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The Motion of a Loosely Fitting Piston in a Closed End Cylinder Containing a Perfect Gas

George W. Stone, George E. Reis, Peter C. Kaestner

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THE MOTION OF A LOOSELY FITTING PISTON IN A CLOSED END CYLINDER CONTAINING A PERFECT GAS

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ABSTRACT

The equations of motion are derived for a piston in a closed end cylinder containing a compressible perfect gas. These equations include the possibility of an arbitrary motion imposed on the cylinder. In the analysis, it is assumed that there is a clearance between the piston and the cylinder which is sufficiently small so that the flow in the gap is controlled by viscous forces. Computer code ØØPS was written which solves the derived equations of motion. Several cases are examined using the ØØPS code; the results are reported in this paper.

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SYMBOLS

A_c	Cross-sectional area of the cylindrical cavity
A_p	Cross-sectional area of the piston
c	Speed of sound in the fluid
f	Friction factor
g	Function describing the velocity profile in the channel
G	External forces acting directly on the piston
h	Channel width, i.e., the clearance between the piston and cylinder
k	Constant exponent in the equation relating the change in viscosity with temperature
ℓ	Length of the two dimensional channel
ℓ_p	Length of the piston
L	Distance measured relative to the cylinder, between the initial position of the piston and its location when impacting the closed end of the cylinder
M	Mach number
m_c	Mass of fluid in the chamber
m_p	Mass of the piston
n	Circumferential length of the piston
P	Pressure in the fluid
R	Gas constant
Rey	Reynolds number based on h, Rey = $\rho\bar{u}h/\mu$
t	Time
T	Temperature of the fluid
u	Velocity of the fluid in the channel with respect to the channel walls
U	Relative velocity of the channel walls
V_c	Total chamber volume
V_D	Chamber dead volume, i.e., the volume in the chamber not occupied by the piston when the piston impacts the closed end of the cylinder
W	Mass flow rate of the fluid out of the chamber
W_1	Mass flow rate of the fluid out of the chamber through a channel of unit depth and width h

W_η	Mass flow rate of the fluid through the channel in the positive direction per unit depth and width h
x	Displacement relative to the cylinder of the piston from its initial position
X	Displacement of the cylinder from its initial position in an inertial frame of reference
y	Distance measured normal to the channel walls from a point midway between the walls to a point in the fluid in the channel
γ	Ratio of the specific heats of the fluid
η	Axial distance along the channel measured positively from the entrance to the exit
μ	Coefficient of viscosity of the fluid
ρ	Density of the fluid
τ	Shear stress
τ_{w_U}	Linear component of the shear stress at the channel wall
τ_{w_g}	Shear stress at channel wall less the quantity τ_{w_U}

Subscripts

c	Refers to chamber properties
e	Refers to properties at the channel exit
i	Refers to initial properties
o	Refers to stagnation properties
p	Refers to the piston
w	Refers to the wall region, or to properties at the channel walls
1 & 2	Refer to the inner and outer channel walls respectively, or separate cross-sections of the channel, in which case station 1 is upstream of station 2.

Superscripts

—	Indicates averages across the channel
^	Indicates averages axially along the channel
*	Indicates choked flow conditions
·	Indicates differentiation with respect to time

THE MOTION OF A LOOSELY FITTING PISTON IN A CLOSED END CYLINDER CONTAINING A PERFECT GAS

Introduction

Although there are reports in the published literature concerning the motion of pistons in cylinders, none have been found which consider the motion of a piston in a closed end cylinder containing a compressible fluid with a gap between the piston and cylinder and where the cylinder is subjected to some arbitrary motion. This particular problem, which has application in the design of accelerometers and velocimeters and in situations where items are inserted into blind holes, is considered in the present report.

The problem of predicting the motion of the piston can be handled in a variety of ways, one of which is to consider the relative piston-cylinder motion as a volume change and to neglect any leakage of the fluid past the piston. However, under some circumstances the fluid leakage can greatly affect the motion of the piston because of the change in the mass of fluid ahead of the piston and through the viscous forces exerted on the sides of the piston. In such instances it becomes necessary to consider the mass flow past the piston. This has been done in Reference 1 for an incompressible fluid and in Reference 2 for a compressible fluid in a cylinder with both ends sealed.

The purpose of the present analysis is to determine the behavior of a piston in a cylinder with one end closed and containing a compressible fluid when the cylinder is subjected to arbitrary motions and where the flow past the piston is viscosity controlled.

The end products of the analysis are the equations of motion of the piston and a computer code ØØPS which solves these equations numerically. The present report does not contain a detailed discussion nor a listing of the ØØPS code. However, this code is operational for use on the CDC6600 computer and is available through the authors. Several representative problems are solved with ØØPS, the results of which are given in the form of tables and plots.

Theory

Consider the piston-cylinder system shown in Figure 1. This diagram gives the coordinate systems and dimensions used to describe the one dimensional axial displacements of the cylinder and piston. X is the displacement of the cylinder with respect to an inertial frame of reference, and x is the displacement of the piston relative to the cylinder. Both of these displacements are measured positive in the direction towards the closed end of the cylinder. The clearance h between the cylinder and the piston is assumed to be uniform and constant both around and along the piston.

The length of the piston is indicated as ℓ_p. η is the axial distance measured positively in the direction of fluid flow in the channel between the piston and cylinder from the channel entrance. The distance L is the maximum distance relative to the cylinder that the piston can travel in the positive x direction from its initial position before bottoming. Any volume in the chamber ahead of the piston when the piston is bottomed out is labelled dead volume. In Figure 1, the dashed lines indicate the initial positions of the piston and cylinder and the solid lines their displaced positions.

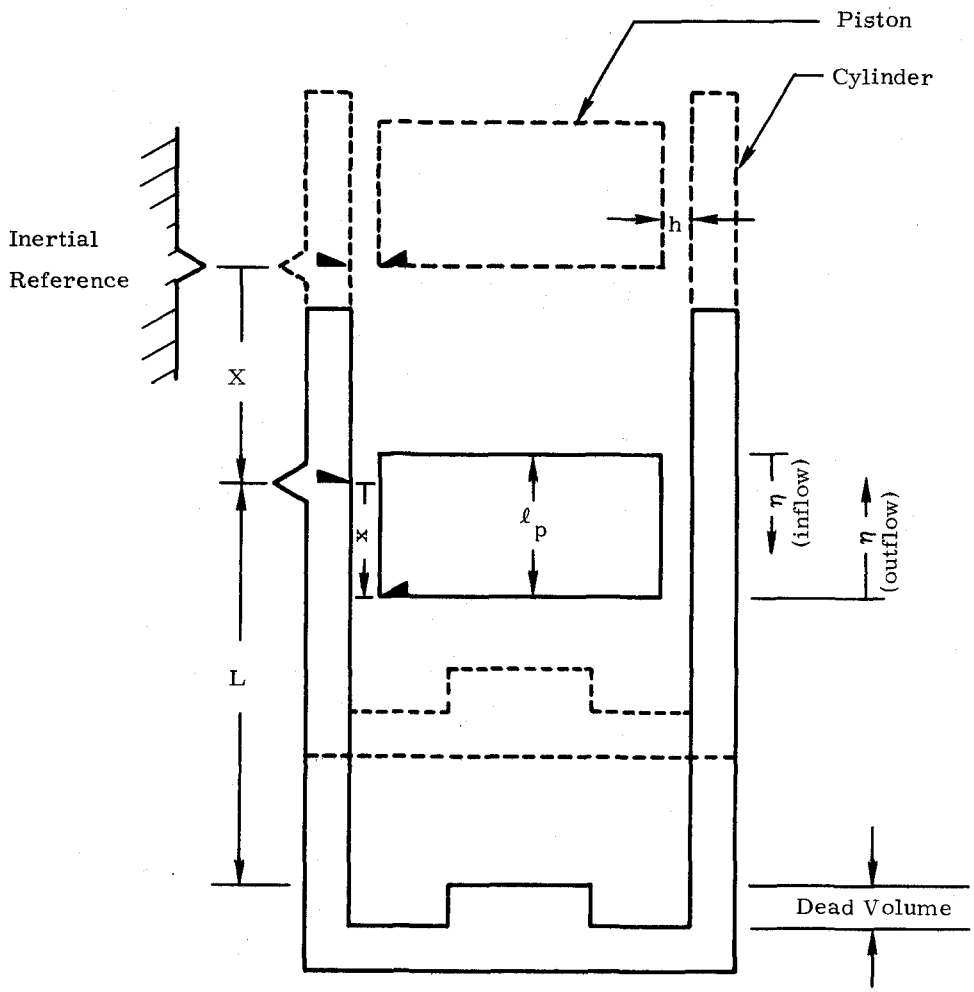

Figure 1. Coordinate systems used in the analysis

With this notation, the equation of motion of the piston in the inertial frame of reference is

$$m_p(\ddot{X}+\ddot{x}) = -(P_c-P_i)\,A_p - \hat{\tau}_{w_p}\,n\ell_p + G \tag{1}$$

where m_p is the mass of the piston, P_c is the pressure in the chamber (the region inside the cylinder and ahead of the piston), P_i is the initial pressure throughout the system, A_p is the cross-sectional area of the piston, n is the length of the perimeter of the piston, and $\hat{\tau}_{w_p}$ is the mean shearing stress

acting on the piston wall. The quantity G represents the external forces acting directly on the piston in the axial direction, e. g. weight, magnetic field, etc.

In this equation, $\hat{\tau}_{w_p}$ is considered positive for shearing forces which tend to move the piston away from the closed end of the cylinder and G is considered positive for forces which tend to move the piston towards the closed end of the cylinder.

In setting up this equation, it has been assumed that: (1) there are no inertial or viscous forces connected with the motion of the fluid either ahead of or behind the piston, (2) the fluid properties are uniform throughout the chamber region ahead of the piston, and (3) the properties of the fluid behind the piston are constant at their initial values.

The two quantities P_c and $\hat{\tau}_{w_p}$ occurring in Eq. (1) are dependent upon the relative motion of the piston and cylinder and are expressible by equations which are coupled to Eq. (1).

An expression for P_c/P_i is derived in Appendix A and for $\hat{\tau}_{w_p}$ in Appendix D. The equations for these two quantities are given in terms of the variables in Eq. (1), the original parameters of the problem, and a new variable W_1, the mass flow rate of the fluid out of the chamber, which in turn is derived in Appendix C.

The analysis produces a system of coupled equations whose exact forms depend on whether the Reynolds number associated with the flow of the fluid in the channel between the piston and the cylinder is greater or less than 1000, whether this flow is or is not choked, and whether the flow is out of ($P_c > P_i$) or into ($P_c < P_i$) the chamber ahead of the piston. The various combinations of these conditions give rise to eight different flow regimes.

