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A Simple Constitutive Description for Cellular Concrete

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ABSTRACT

A simple constitutive description of cellular concrete is presented. It is a non-thermodynamic first approximation and is certainly not meant to be all-inclusive. An elastic-plastic description is used with a yield surface which is a surface of revolution with a planar end cap. The plane cap, normal to the hydrostat in principal stress space, is movable due to volumetric work hardening. Plasticity is handled in two parts; a volumetric part and a deviatoric part. The deviatoric plastic strain rate vector is taken to be normal to the hydrostat in principal stress space.

Introduction

Soils and rocks are common civil engineering materials and are becoming more important as earthquake engineering becomes more important in the eyes of the public. Elastic-plastic models are being applied to these materials. Drucker[1] applied plasticity theory to rocks in 1952. Coon[2] used a hyperelastic model with a yield surface similar to that presented here. Nelson, et al,[3] discusses several methods of modeling concrete and rocks. Among other models, an elastic-plastic model is presented which has a yield or loading surface composed of a paraboloid-like surface of revolution capped by an ellipsoid of revolution which moves. In the present work, the same type of approach is used with some modifications for ease in implementation in a computer program.

This report is in three parts. The first part is a description of the plasticity model used. The second part is a detailed description of how the plasticity equations are solved. The third part is a description of cellular concrete and the application of the present model to the data.

The appendices contain derivations of various pieces of information needed in the model as well as a FORTRAN subroutine for implementing the model in the computer program GOLLY [4], a version of the program by Farhoomand, et al [5].

Plasticity Model

For small strains, both tensile and compressive, cellular concrete is linear elastic but as the strains become larger, nonlinear stress-strain behavior is noted and permanent set occurs. This

permanent set is associated with collapse of the cells. The failure of the cells in compression, however, is accompanied by development of cohesive strength so that a finite shear strength remains while cells are being crushed and this shear strength grows as the pressure increases.

This behavior can be modeled with a yield surface in principal stress space which is a surface of revolution centered on the hydrostat and increasing in radius with increasing pressure. With the meager data available, the approximation of the yield surface can be taken to be a paraboloid* with a planar cap on the normally open end. For generality, a quadratic pressure variation is used.

The yield surface can be expressed as

$$\varphi = (p-f)[J_2 - (a_0+a_1p+a_2p^2)] = 0 \tag{1}$$

where p is pressure, a and h are constants, f is a function of the mean total strain, and J_2 is the second invariant of the stress deviator.**

$$J_2 = \frac{1}{2}\,\underset{\sim}{s} \cdot \underset{\sim}{s} \tag{2}$$

If the stress vector $\underset{\sim}{\sigma}$ is written as $\{\sigma_{11},\sigma_{22},\sigma_{33},\sigma_{12},\sigma_{13},\sigma_{23},\sigma_{21},\sigma_{31},\sigma_{32}\}^T$ and a vector $\underset{\sim}{n}$ is defined as $\{1,1,1,0,0,0,0,0,0\}^T$ then the stress deviator is defined as

$$\underset{\sim}{s} = \underset{\sim}{\sigma} + p\underset{\sim}{n} \tag{3a}$$

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*A general quadratic surface of revolution is actually used in the development although a paraboloid ($a_2 = 0$) fits the data very well and will be used to describe the surface in the text.

**If the stress and strain vectors are expressed in nine-dimensional stress space, then difficulties expressed by Shield and Ziegler [6] are skirted.

where the pressure is related to the stress thru the relation:

$$p = -\mathfrak{n} \cdot \mathfrak{s}/3 \tag{3b}$$

The yield function can be kept in the form stated in Eq. (1), however the theory is easier to explain and solve if two functions are defined, one describing the paraboloid and the other describing a plane which is normal to the hydrostat.

$$\begin{aligned} \varphi_s &= J_2 - (a_0 + a_1 p + a_2 p^2) \\ \varphi_p &= p - f \end{aligned} \tag{4}$$

The strain rate is now decomposed into the mean strain rate, $\dot{\gamma}_T$, and deviatoric strain rate, $\dot{\mathfrak{e}}_T$.

$$\begin{aligned} \dot{\gamma}_T &= \dot{\mathfrak{e}}_T \cdot \mathfrak{n}/3 \\ \dot{\mathfrak{e}}_T &= \dot{\mathfrak{e}}_T - \dot{\gamma}_T \mathfrak{n} \end{aligned} \tag{5}$$

The plasticity theories for the volumetric and deviatoric parts are now taken to be completely independent. The volumetric part is very straightforward and treated first.

