therefore only the fifth row of $\underset{\sim}{D}(\lambda)$ changes. Thus this provides a convenient framework within which to categorize the models. To illustrate this idea consider the following general first order evolution equation for the internal state variable

$$\dot{\phi}_\ell + a_1 \dot{\rho}_g + a_2 \rho'_g + a_3 \dot{m}_g + a_4 m'_g + a_5 \dot{\rho}_\ell$$

$$+ a_6 \rho'_\ell + a_7 \dot{m}_\ell + a_8 m'_\ell + a_9 \phi'_\ell = a_{10} \tag{8}$$

where a_i, $i = 1, 10$ may be functions of $\bar{p}_\xi$, c_ξ, ρ_ξ, m_ξ, and ϕ_ξ but not their derivatives. Replacing eqn. (7) with eqn. (8) causes the fifth row of $D(\lambda)$ to be replaced with

$$(a_1\lambda + a_2, a_3\lambda + a_4, a_5\lambda + a_6, a_7\lambda + a_8, \lambda + a_9) \quad . \tag{9}$$

Now in more generality than this is the general arbitrary order evolution equation for the internal state variable. But even so only the fifth row of $\underset{\sim}{D}(\lambda)$ is altered. So in complete generality let the fifth row of $\underset{\sim}{D}$ be designated simply

$$(D_{51}, D_{52}, D_{53}, D_{54}, D_{55}) \quad .$$

Then $D(\lambda) = (D_{11}D_{22}D_{33}D_{44})D_{55} - (D_{11}D_{22}D_{33}D_{45})D_{54}$

$$+ (D_{11}D_{22}D_{34}D_{45})D_{53} - (D_{11}D_{25}D_{33}D_{44})D_{52} + (D_{12}D_{25}D_{33}D_{44})D_{51} \quad . \qquad (10)$$

Using the facts that $D_{11} = \lambda = D_{33}$ and $D_{25} = -\lambda p_g$ and $D_{45} = \lambda p_\ell$ the expression for the characteristic polynomial of the system reduces to

$$P(\lambda) = (D_{22}D_{44})D_{55} - \lambda(D_{22}p_\ell)D_{54} + (D_{22}p_\ell)D_{53}$$

$$+ \lambda(p_g D_{44})D_{52} - (p_g D_{44})D_{51} \qquad (11)$$

where we have also used the facts $D_{34} = 1 = D_{12}$. Note that

$$D_{22} = (\lambda + u_g)^2 - c_g^2$$

and

$$D_{44} = (\lambda + u_\ell)^2 - c_\ell^2 \quad .$$

To further simplify these equations let us make the following Galilean transformation

$$u_g \to w, \; u_\ell \to -w \text{ where } w = (u_g - u_\ell)/2.$$

Then, the equations

$$D_{22} = (\lambda + w)^2 - c_g^2$$

$$D_{44} = (\lambda - w)^2 - c_\ell^2$$

summarizes all the information needed to generate the characteristic polynomial (eqn. (11)) for the arbitrary evolution equation for the internal state variable.

Next consider the equal-phases case where

$$f_g = f_\ell \text{ for } f = c,\ \rho,\ \phi,\ \bar{p}$$

then

$$D_{22} \to (\lambda + w)^2 - c^2$$

$$D_{44} \to (\lambda - w)^2 - c^2$$

$$\mathscr{P}_g \to -c^2\bar{\rho}$$

$$\mathscr{P}_\ell \to -c^2\bar{\rho}$$

where the assumption has been used that if $\bar{p}_g = \bar{p}_\ell$ then $\hat{p} = \bar{p}_g = \bar{p}_\ell$. Also note that

$$\begin{aligned} P(\lambda) \to\ & [(\lambda + w)^2 - c^2]\ [(\lambda - w)^2 - c^2]\ D_{55} \\ & - [(\lambda + w^2) - c^2]\ c^2\rho\ (D_{53} - \lambda D_{54}) \\ & + c^2\bar{\rho}\ [(\lambda - w)^2 - c^2]\ (D_{51} - \lambda D_{52}) = 0 \end{aligned} \tag{12}$$

Now let's illustrate the use of eqns. (11) and (12) in some examples:

First example: Consider the case where the evolution equation for the internal state variable has the form

$$\dot{\phi}_\ell + a_9\phi'_\ell = a_{10} \tag{13}$$

then this changes the $\underset{\sim}{D}(\lambda)$ of eqn. (2) only in the fifth position of the fifth row where

$$D_{55} = \lambda + \hat{u}_{\ell} \to \lambda + a_9 \quad .$$

Therefore the only requirement on ϕ-laws of the form (13) in order that they lead to real characteristics is that a_9 be real. Thus our categorization procedure quickly takes care of this simple case.