Several equations and quantities have forms which are independent of the flow regime. These are listed in Table I. Other equations have forms which change from one flow regime to another. Table II lists these equations for unchoked flow and Table III for choked flow.

TABLE I

Equations Independent of Flow Regime

Force equation:

$$\ddot{X} + \ddot{x} = -\left(\frac{A_p P_i}{m_p}\right)\left(\frac{P_c}{P_i} - 1\right) - \left(\frac{n\ell_p}{m_p}\right)\hat{\tau}_{w_p} + \left(\frac{G}{m_p}\right) \quad . \tag{2}$$

Equation for the chamber pressure ratio:

$$\frac{1}{\gamma}\left(\frac{P_c}{P_i}\right)^{-1}\frac{d}{dt}\left(\frac{P_c}{P_i}\right) - \frac{1}{\left(L-x+\frac{V_D}{A_c}\right)}\dot{x} = -\frac{nRT_i W_1}{A_c\left(L-x+\frac{V_D}{A_c}\right)P_i}\left(\frac{P_c}{P_i}\right)^{\frac{-1}{\gamma}} \tag{3}$$

TABLE II

Unchoked Flow Equations

	Outflow ($P_c > P_i$)	Inflow ($P_c < P_i$)
Reynolds No.:	$\mathrm{Rey} = \frac{\lvert w_1 \rvert}{\mu_i}\left(\frac{P_c}{P_i}\right)^{-k\left(\frac{\gamma-1}{\gamma}\right)}$	$\mathrm{Rey} = \frac{\lvert w_1 \rvert}{\mu_i}$
Shear Stress:	$\hat{\tau}_{w_p} = \frac{\mu_i}{h}\left(\frac{P_c}{P_i}\right)^{k\left(\frac{\gamma-1}{\gamma}\right)} \dot{x} + \frac{hP_i}{2\ell_p}\left(\frac{P_c}{P_i} - 1\right)\left[1 - \frac{W_1^2 RT_i}{h^2 P_i^2}\left(\frac{P_c}{P_i}\right)^{-\frac{1}{\gamma}}\right]$	$\hat{\tau}_{w_p} = \frac{\mu_i}{h}\dot{x} - \frac{hP_i}{2\ell_p}\left(1 - \frac{P_c}{P_i}\right)\left[1 - \frac{W_1^2 RT_i}{h^2 P_i^2}\left(\frac{P_c}{P_i}\right)^{-1}\right]$
Mass Flow Rate: When Rey ≤ 1000:	$\lvert W_1 \rvert = \left[\frac{8\mu_i \ell_p \left(\frac{P_c}{P_i}\right)^{k\left(\frac{\gamma-1}{\gamma}\right)}}{h \ln\left(\frac{P_c}{P_i}\right)}\right]\left[-1 + \left\{1 + \frac{\left(\frac{h^4 P_i^2}{RT_i}\right)\left[\left(\frac{P_c}{P_i}\right)^2 - 1\right]\ln\left(\frac{P_c}{P_i}\right)}{128\mu_i^2 \ell_p^2 \left(\frac{P_c}{P_i}\right)^{(2K+1)\frac{\gamma-1}{\gamma}}}\right\}^{1/2}\right]$	$\lvert W_1 \rvert = \left[\frac{8\mu_i \ell_p}{h \ln\left(\frac{P_c}{P_i}\right)}\right]\left[1 - \left\{1 - \frac{\left(\frac{h^4 P_i^2}{RT_i}\right)\left[1 - \left(\frac{P_c}{P_i}\right)^2\right]\ln\left(\frac{P_c}{P_i}\right)}{128\mu_i^2 \ell_p^2}\right\}^{1/2}\right]$
When Rey ≥ 1000:	$\lvert W_1 \rvert = \left\{\frac{h^2 P_i^2}{2RT_i}\left(\frac{P_c}{P_i}\right)^{-\left(\frac{\gamma-1}{\gamma}\right)} \frac{\left[\left(\frac{P_c}{P_i}\right)^2 - 1\right]}{\left[0.016\left(\frac{\ell_p}{h}\right) + \ln\left(\frac{P_c}{P_i}\right)\right]}\right\}^{1/2}$	$\lvert W_1 \rvert = \left\{\frac{h^2 P_i^2}{2RT_i} \frac{\left[1 - \left(\frac{P_c}{P_i}\right)^2\right]}{\left[0.016\left(\frac{\ell_p}{h}\right) - \ln\left(\frac{P_c}{P_i}\right)\right]}\right\}^{1/2}$

TABLE III

Choked Flow Equations

	Outflow ($P_c > P_i$)	Inflow ($P_c < P_i$)
Reynolds No.:	$Rey = \frac{\lvert W_1 \rvert}{\mu_i}\left(\frac{P_c}{P_i}\right)^{-k\left(\frac{\gamma-1}{\gamma}\right)}$	$Rey = \frac{\lvert W_1 \rvert}{\mu_i}$
Exit Mach No.:	$(M^*)^2 \rightarrow \frac{1}{\gamma}$	$(M^*)^2 \rightarrow \frac{1}{\gamma}$
Shear Stress:	$\hat{\tau}_{w_p} = \frac{\mu_i}{h}\left(\frac{P_c}{P_i}\right)^{k\left(\frac{\gamma-1}{\gamma}\right)} \dot{x} + \frac{hP_i}{2\ell_p}\left(\frac{P_c}{P_i}\right)\left(1 - \frac{P_e^*}{P_c}\right)\left[1 - \frac{W_1^2 RT_i}{h^2 P_i^2} \frac{\left(\frac{P_c}{P_i}\right)^{-\left(\frac{\gamma+1}{\gamma}\right)}}{\left(\frac{P_e^*}{P_c}\right)}\right]$	$\hat{\tau}_{w_p} = \frac{\mu_i}{h}\dot{x} - \frac{hP_i}{2\ell_p}\left(1 - \frac{P_e^*}{P_i}\right)\left[1 - \frac{W_1^2 RT_i}{h^2 P_i^2}\left(\frac{P_e^*}{P_i}\right)^{-1}\right]$
Exit Pressure:	$\ln\left[\left(\frac{P_e^*}{P_c}\right)^2\right] - 1 + \frac{1}{\left(\frac{P_e^*}{P_c}\right)^2} = 2\hat{f}\left(\frac{\ell_p}{h}\right)$	$\ln\left[\left(\frac{P_e^*}{P_i}\right)^2\right] - 1 + \frac{1}{\left(\frac{P_e^*}{P_i}\right)^2} = 2\hat{f}\left(\frac{\ell_p}{h}\right)$
Friction Factor:	$\hat{f} = \frac{16}{Rey}$ for $Rey \leq 1000$	$\hat{f} = \frac{16}{Rey}$ for $Rey \leq 1000$
	$\hat{f} = 0.016$ for $Rey > 1000$	$\hat{f} = 0.016$ for $Rey > 1000$
Mass Flow Rate	$\lvert W_1 \rvert = \left\{\frac{h^2 P_i^2 \left(\frac{P_c}{P_i}\right)^{\frac{\gamma+1}{\gamma}}}{2RT_i} \frac{\left[1 - \left(\frac{P_e^*}{P_c}\right)^2\right]}{\left[\hat{f}\left(\frac{\ell_p}{h}\right) - \ln\left(\frac{P_e^*}{P_c}\right)\right]}\right\}^{1/2}$	$\lvert W_1 \rvert = \left\{\frac{h^2 P_i^2}{2RT_i} \frac{\left[1 - \left(\frac{P_e^*}{P_i}\right)^2\right]}{\left[\hat{f}\left(\frac{\ell_p}{h}\right) - \ln\left(\frac{P_e^*}{P_i}\right)\right]}\right\}^{1/2}$

Calculations

Since these equations are highly nonlinear and change form depending upon the various flow regimes, there seems to be little hope of developing useful analytic solutions to the system of equations. However, the behavior of systems which are represented approximately by the mathematical model used here can be determined by means of numerical calculations.

Computer code ØØPS was written and used on the CDC 6600 computer. Several representative problems were examined using ØØPS as illustrations of the type of results which can be obtained with this code.

One problem is concerned with dropping a large heavy cylinder into a hole in the ground. A second example considers the behavior of a small mass floating in a cylinder with gravity not acting on the piston but with the cylinder subjected to a sinusoidal motion. The last example is the same as the second except that gravity does act on the piston and the cylinder is subjected to a ramp motion. The data for these three cases are listed in Table IV, V, and VI. The results are shown in Figures 2, 3, and 4.

The accuracy of these calculations can be determined only by comparison with experiments which have not been conducted as yet.

TABLE IV

Calculated Data for a 900-kg Cylinder Falling in a Hole in the Ground

Piston length	= 6.0960 meters	20.0 feet
Piston diameter	= 0.8382 meters	33.0 inches
Piston mass	= 907.19 kilograms	62.16199 slugs
Chamber diameter	= 0.8763 meters	34.5 inches
Initial chamber length	= 48.768 meters	160.0 feet
Dead volume	= 3.1275 meters^3	110.4466 feet^3
Gap thickness	= 1.9050 centimeters	0.75 inches
Piston acceleration due to external forces	= 9.8067 meters/second^2	32.174 feet/second^2
Forced cylinder motion	= (none)	(none)

The working fluid is assumed to be air at sea level with the following properties:

Ratio of specific heats, γ	= 1.4	1.4
Coefficient of viscosity, μ	= 5.7552×10^{-3} $\frac{\text{Newton seconds}}{\text{meter}^2}$	1.202×10^{-5} $\frac{\text{pound second}}{\text{foot}^2}$
Initial chamber temp, T_i	= 14.99° Centigrade	518.67° Rankine
Gas constant, R	= 286.05 $\frac{\text{Joules}}{^\circ\text{Kelvin-kilogram}}$	1716.0 $\frac{\text{foot-lbs}}{\text{Slug-}^\circ\text{Rankine}}$
Initial pressure, P_i	= 1.0136×10^5 Newtons/meter^2	2117.0 pounds/foot^2
Exponent in μ-T equation	= 0.73	0.73

TABLE V

Calculated Data for a 4-Gram Piston Moving in a Case Which is Subjected to a Sinusoidal Acceleration

Piston length	= 1.00 centimeter	0.03281 feet
Piston diameter	= 1.00 centimeter	0.03281 feet
Piston mass	= 4.00 grams	2.741×10^{-4} slugs
Chamber diameter	= 1.01 centimeters	0.03314 feet
Initial chamber length	= 3.00 centimeters	0.09843 feet
Dead volume	= 244.2 centimeters^3	0.008624 feet^3
Gap thickness	= 0.05 centimeters	0.001640 feet
Piston acceleration due to external forces	= 0.0 centimeters/second^2	0.0 feet/second^2
Forced cylinder motion	= 5.0 Sin (6.2832t) centimeters	0.16404 Sin (6.2832t) feet

The working fluid is assumed to be air with the constants given in Table IV.