The first requirement to insure that the stress state is inside the yield surface is that tensile fracture has not occurred.

$$p \geq h \tag{5}$$

Now the next step is to find φ_p. If

$$\varphi_p = 0 \text{ and } \dot{\varphi}_p = 0 \tag{6}$$

then volumetric plasticity is occurring.

The volumetric plasticity can now be handled with either incremental or deformation theory plasticity. In either case, the response is elastic if Eqs. (6) are not satisfied so that Eqn. (7) is written as

$$\dot{p} = -3K_0\dot{\gamma}_T \tag{7}$$

If Eqs. (6) are satisfied, then Eqn. $(4)_2$ is differentiated to give:

$$\left.\begin{aligned} \dot{p} &= f'\dot{\gamma}_T && \text{for } \varphi_p = 0,\ \dot{p} > 0 \\ &= -3K_0\dot{\gamma}_T && \text{for } \varphi_p < 0 \\ & && \text{or } \varphi_p = 0,\ \dot{p} < 0 \\ &= 0 && \text{for } p = h,\ \dot{p} < 0 \end{aligned}\right\} \tag{8}$$

Time integration of $\dot{p}$ gives the pressure for this incremental theory.

Deformation theory plasticity can also be stated and is exactly equivalent to the above incremental theory.

$$\left.\begin{aligned} p &= f(\gamma_T) && \text{for } \varphi_p = 0,\ \dot{\gamma}_T \le 0 \\ &= p(t_u) - 3K_0\int_{t_u}^{t}\dot{\gamma}_T dt && \text{for } \varphi_p < 0 \\ &= h && \text{for } p = h,\ \dot{\gamma}_T > 0 \end{aligned}\right\} \tag{9}$$

where t_u is the most recent time that the conditions of Eqn. (9) were met. Initially, t_u is zero. Condition $(9)_2$ is identical to Eqn. (7).

The equivalence of Eqs. (8) and (9) is shown by differentiation of Eqs. (9). The equivalence of incremental and deformation theories for proportional loading is also discussed by Budiansky[4]. In a practical sense, the deformation theory requires time integration of $\dot{\gamma}_T$, while the incremental theory requires time integration of $f'\dot{\gamma}_T$. Since raw data for a user is in the form of $f(\gamma_T)$, the incremental theory will also require differentiation of this experimental data in order to find f'.

The deviatoric part uses almost conventional plasticity theory. If $\varphi_s < 0$ or $\varphi_s = 0$ and $\dot{\varphi}_s < 0$, then the material is assumed to behave elastically in the deviatoric mode, i.e.,

$$\dot{\underset{\sim}{s}} = 2G\,\dot{\underset{\sim}{\varepsilon}}$$

If however the yield surface is reached and loading is taking place, i.e., if

$$\varphi_s = 0$$

$$\text{and} \qquad \frac{\partial \varphi_s}{\partial \underset{\sim}{s}} \cdot \dot{\underset{\sim}{s}} \geq 0 \tag{10}$$

then a deviatoric plasticity process must be described.

The elastic deviatoric strainrate vector is defined as

$$\dot{\varepsilon}_e = \dot{\underset{\sim}{s}}/2G \tag{11}$$

The plastic deviatoric strainrate vector is defined as

$$\dot{\varepsilon}_p = \dot{\varepsilon}_T - \dot{\varepsilon}_e \tag{12}$$

The plastic strainrate vector is now assumed to lie in a direction normal to the yield surface φ_s in the eight dimensional <u>deviatoric</u>

stress space, a subspace of the nine-dimensional stress space. This is stated in equation form as

$$\dot{\underset{\sim}{\varepsilon}} = \lambda \frac{\partial \varphi_s}{\partial \underset{\sim}{s}} \tag{13}$$

where λ is a constant to be determined using the loading condition of Eqn. $(10)_2$ where Eqn. (4) is used.

$$\dot{\varphi}_s = \frac{\partial \varphi_s}{\partial \underset{\sim}{s}} \cdot \dot{\underset{\sim}{s}} + \frac{\partial \varphi_s}{\partial p} \dot{p} = 0$$