<u>Second Example</u>: We wish to emphasize this point: The Equal-Pressures Model is also expressible in the form of eqns. (3), (4), (5), (6), and (7) being replaced by the following ϕ-law:

$$\dot{\phi}_{\ell} - C_{\rho\ell}\dot{\rho}_{\ell} + C_{\rho g}\dot{\rho}_{g} = 0 \tag{14}$$

where

$$C_{\rho\xi} = \rho_m c_m^2 \phi_{\xi} r_{\xi} \phi_{\xi} / \bar{\rho}_{\xi} \tag{15}$$

for $\xi = g,\ell$ while $\xi = \ell,g$. That the Equal-Pressures Model may be put in this form is not entirely obvious; see [6] for a derivation of (14). Replacing (7) with (14) leads to $D_{55} = \lambda$, $D_{51} = C_{\rho g}\lambda$, $D_{53} = -C_{\rho\ell}\lambda$ and $D_{52} = D_{54} = 0$. Substituting these into (12) and cancelling out a λ leads to

$$P_4(\lambda) = [(\lambda + w)^2 - c^2]\,[(\lambda - w)^2 - c^2]$$

$$+ c^2\bar{\rho}C_{\rho}\,[(\lambda + w)^2 - 2c^2 + (\lambda - w)^2] \tag{16}$$

where $C_{\rho} = C_{\rho g} = C_{\rho\ell}$ in accordance with the equal-phase case.
Now by the symmetry of $P_4(\lambda)$ about $\lambda = 0$, in order for $P_4(\lambda)$ to have four real roots it is necessary that $P_4(0) > 0$ and this is equivalent to

$$(w^2 - c^2)^2 + 2c^2\bar{\rho}C_\rho\ (w^2 - c^2) > 0 \tag{17}$$

and this holds iff

$$1 - 2\bar{\rho}C_\rho < w^2/c^2 < 1 \tag{18}$$

Thus if (18) is violated then the models with a ϕ-law of the form

$$\dot{\phi}_\ell = C_{\rho\ell}\dot{\rho}_\ell - C_{\rho g}\dot{\rho}_g \tag{19}$$

has complex characteristics. Thus our categorization procedure quickly eliminates any ϕ-law of the form (19) from further consideration with respect to ϕ-laws which lead to models with real sound speeds.

<u>Third Example</u>: Consider a case where the evolution equation for the internal state variable involves higher order derivatives. For example, let the evolution equation be

$$\ddot{\phi}_\ell + a_1\dot{\phi}'_\ell + a_2\phi''_\ell + a_3\dot{\phi}_\ell + a_4\phi'_\ell = a_5 \tag{20}$$

where the a_i, $1 < i < 5$, are functions of p_ξ, ρ_ξ, m_ξ, and ϕ_ξ but not of their derivatives.

In doing a von Neumann stability analysis we consider solutions of the form

$$\underset{\sim}{U} = \underset{\sim}{U}_0 \exp\ (ikx + \alpha t) \tag{21}$$

so that $\dot{\underset{\sim}{U}} \to \alpha\underset{\sim}{U}$ and $U' \to ik\underset{\sim}{U}$ and α and ik are related to λ by $\lambda = \alpha/ik$. Thus $D(\lambda) \to D(\alpha, ik)$ where

$$D(\alpha,ik) = \begin{bmatrix} \alpha & ik & 0 & 0 & 0 \\ 0 & \begin{matrix}(\alpha + iku_g)^2 \\ -(ikc_g)^2\end{matrix} & 0 & 0 & \mathscr{P}_g \\ 0 & 0 & \alpha & ik & 0 \\ 0 & 0 & 0 & \begin{matrix}(\lambda + iku_\ell)^2 \\ -c_\ell^2\end{matrix} & \mathscr{P}_\ell \\ 0 & 0 & 0 & 0 & D_{55}(\alpha,ik) \end{bmatrix} \tag{22}$$

where

$$D_{55}(\alpha,ik) = \alpha^2 + a_1\alpha ik + a_2(ik)^2 + a_3\alpha + a_4(ik) \tag{23}$$

follows from (20) and (21).

Let $\alpha_i(k)$ be the roots of the polynomial det (D). By Theorem 1, stability requires that there exist a finite constant number B such that

$$\max \mathrm{Re}(\alpha_i(k)) < B \tag{24}$$

for all k. From the structure of D, the upper 4x4 block of D will yield only $\alpha_i(k)$ which are purely imaginary and therefore satisfy (24). Thus we have an easy test which provides a necessary condition to decide whether the form (20) leads to stable models according to whether or not the roots of $D_{55}(\alpha,ik)$ satisfy (24).

3. CONCLUDING REMARKS

Herein, the derivation procedure for a simple case of two-phase flow, namely stratified flow, was carefully studied. All of the approximations and assumptions made in constructing the model were enumerated and analyzed for their effect on the sound speeds (or wave speeds) of the model. This led to a conclusion as to which of the assumptions causes the sound speeds to become complex or imaginary: namely, the equal-pressures assumption. The relaxation of the requirement that the pressures in the different phases be equal led to a model (namely, the Unequal-Pressures model) with real sound speeds and this condition is necessary for stability in the sense of von Neumann. The analysis also suggests that the best way to categorize these models is by their evolution equations for their internal state variables. Categorizing the models this way leads to a rather simple and easy to check condition which is necessary for these models to be stable in the sense of von Neumann. Also a model with real sound speeds for two-phase flow of a bubbly liquid is suggested by an investigation of the extension of the Unequal-Pressures model for single layered flow to multi-layered flow.

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