TABLE VI

Calculated Data for a 4-Gram Piston Moving in a Case Which is Subjected to a Ramp Displacement

Piston length	= 1.00 centimeter	0.03281 feet
Piston diameter	= 1.00 centimeter	0.03281 feet
Piston mass	= 4.00 grams	2.741×10^{-4} slugs
Chamber diameter	= 1.01 centimeters	0.03314 feet
Initial chamber length	= 3.00 centimeters	0.09843 feet
Dead volume	= 244.2 centimeters^3	0.008624 feet^3
Gap thickness	= 0.05 centimeters	0.0016404 feet
Piston acceleration due to external forces	= 9.8067 meters/second^2	32.174 feet/second^2
Forced cylinder motion	= 37.5 t centimeters	1.2303 t feet

The working fluid is assumed to be air with the same constants given in Table IV.

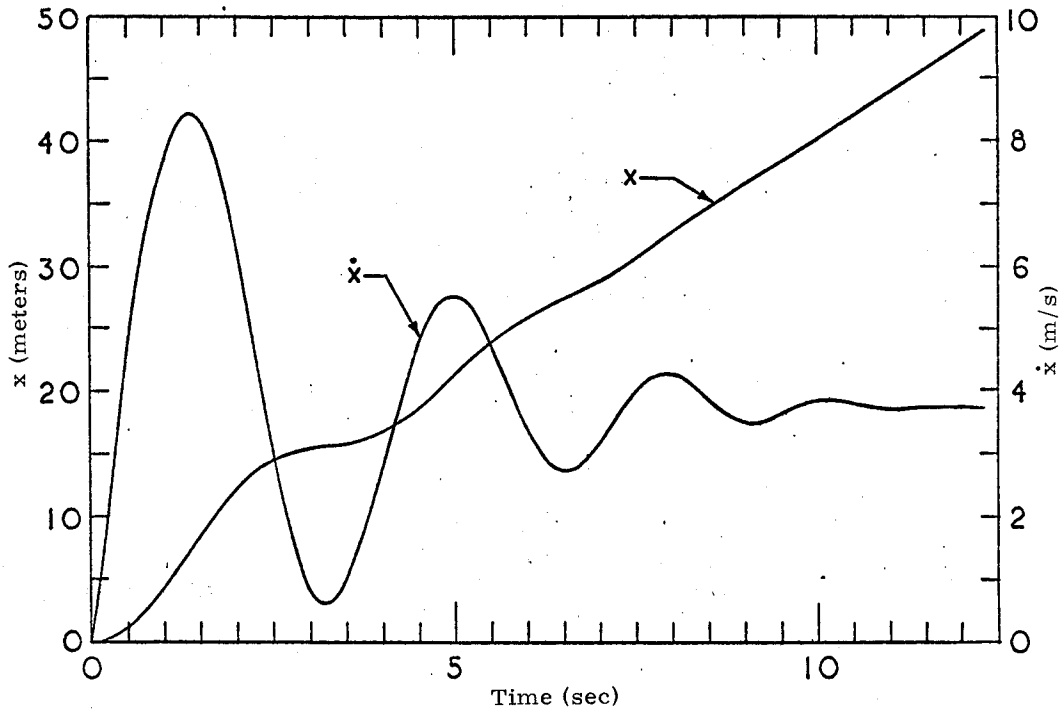

Figure 2. Variation of the displacement with time for a 900-kg cylinder falling in a hole in the ground

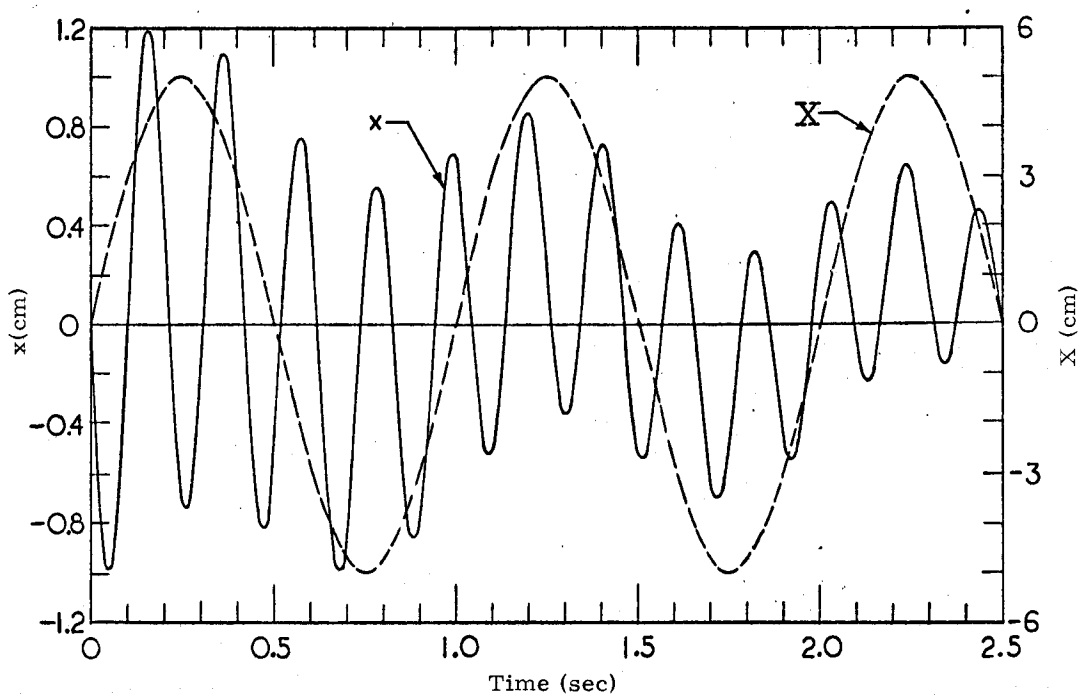

Figure 3. Variation of the displacement with time for a 4-gram piston moving in a cylinder which is subjected to sinusoidal motion

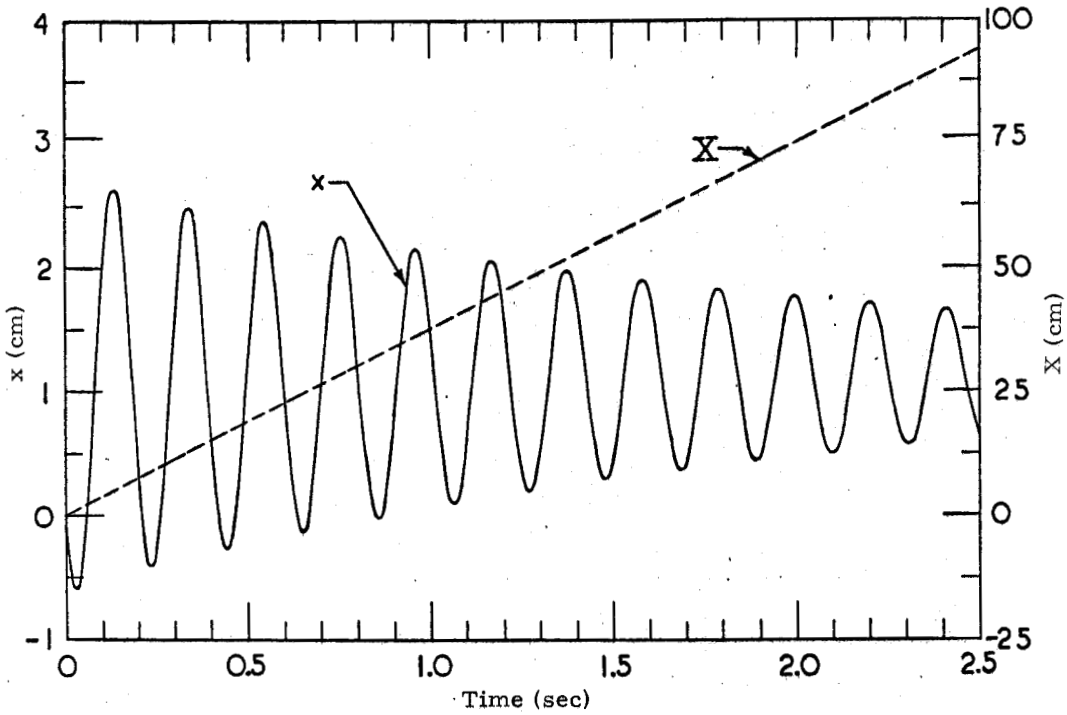

Figure 4. Variation of the displacement with time for a 4-gram piston moving in a cylinder which is subjected to a ramp displacement

APPENDIX A

ANALYSIS OF THE COMPRESSION IN THE CHAMBER

The fluid in the piston-cylinder device considered in this report is assumed to be a perfect gas with the equation of state

$$P = \rho RT \tag{A-1}$$

where P, ρ, R, and T are the pressure, density, gas constant, and temperature, respectively. If the subscript c is used to indicate the chamber properties, Eq. (A-1) is

$$P_c = \rho_c RT_c \quad . \tag{A-2}$$

Multiplying both sides of Eq. (A-2) by V_c, the volume of the chamber,

$$P_c V_c = (\rho_c V_c)\, RT_c = m_c RT_c \tag{A-3}$$

where m_c, the mass in the chamber, is

$$m_c = \rho_c V_c \quad . \tag{A-4}$$

Taking the natural logarithms of Eq. (A-3) and differentiating the result with respect to time gives

$$\frac{1}{P_c}\frac{dP_c}{dt} + \frac{1}{V_c}\frac{dV_c}{dt} = \frac{1}{m_c}\frac{dm_c}{dt} + \frac{1}{T_c}\frac{dT_c}{dt} \quad . \tag{A-5}$$

The term dm_c/dt in Eq. (A-5) is the mass flow rate into the chamber, which for convenience will be written as

$$\frac{dm_c}{dt} = -W = -nW_1 \tag{A-6}$$

where W is the mass flow rate whose positive direction corresponds to flow out of the chamber, n is the length of the perimeter of the piston, and W_1 is the mass flow rate out of the chamber per unit circumferential length of the piston. Substituting from Eqs. (A-3) and A-6) into (A-5) gives

$$\frac{1}{P_c}\frac{dP_c}{dt} + \frac{1}{V_c}\frac{dV_c}{dt} = -\frac{RT_c}{P_c V_c} W + \frac{1}{T_c}\frac{dT_c}{dt} \quad . \tag{A-7}$$