Equation (2) is used to give

$$\frac{\partial \varphi_s}{\partial \underset{\sim}{s}} = \frac{\partial J_2}{\partial \underset{\sim}{s}} = \underset{\sim}{s} \tag{14}$$

This result is used together with Eqn. $(4)_1$ to give:

$$\underset{\sim}{s} \cdot \dot{\underset{\sim}{s}} = (a_1 + 2a_2 p)\dot{p} \tag{15}$$

An expression for $\dot{\underset{\sim}{s}}$ is now found from Eqns. (11), (12), and (13) as

$$\dot{\underset{\sim}{s}} = 2G(\underset{\sim}{\varepsilon}_T - \lambda \underset{\sim}{s}) \tag{16}$$

This is substituted into Eqn. (15) and the result solved for λ.

$$\lambda = \frac{\underset{\sim}{s} \cdot \dot{\underset{\sim}{\varepsilon}} - (a_1 + 2a_2 p)\frac{\dot{p}}{2G}}{\underset{\sim}{s} \cdot \underset{\sim}{s}} \tag{17}$$

Equation (17) can be substituted into Eqn. (16) to find the deviatoric stress rate.

$$\dot{\underset{\sim}{s}} = 2G\,\dot{\underset{\sim}{\varepsilon}}_T - \left[\frac{2G\,\underset{\sim}{s} \cdot \dot{\underset{\sim}{\varepsilon}}_T - (a_1 + 2a_2 p)\dot{p}}{\underset{\sim}{s} \cdot \underset{\sim}{s}}\right]\underset{\sim}{s} \tag{18}$$

Discussion

One rather disturbing aspect of Eqs. (17) and (18) is that the deviatoric stress rate depends not only upon the deviatoric strainrate $\dot{\underset{\sim}{\varepsilon}}_T$ but also on the pressure rate. This, however, can be explained geometrically as follows. The stress state is initially at some point A on the yield surface and remote from the end cap. This is illustrated in Figure 1. The stress rate might be such as to move the state to point B. This requires a change in pressure $-(\Delta p)\underset{\sim}{n}$ and a change in deviatoric stress $\underset{\sim}{\Delta s}$. The yield surface has a larger radius at B (measured on a pi plane thru B) than at A and, hence, the deviatoric process is partially elastic in order to allow for the expanding deviatoric yield surface. Conversely, a decreasing pressure will result in more deviatoric plasticity than ordinarily encountered.

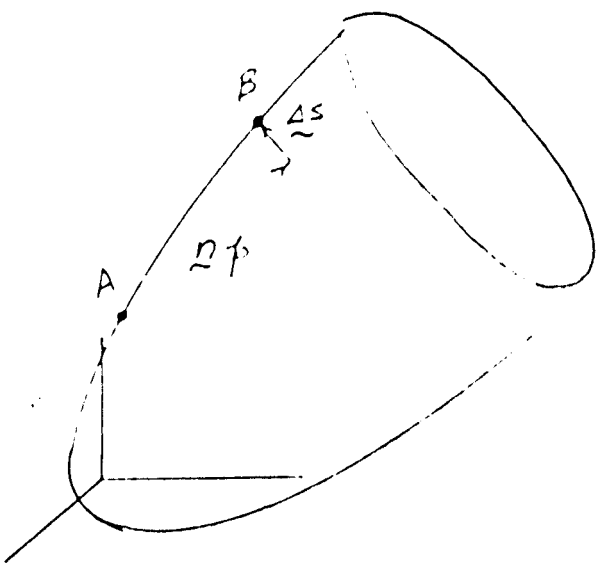

Figure 1. Yield surface in principal stress space.

This movement along the deviatoric yield surface is not accompanied by permanent volume change. This is shown in Appendix I. Permanent volume change only occurs during the volumetric plasticity. This behavior is accomplished by making the deviatoric plastic strainrate vector normal to the yield surface in the deviatoric subspace.

If it were taken to be normal to the yield surface in the full stress space, then permanent volumetric change would occur and would always be such as to increase in volume no matter which direction the loading is taking place. This is also shown in Appendix I.

Thus, Eqns. (17) and (18) appear to be reasonable and not inconsistent with physical behavior.

Details of Implementation in a Code

The mathematical model which has been presented must be implemented to be useful. There are many details in the implementation which are almost as important as the model itself. In this section an attempt is made to cover some of these details. A listing of a FORTRAN subroutine is given in Appendix II which contains the suggestions outlined here. It is for the program GOLLY[4,5] where $\dot{\epsilon}_{13} = \dot{\epsilon}_{23} = 0$ and engineering shear strain is used.