The volume in the chamber is

$$V_c = A_c(L-x) + V_D \tag{A-8}$$

where A_c is the cross-sectional area of the chamber, L is the maximum length of travel of the piston in the positive x direction relative to the cylinder, x is displacement of the piston relative to the cylinder, and V_D is the chamber dead volume. Substituting this expression for V_c into Eq. (A-7) yields

$$\frac{1}{P_c}\frac{dP_c}{dt} - \frac{A_c}{A_c(L-x)+V_D}\frac{dx}{dt} = -\frac{RT_cW}{[A_c(L-x)+V_D]P_c} + \frac{1}{T_c}\frac{dT_c}{dt} \quad . \tag{A-9}$$

For the analysis being presented here, it is assumed that the gas expands and compresses isentropically from its initial conditions whenever it is in the chamber. The pressure-temperature relationship for this process is given by

$$\frac{T_c}{T_i} = \left(\frac{P_c}{P_i}\right)^{\frac{\gamma-1}{\gamma}} \tag{A-10}$$

where T_i and P_i are the initial temperature and pressure of the system and γ is the ratio of the specific heats of the gas. The logarithmic time derivative of Eq. (A-10) is

$$\frac{1}{T_c}\frac{dT_c}{dt} = \frac{\gamma-1}{\gamma}\frac{1}{P_c}\frac{dP_c}{dt} \quad . \tag{A-11}$$

Substituting from this last equation into Eq. (A-9) gives

$$\frac{1}{P_c}\frac{dP_c}{dt} - \frac{A_c}{A_c(L-x)+V_D}\frac{dx}{dt} = -\frac{RT_iW}{[A_c(L-x)+V_D]P_c}\left(\frac{P_c}{P_i}\right)^{\frac{\gamma-1}{\gamma}} + \left(\frac{\gamma-1}{\gamma}\right)\frac{1}{P_c}\frac{dP_c}{dt} \tag{A-12}$$

Which can be rearranged to yield, using Eqs. (A-6) and A-10),

$$\frac{1}{\gamma}\left(\frac{P_c}{P_i}\right)^{-1}\frac{d}{dt}\left(\frac{P_c}{P_i}\right) - \frac{1}{\left(L-x+\frac{V_D}{A_c}\right)}\frac{dx}{dt} = -\frac{nRT_iW_1}{A_c\left(L-x+\frac{V_D}{A_c}\right)P_i}\left(\frac{P_c}{P_i}\right)^{\frac{-1}{\gamma}} \tag{A-13}$$

APPENDIX B

THE MOMENTUM EQUATION FOR FLOW PAST THE PISTON

In order to analyze the flow past the piston it is assumed that the channel width between the piston and the cylinder is small (usually less than one part in fifty) when compared to the other geometric dimensions of the system, especially the length of the piston. Under this assumption, the flow of the fluid in the channel is dominated by viscous forces and not by inertial forces. When this is the case, the viscous forces can be quite large and as a result the velocity profile across the channel changes slowly with distance along the channel.

When these conditions prevail, the flow past the piston can be considered from a one dimension flow viewpoint.[3] The flow leakage is also assumed to vary slowly with time so that the thermodynamic processes in the channel can be considered as being quasi-static. This implies that the "characteristic times" of any forced motion of the case must be several times longer than the time required for a disturbance to propagate through the fluid from one end of the system to the other. The transit time of such a disturbance is on the order of

$$\left[\ell_p + L + \frac{A_p}{\pi}\right]^{1/2} \Big/ c \tag{B-1}$$

where c is the velocity of sound in the fluid along the path followed by the disturbance but for practical purposes can be chosen as the ambient sound velocity when calculating the transit time given by this expression.

Assume a channel of constant width such as that shown in Figure B-1. It should be noted that when the piston moves inside the cylinder, which for problems of interest here is the usual case, one wall of the channel will be moving with respect to the other.

The momentum equation for the control volume shown in Figure B-1 can be written as

$$W_\eta d\bar{u} + h d\bar{P} + \left(\tau_{w_1} + \tau_{w_2}\right) d\eta = 0 \tag{B-2}$$

where τ_{w_1} and τ_{w_2} are the wall shearing stresses at the piston and the cylinder, respectively, and are not equal unless U, the velocity of the cylinder with respect to the piston, is zero. η is the axial distance along the channel measured positively from the channel entrance in the direction of flow in the channel; $\bar{u}$ and $\bar{p}$ are the mean velocity and pressure, respectively, of the flow in the channel averaged across the channel width h with $\bar{u}$ measured positively in the positive η direction; and W_η

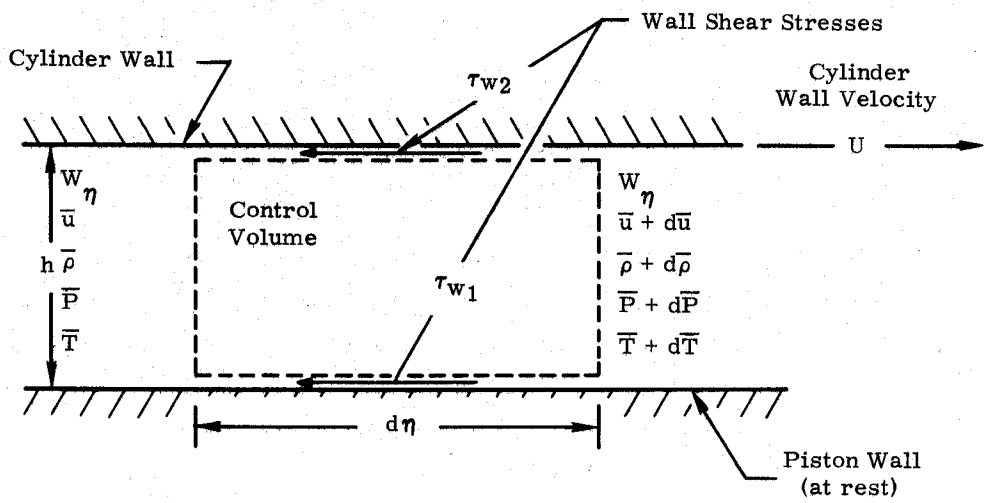

Figure B-1. Control volume used for the one-dimensional analysis of the flow in the channel

is the mass flow rate in the positive η direction per unit channel depth. The equation of continuity is then

$$W_\eta = \overline{\rho u} h \quad . \tag{B-3}$$

We define an average wall shear stress as

$$\bar{\tau}_w = \frac{\tau_{w_1} + \tau_{w_2}}{2} \tag{B-4}$$

By using Eqs. (B-3) and (B-4), Eq. (B-2) can be written

$$\bar{\rho}\bar{u}\bar{h}\, d\bar{u} + h\, d\bar{P} + 2\bar{\tau}_w\, d\eta = 0 \quad . \tag{B-5}$$

Taking the natural logarithm of each side of Eq. (B-3),

$$\frac{d\bar{\rho}}{\bar{\rho}} = -\frac{d\bar{u}}{\bar{u}} \quad . \tag{B-6}$$

Substituting from this last equation into Eq. (B-5) gives

$$-h\, u^{-2}\, d\bar{\rho} + h d\bar{P} + 2\bar{\tau}_w\, d\eta = 0 \quad , \tag{B-7}$$

and on substituting from Eq. (B-3) into (B-7),

$$-\frac{W_\eta^2}{h}\frac{d\bar{\rho}}{\bar{\rho}^2} + hd\bar{P} + 2\bar{\tau}_w\, d\eta = 0 \quad . \tag{B-8}$$

The magnitude of W_η is the same as the magnitude of W_1 given by Eq. (A-6) so that W_η^2 can be replaced by W_1^2 in Eq. (B-8). If it is assumed also that the density and pressure are constant across the channel, the bars over $\bar{\rho}$ and $\bar{P}$ can be dropped. Equation (B-8) then reduces to

$$-\frac{W_1^2}{h}\frac{d\rho}{\rho^2} + hdP + 2\bar{\tau}_w\, d\eta = 0 \quad . \tag{B-9}$$

In general, for engineering purposes the flow through ducts when the flow is controlled by viscosity involves the use of an experimentally determined friction factor f which does not appear explicitly in any of the foregoing equations. It is directly related to the wall shear stress $\bar{\tau}_w$ by the definition

$$\bar{\tau}_w = \left(\frac{1}{2}\bar{\rho}\bar{u}^2\right)f \quad . \tag{B-10}$$

APPENDIX C

FRICTION FACTOR AND MASS FLOW ANALYSIS

This part of the analysis begins with the momentum equation for the flow in the channel given by Eq. (B-8)

$$-\frac{W_1^2}{h}\frac{d\rho}{\rho^2} + hdP + 2\bar{\tau}_w d\eta = 0 \tag{C-1}$$

and the equation defining the friction factor, Eq. (B-10)

$$\bar{\tau}_w = \frac{1}{2}\rho u^2 \bar{f} \quad . \tag{C-2}$$

Substitution of Eq. (C-2) into Eq. (C-1) gives

$$-\frac{W_1^2}{h}\frac{d\rho}{\rho^2} + hdP + \rho u^2 \bar{f} d\eta = 0 \quad . \tag{C-3}$$

Multiplying Eq. (C-3) by ρ and substituting $(W_1/h)^2$ for $(\rho u)^2$ from Eq. (B-3), with $W_\eta^2 = W_1^2$ as in Eq. (B-9), gives

$$-\frac{W^2}{h}\frac{d\rho}{\rho} + h\rho dP + \frac{W_1^2}{h^2}\bar{f} d\eta = 0 \quad . \tag{C-4}$$

Divide by W_1^2/h to get

$$-\frac{d\rho}{\rho} + \frac{h^2}{W_1^2}\rho dP + \bar{f}\frac{d\eta}{h} = 0 \quad . \tag{C-5}$$

In order to obtain a pressure-density relation with which to integrate Eq. (C-5), it is necessary to assume an energy process for the flow. The two most common processes which are assumed for tube and channel flow are the isothermal and adiabatic behavior of the fluid. The choice between these two can be resolved as follows.

The isothermal assumption can be expressed as

$$\frac{T_0}{T} = 1 \tag{C-6}$$

where T_0 is the stagnation temperature of the fluid. This assumption is generally valid in long tubes and ducts when the fluid velocities are rather small.

The adiabatic assumption is generally applied to flows where the fluid velocity is large, for example, near the speed of sound in the fluid.

The adiabatic energy equation can be written in terms of the Mach number as

$$\frac{T_0}{T} = 1 + \frac{\gamma-1}{2} M^2 \tag{C-7}$$

where

$$M = \frac{U}{c} \tag{C-8}$$

and the velocity of sound c for a perfect gas is given by

$$c^2 = \gamma RT = \gamma \frac{P}{\rho} \quad . \tag{C-9}$$

It can be shown that the maximum mach number a flow can attain in adiabatic flow is unity at the exit for a flow that accelerates from rest through a constant area duct with no throats or nozzles. Applying this condition to Eq. (C-7) yields

$$\left.\frac{T_0}{T}\right]_{max} = \frac{\gamma+1}{2} \tag{C-10}$$

which for air with $\gamma = 1.4$ gives

$$\left.\frac{T_0}{T}\right]_{max} = 1.2 \quad . \tag{C-11}$$

A comparison of Eqs. (C-6) and (C-11) shows that the maximum difference in the temperature calculated with these two assumptions is only 20 percent in the extreme condition of choked flow. Because this difference is not severe and since physically the flow is neither isothermal nor adiabatic but somewhere in between, it will be assumed that the more easily calculated isothermal process is adequate for determining the mass flow in the channel.

Using the equation of state for the isothermal process

$$\frac{P}{\rho} = RT = \text{Constant} \tag{C-12}$$

to eliminate ρ and $d\rho$ from Eq. (C-5)

$$-\frac{dP}{P} + \frac{h^2}{W_1^2}\frac{P}{RT}\,dP + \frac{1}{h}\bar{f}\,d\eta = 0 \quad . \tag{C-13}$$

This equation can be integrated between the channel inlet and exit (stations 1 and 2 respectively) to get

$$-\ln\left(\frac{P_2}{P_1}\right) + \frac{h^2}{2W_1^2RT_1}\left(P_2^2 - P_1^2\right) + \hat{\bar{f}}\frac{\ell}{h} = 0 \tag{C-14}$$

where the temperature of the gas in the channel is assumed to be the temperature of the gas in the inlet reservoir, the average friction factor $\hat{\bar{f}}$ is defined as

$$\hat{\bar{f}} = \frac{1}{\ell}\int_0^{\ell} \bar{f}\,d\eta \quad , \tag{C-15}$$

and where ℓ is the length of the channel. Setting $\ell = \ell_p$ and solving Eq. (C-14) for W_1^2,

$$W_1^2 = \frac{\frac{h^2}{2RT_1}\left(P_1^2 - P_2^2\right)}{\hat{\bar{f}}\left(\frac{\ell_p}{h}\right) + \ln\left(\frac{P_1}{P_2}\right)} \quad . \tag{C-16}$$

As before our unchoked outflow ($P_c > P_i$), $P_1 = P_c$, $P_2 = P_i$, $T_1 = T_c$. If these conditions are applied to Eq. (C-16), then

$$|W_1| = \left\{\left(\frac{h^2}{2RT_c}\right)\frac{P_c^2 - P_i^2}{\left[\hat{\bar{f}}\left(\frac{\ell_p}{h}\right) + \ln\left(\frac{P_c}{P_i}\right)\right]}\right\}^{1/2} = \left\{\frac{h^2P_i^2\left[\left(\frac{P_c}{P_i}\right)^2 - 1\right]}{2RT_i\left(\frac{T_c}{T_i}\right)\left[\hat{\bar{f}}\left(\frac{\ell_p}{h}\right) + \ln\left(\frac{P_c}{P_i}\right)\right]}\right\}^{1/2}$$

$$= \left\{ \frac{h^2 P_1^2}{2RT_i \left(\frac{P_c}{P_i}\right)^{\frac{\gamma-1}{\gamma}}} \frac{\left[\left(\frac{P_c}{P_i}\right)^2 - 1\right]}{\left[\hat{f}\left(\frac{\ell_p}{h}\right) + \ln\left(\frac{P_c}{P_i}\right)\right]} \right\}^{1/2} . \tag{C-17}$$

For unchoked inflow ($P_c < P_i$), $P_1 = P_i$, $P_2 = P_c$ and $T_1 = T_i$, when they are inserted into Eq. (C-16)

$$|W_1| = \left\{ \left(\frac{h^2}{2RT_i}\right) \frac{\left(P_i^2 - P_c^2\right)}{\left[\hat{f}\left(\frac{\ell_p}{h}\right) + \ln\left(\frac{P_i}{P_c}\right)\right]} \right\}^{1/2} - \left\{ \frac{h^2 P_i^2}{2RT_i} \frac{\left[1 - \left(\frac{P_c}{P_i}\right)^2\right]}{\left[\hat{f}\left(\frac{\ell_p}{h}\right) - \ln\left(\frac{P_c}{P_i}\right)\right]} \right\}^{1/2} \tag{C-18}$$

For choked outflow ($P_c > P_i$), $P_1 = P_c$, $P_2 = P_e^*$ and $T_1 = T_c$. Then Eq. (C-16) becomes

$$|W_1| = \left\{ \left(\frac{h^2}{2RT_c}\right) \frac{\left(P_c^2 - P_e^{*2}\right)}{\left[\hat{f}\left(\frac{\ell_p}{h}\right) + \ln\left(\frac{P_c}{P_e^*}\right)\right]} \right\}^{1/2} = \left\{ \frac{h^2 P_i^2 \left(\frac{P_c}{P_i}\right)^2 \left[1 - \left(\frac{P_e^*}{P_c}\right)^2\right]}{2RT_i \left(\frac{T_c}{T_i}\right) \left[\hat{f}\left(\frac{\ell_p}{h}\right) - \ln\left(\frac{P_e^*}{P_c}\right)\right]} \right\}^{1/2}$$

$$= \left\{ \frac{h^2 P_i^2 \left(\frac{P_c}{P_i}\right)^{\frac{\gamma+1}{\gamma}}}{2RT_i} \frac{\left[1 - \left(\frac{P_e^*}{P_c}\right)^2\right]}{\left[\hat{f}\left(\frac{\ell_p}{h}\right) - \ln\left(\frac{P_e^*}{P_c}\right)\right]} \right\}^{1/2} . \tag{C-19}$$

For choked inflow ($P_c < P_i$), $P_1 = P_i$, $P_2 = P_e^*$ and $T_1 = T_i$. Then Eq. (C-16) is

$$|W_1| = \left\{ \left(\frac{h^2}{2RT_i}\right) \frac{\left(P_i^2 - P_e^{*2}\right)}{\left[\hat{f}\left(\frac{\ell_p}{h}\right) + \ln\left(\frac{P_i}{P_e^*}\right)\right]} \right\}^{1/2} = \left\{ \frac{h^2 P_i^2}{2RT_i} \frac{\left[1 - \left(\frac{P_e^*}{P_i}\right)^2\right]}{\left[\hat{f}\,\frac{\ell_p}{h} - \ln\left(\frac{P_e^*}{P_i}\right)\right]} \right\}^{1/2} \tag{C-20}$$

These equations have used the isentropic temperature-pressure relationship in the reservoirs at the ends of the channel, indicated by Eq. (A-10), to eliminate the temperature ratios. Expressions which relate P_e^*/P_c and P_e^*/P_i to the friction factor are given by Eqs. (E-26) and (E-27).

For unchoked flow, $\hat{\bar{f}}$ can be eliminated from Eqs. (C-17) and (C-18) by substituting from Eqs. (F-18), (F-19), F-23), and (F-24) under the appropriate conditions.

When the flow is out of the chamber and Rey ≤ 1000, Eqs. (F-18) and (F-24) apply. Then

$$\hat{\bar{f}} = \frac{16}{\mathrm{Rey}} \frac{16\mu_i}{W_1}\left(\frac{P_c}{P_i}\right)^{k\left(\frac{\gamma-1}{\gamma}\right)} \tag{C-21}$$

when this is inserted into Eq. (C-17) the equation for the mass flow rate becomes

$$\ln\left(\frac{P_c}{P_i}\right) |W_1|^2 + \left[\frac{16\mu_i \ell_p}{h}\left(\frac{P_c}{P_i}\right)^{k\left(\frac{\gamma-1}{\gamma}\right)}\right]|W_1| - \left(\frac{h^2 P_i^2}{2RT_i}\right)\left(\frac{P_c}{P_i}\right)^{-\left(\frac{\gamma-1}{\gamma}\right)}\left[\left(\frac{P_c}{P_i}\right)^2 - 1\right] = 0 \quad . \tag{C-22}$$

Solving this for $|W_1|$,

$$|W_1| \left[\frac{8\mu_i \ell_p \left(\frac{P_c}{P_i}\right)^{k\left(\frac{\gamma-1}{\gamma}\right)}}{h \ln\left(\frac{P_c}{P_i}\right)}\right]\left[-1 + \left\{1 + \frac{\left(\frac{h^4 P_i^2}{R T_i}\right)\left[\left(\frac{P_c}{P_i}\right)^2 - 1\right]\ln\left(\frac{P_c}{P_i}\right)}{128\mu_i^2 \ell_p^2 \left(\frac{P_c}{P_i}\right)^{(2K+1)\left(\frac{\gamma-1}{\gamma}\right)}}\right\}^{1/2}\right] . \tag{C-23}$$

If the flow is into the chamber and Rey ≤ 1000, Eqs. (F-18) and (F-23) apply. Where these are inserted into Eq. (C-18),

$$-\ln\left(\frac{P_c}{P_i}\right) |W_1|^2 + \left(\frac{16\mu_i \ell_p}{h}\right) |W_1| - \left(\frac{h^2 P_i^2}{2R T_i}\right)\left[1 - \left(\frac{P_c}{P_i}\right)^2\right] = 0 \quad . \tag{C-24}$$

From this

$$|W_1| = \left[\frac{8\mu_i \ell_p}{h \ln\left(\frac{P_c}{P_i}\right)}\right]\left[1 - \left\{1 - \frac{\left(\frac{h^4 P_i^2}{RT_i}\right)\left[1 - \left(\frac{P_c}{P_i}\right)^2\right]\ln\left(\frac{P_c}{P_i}\right)}{128\,\mu_i^2 \ell_p^2}\right\}^{1/2}\right]. \tag{C-25}$$