Assume that some stress state ${}^{o}\underset{\sim}{\sigma}$ and a strainrate $\dot{\underset{\sim}{\epsilon}}_T$ exist at some time together with the value of Δt. The new stress state ${}'\underset{\sim}{\sigma}$ is to be found at the time $t_1 = t_0 + \Delta t$.

The first step is to decompose the stress and strainrate into deviatoric and spherical parts using Eqs. (3b) and (3a) and Eqn. (5).

$$ {}^{o}p = -({}^{o}\sigma_{11} + {}^{o}\sigma_{22} + {}^{o}\sigma_{33})/3 \tag{19a} $$

$$ \left.\begin{array}{ll} {}^{o}s_{11} = {}^{o}\sigma_{11} + {}^{o}p \;\; ; & {}^{o}s_{12} = {}^{o}s_{21} = {}^{o}\sigma_{12} \\ {}^{o}s_{22} = {}^{o}\sigma_{22} + {}^{o}p \;\; ; & {}^{o}s_{13} = {}^{o}s_{31} = {}^{o}\sigma_{13} \\ {}^{o}s_{33} = {}^{o}\sigma_{33} + {}^{o}p \;\; ; & {}^{o}s_{23} = {}^{o}s_{32} = {}^{o}\sigma_{23} \end{array}\right\} \tag{19b} $$

$$\dot{\gamma}_T = (\dot{e}_{11} + \dot{e}_{22} + \dot{e}_{33}) \tag{20a}$$

$$\left.\begin{aligned} \dot{\epsilon}_{T_{11}} &= \dot{e}_{T_{11}} - \dot{\gamma}_T/3 & \dot{\epsilon}_{T_{12}} &= \dot{\epsilon}_{T_{21}} = \dot{e}_{T_{12}} \\ \dot{\epsilon}_{T_{22}} &= \dot{e}_{T_{22}} - \dot{\gamma}_T/3 & \dot{\epsilon}_{T_{13}} &= \dot{\epsilon}_{T_{31}} = \dot{e}_{T_{13}} \\ \dot{\epsilon}_{T_{33}} &= \dot{e}_{T_{33}} - \dot{\gamma}_T/3 & \dot{\epsilon}_{T_{23}} &= \dot{\epsilon}_{T_{32}} = \dot{e}_{T_{23}} \end{aligned}\right\} \tag{20b}$$

The volumetric part is handled first. A time integration of the volumetric strainrate is made using a simple integrator.

$$'\gamma_T = {}^o\gamma_T + \dot{\gamma}_T \, \Delta t \tag{21}$$

The volumetric process is assumed to be elastic which would give, using Eqn. (7):

$$'\bar{p} = {}^o p - K_o \dot{\gamma}_T \Delta t \tag{22}$$

A check is first made to find whether Eqn. (5) has been violated.

$$\text{If } '\bar{p} \geq h \text{ then } '\bar{p} = h \tag{23a}$$

The stress state is then at the tip of the surface of revolution and the deviatoric stress must be zero.*

$$'s_{ij} = 0 \quad , \quad \text{all } i,j = 1, 2, 3 \tag{23b}$$

For this case the stiffness matrix is defined as:

$$E_{11} = E_{12} = E_{13} = E_{21} = E_{22} = E_{23} = E_{31} = E_{32} = E_{33} = \frac{{}^o p - {}'p}{\dot{\gamma}_T \Delta t} \tag{23c}$$

---

*This simple fracture model certainly leaves much to be desired. In a gross sense it models the stress release and subsequent healing which are thought to exist in the real material.

and all other $E_{ij} = 0$.

This is taken from Eqn. (II.6) of Appendix II.

If the inequality of Eqn. (23a) is not satisfied, then a test is made to find whether the stress state is on the planar end cap.

$$\text{If } {}'\gamma_T \geq \gamma_u \tag{24a}$$

where γ_u is the total mean strain value at the time when the end cap was last reached, then the volumetric process is elastic. The new pressure state and volumetric part of the stiffness matrix are found using Eqs. (22) and (II.6).

$$'p = '\bar{p} \tag{24b}$$

$$\check{E}_{11} = \check{E}_{12} = \check{E}_{13} = \check{E}_{21} = \check{E}_{22} = \check{E}_{23} = \check{E}_{31} = \check{E}_{32} = \check{E}_{33} = K_o \tag{24c}$$

$$\check{E}_{ij} = 0, \text{ all other } ij$$

The deviatoric stress is then found.