When the flow is out of the chamber and Rey > 1000, Eq. (F-19) applies and when inserted into Eq. (C-17),

$$|W_1| = \left\{\frac{h^2 P_i^2}{2RT_i}\left(\frac{P_c}{P_i}\right)^{-\left(\frac{\gamma-1}{\gamma}\right)} \frac{\left[\left(\frac{P_c}{P_i}\right)^2 - 1\right]}{\left[0.016\left(\frac{\ell_p}{h}\right) + \ln\left(\frac{P_c}{P_i}\right)\right]}\right\}^{1/2}. \tag{C-26}$$

When the flow is into the chamber and Rey > 1000, $\hat{\bar{f}}$ can be substituted as in Eq. (C-26) from Eq. (F-19), giving

$$|W_1| = \left\{\frac{h^2 P_i^2}{2RT_i} \frac{\left[1 - \left(\frac{P_c}{P_i}\right)^2\right]}{\left[0.016\left(\frac{\ell_p}{h}\right) - \ln\left(\frac{P_c}{P_i}\right)\right]}\right\}^{1/2}. \tag{C-27}$$

APPENDIX D

Shear Stress Flow Analysis

The momentum equation for flow in the channel is, from Eq. (B-9),

$$-\frac{W_1^2}{h}\frac{d\rho}{\rho^2} + hdP + 2\bar{\tau}_w\, d\eta = 0 \quad . \tag{D-1}$$

This can be integrated directly between two sections of the channel located a distance ℓ apart as follows.

$$-\frac{W_1^2}{h}\int_{\rho_1}^{\rho_2}\frac{d\rho}{\rho^2} + h\int_{P_1}^{P_2} dP + 2\int_0^{\ell}\bar{\tau}_w\, d\eta = 0 \tag{D-2}$$

so that

$$\frac{W_1^2}{h}\left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right) + h(P_2 - P_1) + 2\hat{\bar{\tau}}_w\ell = 0 \tag{D-3}$$

where the quantity $\hat{\bar{\tau}}_w$ is defined as

$$\bar{\tau}_w = \frac{1}{\ell}\int_0^{\ell}\bar{\tau}_w\, d\eta \quad . \tag{D-4}$$

From Eq. (D-3)

$$W_1^2 = \left[h^2(P_2 - P_1) + 2\hat{\bar{\tau}}_w h\ell\right]\left(\frac{\rho_2\rho_1}{\rho_2 - \rho_1}\right) \quad . \tag{D-5}$$

It should be noted that Eq. (D-5) is exact within the limits of the one-dimensional analysis being used here.

This last equation can also be written as

$$\hat{\bar{\tau}}_w = \frac{1}{2h\ell}\left[h^2(P_1-P_2) - W_1^2\left(\frac{\rho_1-\rho_2}{\rho_1\rho_2}\right)\right] \quad . \tag{D-6}$$

On replacing the ρ's by means of Eq. (A-1),

$$\hat{\bar{\tau}}_w = \frac{1}{2h\ell}\left[h^2(P_1-P_2) - W_1^2R\left(\frac{T_2}{P_2} - \frac{T_1}{P_1}\right)\right] \quad . \tag{D-7}$$

By making the assumption that the flow in the channel is isothermal ($T_2 = T_1$), Eq. (D-7) can be written as

$$\bar{\tau}_w = \frac{h}{2\ell}(P_1-P_2)\left[1 - \frac{W_1^2RT_1}{h^2P_1P_2}\right] \quad . \tag{D-8}$$

In these equations, station 1 is upstream of station 2.

For unchoked flow out of the chamber

$$\left.\begin{aligned} P_1 &= P_c \\ P_2 &= P_i \end{aligned}\right\} \quad . \tag{D-9}$$

By using the isentropic pressure-temperature relationship given by Eq. (A-10) for the gas in the chamber, the temperature T_1 can be written as

$$T_1 = T_c = T_i\left(\frac{P_c}{P_i}\right)^{\frac{\gamma-1}{\gamma}} \quad . \tag{D-10}$$

When these conditions are inserted into Eq. (D-8), noting also that ℓ is the length of the piston,

$$\hat{\bar{\tau}}_{w_{out}} = \frac{hP_i}{2\ell_p}\left(\frac{P_c}{P_i} - 1\right)\left[1 - \frac{W_1^2RT_i}{h^2P_i^2}\left(\frac{P_c}{P_i}\right)^{-\frac{1}{\gamma}}\right] \quad . \tag{D-11}$$

For unchoked flow into the chamber,

$$\left.\begin{aligned} P_1 &= P_i \\ P_2 &= P_c \\ T_1 &= T_i \end{aligned}\right\} \quad . \tag{D-12}$$

These conditions reduce Eq. (D-8) to

$$\hat{\tau}_{w_{in}} = \frac{hP_i}{2\ell_p}\left(1 - \frac{P_c}{P_i}\right)\left[1 - \frac{W_1^2 RT_i}{h^2 P_i^2}\left(\frac{P_c}{P_i}\right)^{-1}\right] \quad . \tag{D-13}$$

When the flow is choked, the exit pressure P_e^* is independent of the pressure downstream of the exit. Appendix E gives equations relating P_e^*/P_c for flow out of the chamber and P_e^*/P_i for flow into the chamber with the friction factor. Equation (D-8) can be put in terms of these pressure ratios as follows.

For choked flow out of the chamber,

$$\left.\begin{aligned} P_1 &= P_c \\ P_2 &= P_e^* \\ T_1 &= T_c \end{aligned}\right\} \tag{D-14}$$

so that Eq. (D-8) becomes

$$\hat{\tau}_{w_{out}} = \frac{h}{2\ell_p}\left(P_c - P_e^*\right)\left[1 - \frac{W_1^2 RT_c}{h^2 P_c P_e^*}\right] \quad . \tag{D-15}$$

If T_c is eliminated by means of Eq. (A-10)

$$\hat{\tau}_{w_{out}} = \frac{hP_i}{2\ell_p}\left(\frac{P_c}{P_i}\right)\left(1 - \frac{P_e^*}{P_c}\right)\left[1 - \frac{W_1^2 RT_i}{h^2 P_i^2}\,\frac{\left(\frac{P_c}{P_i}\right)^{-\left(\frac{\gamma+1}{\gamma}\right)}}{\left(\frac{P_e^*}{P_c}\right)}\right] . \tag{D-16}$$

For choked flow into the chamber

$$\left.\begin{aligned} P_1 &= P_i \\ P_2 &= P_e^* \\ T_1 &= T_i \end{aligned}\right\} \quad . \tag{D-17}$$

With these conditions, Eq. (D-8) becomes

$$\hat{\bar{\tau}}^*_{w_{in}} = \frac{hP_i}{2\ell_p}\left(1 - \frac{P_e^*}{P_i}\right)\left[1 - \frac{W_1^2 RT_i}{h^2 P_i^2}\left(\frac{P_e^*}{P_i}\right)^{-1}\right] \quad . \tag{D-18}$$

These expressions for $\hat{\bar{\tau}}_w$ given by Eqs. (D-11), (D-13), (D-16) and (D-18) all give rise to positive values of $\hat{\bar{\tau}}$ and represent the magnitude of shear force per unit wall area exerted on the fluid in the channel by the piston. The directions in which these forces act must be inferred from the direction of fluid flow in the channel.

The force exerted on the fluid by the piston due to the relative motion of the piston and cylinder walls is derived in Appendix F and is given by the first term on the right-hand-side of Eq. (F-15). In this equation, U is considered positive when the cylinder wall is moving in the same direction as the fluid flow in the channel if the piston is assumed to be at rest. Therefore, for the problem here

$$U = \dot{x} \tag{D-19}$$

when the flow is out of the chamber and

$$U = -\dot{x} \tag{D-20}$$

when the flow is into the chamber. When these two equations are combined with the appropriate equation for $\hat{\bar{\tau}}_w$ for choked or unchoked inflow or outflow, the forces exerted in the positive x direction by the piston wall on the fluid in the channel are as follows.

For unchoked outflow ($P_c > P_i$):

$$\hat{\tau}_{w_p} = \frac{\mu_w U}{h} + \hat{\bar{\tau}}_w = \frac{\mu_w \dot{x}}{h} + \hat{\bar{\tau}}_w = \frac{\mu_i}{h}\left(\frac{P_c}{P_i}\right)^{k\left(\frac{\gamma-1}{\gamma}\right)} \dot{x} + \hat{\bar{\tau}}_w$$

$$= \frac{\mu_i}{h}\left(\frac{P_c}{P_i}\right)^{k\left(\frac{\gamma-1}{\gamma}\right)} \dot{x} + \frac{hP_i}{2\ell_p}\left(\frac{P_c}{P_i} - 1\right)\left[1 - \frac{W_1^2 RT_i}{h^2 P_i^2}\left(\frac{P_c}{P_i}\right)^{-\frac{1}{\gamma}}\right] . \tag{D-21}$$

For unchoked inflow ($P_c < P_i$):

$$\hat{\tau}_{w_p} = -\left(\frac{\mu_w U}{h} + \hat{\bar{\tau}}_w\right) = \frac{\mu_i}{h}\dot{x} - \frac{hP_i}{2\ell_p}\left(1 - \frac{P_c}{P_i}\right)\left[1 - \frac{W_1^2 RT_i}{h^2 P_i^2}\left(\frac{P_c}{P_i}\right)^{-1}\right] . \tag{D-22}$$

For choked outflow ($P_c > P_i$):

$$\hat{\tau}_{w_p} = \frac{\mu_w U}{h} + \hat{\bar{\tau}}_w = \frac{\mu_i}{h}\left(\frac{P_c}{P_i}\right)^{k\left(\frac{\gamma-1}{\gamma}\right)} \dot{x}$$

$$+ \frac{hP_i}{2\ell_p}\left(\frac{P_c}{P_i}\right)\left(1 - \frac{P_e^*}{P_c}\right)\left[1 - \frac{W_1^2 RT_i}{h^2 P_i^2} \frac{\left(\frac{P_c}{P_i}\right)^{-\frac{\gamma-1}{\gamma}}}{\left(\frac{P_e^*}{P_c}\right)}\right] . \tag{D-23}$$

For choked inflow ($P_c < P_i$):

$$\hat{\tau}_{w_p} = -\left(\frac{\mu_w U}{h} + \hat{\bar{\tau}}_w\right) = \frac{\mu_i}{h}\dot{x} - \frac{hP_i}{2\ell_p}\left(1 - \frac{P_e^*}{P_i}\right)\left[1 - \frac{W_1^2 RT_i}{h^2 P_i^2}\left(\frac{P_e^*}{P_i}\right)^{-1}\right] . \tag{D-24}$$

In arriving at these final forms, μ_w has been put in terms of μ_i by substituting from Eqs. (F-21) and F-22).

APPENDIX E

Channel Entrance and Exit Conditions

Entrance Conditions

The fluid in the chamber is essentially at rest with respect to the cylinder and piston and must therefore accelerate while entering the channel due to the difference in pressures ahead of and behind the piston. However, it will be assumed that the velocity of the fluid just before entering the channel is sufficiently small that the properties of the fluid entering the channel are essentially those of the fluid in the chamber for flow out of the chamber and those of the fluid behind the piston for flow into the chamber. This is equivalent to assuming that there are no entrance losses. For this to be essentially the case, the ratio of the gap width to the piston radius should be small, say on the order of 1/10 or less. Under these conditions and using the subscript 1 to indiate inlet properties, for outflow ($P_c > P_i$),

$$P_1 = P_c \quad ; \qquad \rho_1 = \rho_c \quad ; \qquad \text{and} \qquad T_1 = T_c \quad . \tag{E-1}$$

and for inflow ($P_c < P_i$),

$$P_1 = P_i \quad ; \qquad \rho_1 = \rho_i \quad ; \qquad \text{and} \qquad T_1 = T_i \quad . \tag{E-2}$$

Exit Conditions

The exit conditions are not as simple as the input conditions since it is necessary to consider both the choked and unchoked flow regimes. The distinction between these regimes will be presented in terms of the critical exit Mach number M_e^*, i.e., the maximum Mach number which the flow can attain at the channel exit.

The momentum equation for the assumed isothermal flow through the channel is (Eq. (C-13)):

$$-\frac{dP}{P} + \frac{h^2}{W_1^2}\frac{P}{RT}\,dP + \frac{1}{h}\,\bar{f}\,d\eta = 0 \quad . \tag{E-3}$$

The isentropic speed of sound in a perfect gas is given by

$$c^2 = \gamma RT = \frac{\gamma P}{\rho} \tag{E-4}$$

which is a constant for isothermal flow.

The continuity equation for quasi-steady flow is (Eq. (B-3)):

$$\rho u = \frac{W_1}{h} = \text{constant} \tag{E-5}$$

and therefore

$$\rho^2 u^2 = \frac{W_1^2}{h^2} = \text{constant} \quad . \tag{E-6}$$

Dividing Eq. (E-6) by c^2 and substituting from Eq. (E-4) gives

$$\frac{\rho^2 u^2}{c^2} = \frac{W_1^2}{h^2 c^2} = \frac{W_1^2}{h^2 \gamma RT} = \text{constant} \tag{E-7}$$

which can be written, in terms of the Mach number M, as

$$\rho^2 M^2 = \frac{W_1^2}{\gamma h^2 RT} = \text{constant} \quad . \tag{E-8}$$

Again, using the isothermal equation of state in the form

$$\frac{P}{\rho} = RT = \text{constant} \tag{E-9}$$

to eliminate the density from Eq. (E-8) yields finally

$$P^2 M^2 = \frac{RT}{\gamma h^2} W_1^2 = \text{constant} \quad . \tag{E-10}$$

Taking the logarithmic derivative of Eq. (E-10) gives also

$$\frac{dP}{P} = -\frac{1}{2M^2} d(M^2) \tag{E-11}$$

The substitution of Eqs. (E-10) and (E-11) into Eq. (E-3) yields the momentum equation in terms of the Mach number in the form

$$\left(\frac{1}{\gamma M^4} - \frac{1}{M^2}\right) d(M^2) = 2\bar{f}\,\frac{d\eta}{h} \quad . \tag{E-12}$$

If Eq. (E-12) is divided through by $d\eta$, the right-hand-side of the resulting equation is always positive. This required that

$$\left(\frac{1}{\gamma M^4} - \frac{1}{M^2}\right) \quad \text{and} \quad \frac{d(M^2)}{d\eta} \quad \text{must have the same sign.}$$

This in turn implies that if

$$\left(\frac{1}{\gamma M^4} - \frac{1}{M^2}\right) < 0 \tag{E-13}$$

then

$$M^2 > \frac{1}{\gamma} \text{ and } \frac{d(M^2)}{d\eta} < 0 \quad , \tag{E-14}$$

and if

$$\left(\frac{1}{\gamma M^4} - \frac{1}{M^2}\right) > 0 \tag{E-15}$$

then

$$M^2 < \frac{1}{\gamma} \text{ and } \frac{d(M^2)}{d\eta} > 0 \quad . \tag{E-16}$$

From Eqs. (E-14) and (E-16), it can be seen that as η increases, M^2 asymptotically approaches the value $\frac{1}{\gamma}$.

This singular point of Eq. (E-12) yields the "choking" condition, indicated by the superscript *, that the Mach number at the exit

$$(M^*)^2 \rightarrow \frac{1}{\gamma}$$

When Eq. (E-12) is integrated over the length of the channel, where the subscripts 1 and 2 refer to the inlet and exit planes respectively, i. e.,

$$\frac{1}{\gamma}\int_{M_1^2}^{M_2^2} \frac{d(M^2)}{M^4} - \int_{M_1^2}^{M_2^2} \frac{d(M^2)}{M^2} = \frac{2}{h}\int_o^{\ell_p} \bar{f}\, d\eta \quad , \tag{E-18}$$

the result is

$$-\frac{1}{\gamma}\left[\frac{1}{M_2^2} - \frac{1}{M_1^2}\right] - \ln\left[\frac{M_2^2}{M_1^2}\right] = \frac{2}{h}\hat{\bar{f}}\,\ell_p \tag{E-19}$$

where $\hat{\bar{f}}$ is the average value of the friction factor along the channel given by

$$\hat{\bar{f}} = \frac{1}{\ell_p}\int_o^{\ell_p} \bar{f}\, d\eta \quad . \tag{E-20}$$

When the exit flow is choked

$$M_2 \approx M_2^* = \left(\frac{1}{\gamma}\right)^{1/2} \quad . \tag{E-21}$$

Substitution of this last equation into Eq. (E-19) yields

$$\ln\left[\gamma\left(M_1^*\right)^2\right] - 1 + \frac{1}{\gamma\left(M_1^*\right)^2} = 2\hat{\bar{f}}\left(\frac{\ell_p}{h}\right) \tag{E-22}$$

where M_1^* is the Mach number at the entrance for choked flow.

Inlet - Exit Pressure Ratios

Choked Flow

From Eq. (E-10)

$$P_1^2 M_1^2 = P_2^2 M_2^2 \tag{E-23}$$

which for the inlet and exit in choked flow yields

$$\left(\frac{P_2^*}{P_1^*}\right)^2 = \left(\frac{M_1^*}{M_2^*}\right)^2 = \gamma\left(M_1^*\right)^2 \quad . \tag{E-24}$$

When the Mach number is eliminated from Eq. (E-22) using Eq. (E-24),

$$\ln\left[\left(\frac{P_2^*}{P_1^*}\right)^2\right] - 1 + \frac{1}{\left(\frac{P_2^*}{P_1^*}\right)^2} = 2\hat{f}\left(\frac{\ell_p}{h}\right) \quad . \tag{E-25}$$

If $P_c > P_i$, the flow will be out of the chamber and the pressure at the inlet is essentially the chamber pressure P_c. If P_i^* is set equal to P_c and P_e^* is written for P_2^*, Eq. (E-25) becomes

$$\ln\left(\frac{P_e^*}{P_c}\right)^2 - 1 + \frac{1}{\left(\frac{P_e^*}{P_c}\right)^2} = 2\hat{f}\left(\frac{\ell_p}{h}\right) \quad . \tag{E-26}$$

If $P_c < P_i$, the flow will be into the chamber and the pressure at the inlet is essentially the initial pressure P_i. If P_1^* is set equal to P_i and P_e^* is written for P_2^*, Eq. (E-25) becomes

$$\ln\left[\left(\frac{P_e^*}{P_i}\right)^2\right] - 1 + \frac{1}{\left(\frac{P_e^*}{P_i}\right)^2} = 2\hat{f}\left(\frac{\ell_p}{h}\right) \quad . \tag{E-27}$$

Unchoked Flow

With regard to unchoked flow in the channel, it should be noted that as the piston begins to move inside the cylinder the pressure in the chamber varies slowly. This produces a flow in the channel whose initial Mach number is lower than M_e^*. Reference 4 has shown that under these conditions, i.e., when the flow is not choked, the exit pressure is essentially the back pressure. Therefore, for outflow ($P_c > P_i$)

$$\left.\begin{aligned} P_1 &\approx P_c \\ P_2 &= P_e = P_i \end{aligned}\right\} \tag{E-28}$$

and for inflow ($P_c < P_i$)

$$\left.\begin{aligned} P_1 &\approx P_i \\ P_2 &= P_e = P_c \end{aligned}\right\} . \tag{E-29}$$

APPENDIX F

The Effects of Friction

Qualitative Two-Dimensional Shear Stress Analysis

Consider the constant width channel shown in Fig. F-1.

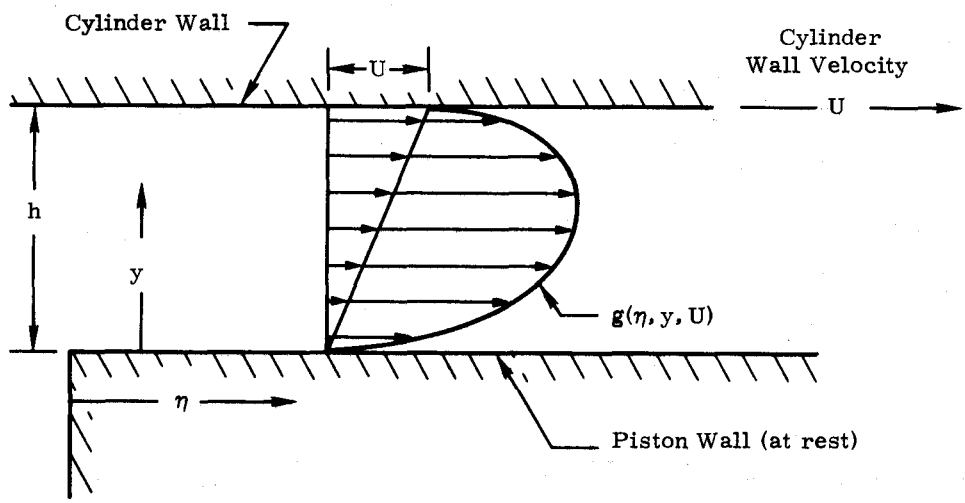

Figure F-1. Velocity profile for a one-dimensional flow in the channel

The velocity profile can be written as

$$u = \frac{Uy}{h} + g(\eta, y, U) \tag{F-1}$$

where $g(\eta, y, U)$ is some unknown function and from a consideration of continuity for compressible flow it must depend on U, the relative velocity of wall 2 with respect to wall 1.

The local wall shearing stress for a newtonian fluid is given by

$$\tau_w = \pm \mu_w \left.\frac{\partial u}{\partial y}\right]_w \tag{F-2}$$

where the $\pm$ sign depends upon where the velocity gradient is evaluated. Differentiating Eq. (F-1) with respect to y when U is constant gives

$$\frac{\partial u}{\partial y} = \frac{U}{h} + \frac{\partial g}{\partial y} \quad . \tag{F-3}$$

Substituting for $\partial u/\partial y$ from this last equation into the preceding one,

$$\tau_{w_1} = \mu_w \frac{U}{h} + \mu_w \left.\frac{\partial g}{\partial y}\right]_{y=0} \quad . \tag{F-4}$$

and

$$\tau_{w_2} = -\mu_w \frac{U}{h} - \mu_w \left.\frac{\partial g}{\partial y}\right]_{y=h} \quad . \tag{F-5}$$

If U=0, the velocity profile $g(\eta, y, 0)$ is symmetric about the channel centerline, $y = \frac{h}{2}$. Then, under this condition,

$$\left.\frac{\partial g(\eta, y, 0)}{\partial y}\right]_{y=0} = -\left.\frac{\partial g(\eta, y, 0)}{\partial y}\right]_{y=h} \quad . \tag{F-6}$$

It will be assumed that this symmetric property holds approximately for $U \neq 0$, i.e.,

$$\left.\frac{\partial g}{\partial y}\right]_{y=0} = -\left.\frac{\partial g}{\partial y}\right]_{y=h} \tag{F-7}$$

for all values of U.

We now define two components of the shear stress

$$\tau_{w_U} = \mu_w\left(\frac{U}{h}\right) \quad \text{and} \quad \tau_{w_g} = \mu_w \left.\frac{\partial g}{\partial y}\right]_{y=0} \tag{F-8}$$

Using Eq. (F-8), the wall shearing stresses, Eqs. (F-4) and (F-5), can be written

$$\tau_{w_1} = \tau_{w_U} + \tau_{w_g} \tag{F-9}$$

and

$$\tau_{w_2} = -\tau_{w_U} + \tau_{w_g} \quad . \tag{F-10}$$

An average shear stress $\bar{\tau}_w$ can be defined by combining Eqs. (F-9) and (F-10) to obtain

$$\bar{\tau}_w = \frac{\tau_{w_1} + \tau_{w_2}}{2} = \tau_{w_g} \quad . \tag{F-11}$$

The average of $\bar{\tau}_w$ over the length of the channel is

$$\hat{\bar{\tau}}_w = \frac{1}{\ell_p}\int_0^{\ell_p} \bar{\tau}_w d\eta = \frac{1}{\ell_p}\int_0^{\ell_p} \tau_{w_g} d\eta \quad . \tag{F-12}$$

If wall 1 is assumed to be the piston wall, then the mean shear stress acting on the fluid at the piston wall is

$$\hat{\tau}_{w_p} = \frac{1}{\ell_p}\int_0^{\ell_p} \tau_{w_1} d\eta \quad . \tag{F-13}$$

If $\mu_w(U/h)$ from Eq. (F-8) is substituted for τ_{w_U} in Eq. (F-9) and the result then substituted for τ_{w_1} in Eq. (F-13),

$$\hat{\tau}_{w_p} = \frac{1}{\ell_p}\int_0^{\ell_p} \frac{\mu_w U}{h} d\eta + \frac{1}{\ell_p}\int_0^{\ell_p} \tau_{w_g} d\eta \quad . \tag{F-14}$$

Because the flow has been assumed isothermal, μ_w is independent of η and the first integral in Eq. (F-14) can be evaluated directly. The second integral has been defined as $\hat{\bar{\tau}}_w$ from Eq. (F-12). Therefore

$$\hat{\tau}_{w_p} = \left(\frac{\mu_w U}{h}\right) + \hat{\bar{\tau}}_w \quad . \tag{F-15}$$

The dependence of the viscosity on temperature is assumed to be of the form

$$\frac{\mu}{\mu_i} = \left(\frac{T}{T_i}\right)^k \tag{F-16}$$

where k is dependent upon the fluid under consideration and the range of temperatures involved. The quantities μ_i and T_i are reference values of viscosity and temperature and are written here as the initial values of the system.

Experimental Data for the Friction Factor

Due to the mechanical difficulty of moving two flat plates relative to one another with a pressure gradient in the direction of motion for any appreciable time in order to obtain fully developed flow, very little experimental data has appeared in the literature for the flow between moving walls with a pressure gradient. [In fact, the only data known to the authors are concerned with fixed walls.] In order to get around this problem, it is assumed here that the dependence of the function $g(\eta, y, U)$ on U is slight, with the result that the friction factor $\hat{\bar{f}}$ defined by Eq. (F-15) is

$$\hat{\bar{f}}_{U=0} \simeq \bar{f}_{U\neq 0} = \hat{\bar{f}} \quad . \tag{F-17}$$

The experimental data used here were taken from Ref. 5. A plot of the friction factor versus Reynolds number given there is shown in Figure F-2. These data can be roughly approximated by

$$\hat{\bar{f}} = \frac{16}{\text{Rey}} \qquad \text{for} \qquad \text{Rey} \leq 1000 \tag{F-18}$$

and

$$\hat{\bar{f}} = 0.016 \quad \text{for} \quad \text{Rey} > 1000 \quad , \tag{F-19}$$

where the Reynolds number, Rey, is

$$\text{Rey} = \frac{\rho u h}{\mu_w} = \frac{|W_1|}{\mu_w} \quad . \tag{F-20}$$

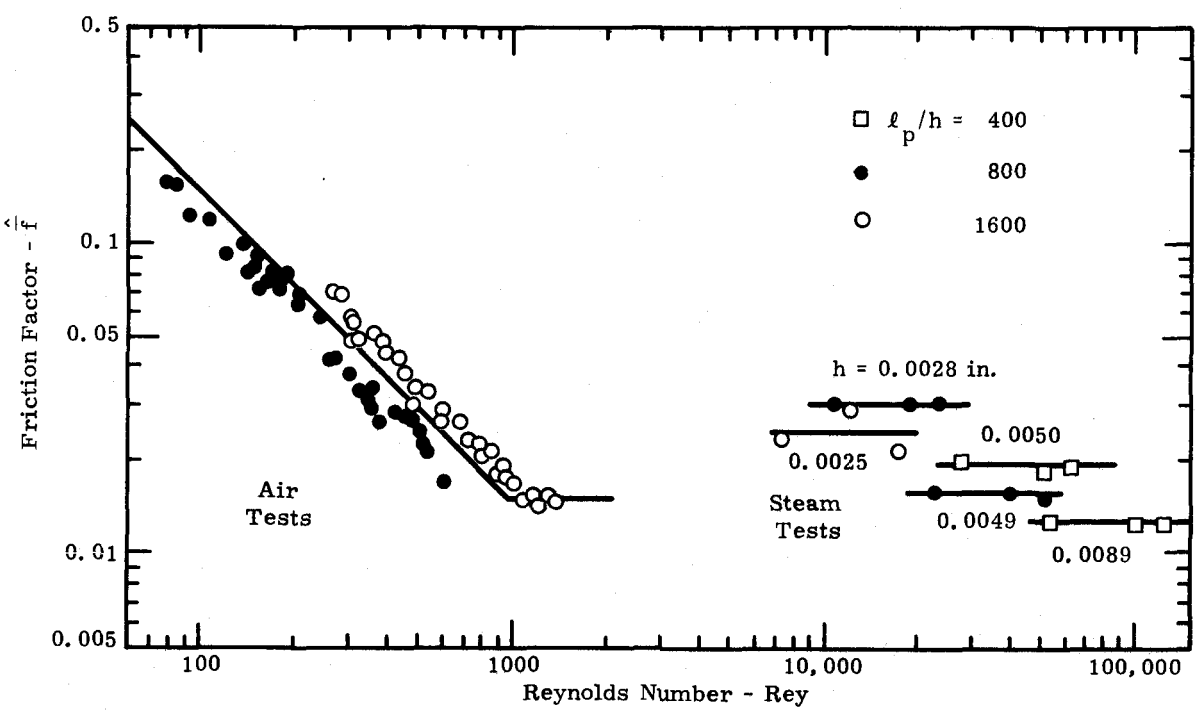

Figure F-2. Variation of friction factor with Reynolds number for channel flow

If it is assumed that the temperature of the walls in the channel are at the temperature of the gas in the volume from which the fluid is flowing, then for flow into the chamber, Eq. (F-16) becomes

$$\frac{\mu_w}{\mu_i} = \left(\frac{T_i}{T_i}\right)^k = 1 \quad . \tag{F-21}$$

If the flow is out of the chamber,

$$\frac{\mu_w}{\mu_i} = \left(\frac{T_c}{T_i}\right)^k = \left(\frac{P_c}{P_i}\right)^{k\left(\frac{\gamma-1}{\gamma}\right)} \tag{F-22}$$

where the adiabatic pressure-temperature relationship has been used to replace the temperature ratio with the pressure ratio. If these last two equations are used to eliminate μ_w from Eq. (F-20), then for inflow

$$\text{Rey} = \frac{|W_1|}{\mu_i} \tag{F-23}$$

and for outflow

$$\text{Rey} = \frac{|W_1|}{\mu_i}\left(\frac{P_c}{P_i}\right)^{-k\left(\frac{\gamma-1}{\gamma}\right)} \quad . \tag{F-24}$$

References

1. Schmidt, G. L., Theoretical Analysis of the Dynamics of the Variable-Orifice Velocimeter, Sandia Corporation SC-4383(WD), Albuquerque, New Mexico, October 1959, Confidential.

2. Bennett, M. D., Analysis of the Motion of the Air-Bearing Velocimeter, A Single-Integrating Accelerometer, Sandia Corporation, SCTM 456-58(51), Albuquerque, New Mexico, January 6, 1959.

3. Kuethe, A. M., and Schetzer, J. D., Foundations of Aerodynamics, John Wiley and Sons Inc., Third Printing, 1957, p. 135.

4. Shapiro, A. H., The Dynamics and Thermodynamics of Compressible Fluid Flow, Ronald Press Co., New York, 1953, Vol. 1, Chapter 6, p. 159.

5. Egli, A., The Leakage of Gases Through Narrow Channels, ASME Transactions, Journal of Applied Mechanics, Volume 4, Number 2, June 1937, p. A-63.

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