If the inequality of Eqn. (24a) is violated, then volumetric plasticity is occurring and the value of γ_u is updated.

$$\gamma_u = {}'\gamma_T \tag{25a}$$

The pressure state is found using Eqn. (9), and the volumetric stiffness from Eqn. (II.6).

$$'p = f('\gamma_T) \tag{25b}$$

$$\check{E}_{11} = \check{E}_{12} = \check{E}_{13} = \check{E}_{21} = \check{E}_{22} = \check{E}_{23} = \check{E}_{31} = \check{E}_{32} = \check{E}_{33} = \frac{{}^{o}p - {}'p}{3\eta_v \Delta t} \tag{25c}$$

and $\check{E}_{ij} = 0$, all other ij

The deviatoric stress is found next.

A trial elastic deviatoric stress vector is found using Eqn. (11) and the yield function $\bar{\varphi}_s$ evaluated* using Eqs. (4) and (2).

$$'\bar{s}_{ij} = {}^{o}s_{ij} + \Delta s_{ij} \quad ; \quad i,j = 1, 2, 3 \tag{26}$$

where:

$$\Delta s_{11} = 2G\dot{\varepsilon}_{11}\Delta t \qquad \Delta s_{12} = G\dot{\varepsilon}_{12}\Delta t$$

$$\Delta s_{22} = 2G\dot{\varepsilon}_{22}\Delta t \qquad \Delta s_{13} = G\dot{\varepsilon}_{13}\Delta t$$

$$\Delta s_{33} = 2G\dot{\varepsilon}_{33}\Delta t \qquad \Delta s_{23} = G\dot{\varepsilon}_{23}\Delta t$$

$$\bar{J}_2 = \frac{1}{2}\left[('\bar{s}_{11})^2 + ('\bar{s}_{22})^2 + ('\bar{s}_{33})^2 + 2('\bar{s}_{12})^2 + 2('\bar{s}_{13})^2 + 2('\bar{s}_{23})^2\right] \tag{27}$$

$$\bar{\varphi}_s = \bar{J}_2 - \frac{1}{2}k^2 \tag{28a}$$

where

$$k^2 = 2(a_0 + a_1{}'p + a_2('p)^2) \tag{28b}$$

A test is made to determine whether the trial stress state lies inside the yield surface.

*Engineering shear strains are used here. If tensor strains were used, then all stress deviator increment components would be

$$\Delta s_{ij} = 2G\dot{\varepsilon}_{ij}\Delta t$$

$$\text{If } \dot{\phi}_s \le 0 \tag{29a}$$

then the process was elastic.

$$'s_{11} = '\bar{s}_{11}\ ,\ 's_{12} = '\bar{s}_{12}\ ,\ \ldots\ ,\ 's_{33} = '\bar{s}_{33} \tag{29b}$$

The deviatoric stiffness matrix* is taken from Appendix II to be

$$\overset{s}{[E]} = \begin{bmatrix} \frac{4}{3}G & -\frac{2}{3}G & -\frac{2}{3}G & 0 & 0 & 0 \\ -\frac{2}{3}G & \frac{4}{3}G & -\frac{2}{3}G & 0 & 0 & 0 \\ -\frac{2}{3}G & -\frac{2}{3}G & \frac{4}{3}G & 0 & 0 & 0 \\ 0 & 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & 0 & G \end{bmatrix} \tag{29c}$$

If the inequality expressed by Eqn. (29a) is violated then the deviatoric process was at least partially plastic. The process is assumed to be partially elastic and partially plastic. The portion ($k\Delta t$) of the time step required to move back to the yield surface along the assumed elastic path is found to be:

---

*If tensor shear strains are used rather than engineering shear strains, the lower diagonal terms become 2G rather than G.

$$k = \frac{{}'\underset{\sim}{s} \cdot \Delta\underset{\sim}{s} - \sqrt{({}'\underset{\sim}{\bar{s}} \cdot \Delta\underset{\sim}{s})^2 - 2\bar{\varphi}_s \Delta\underset{\sim}{s} \cdot \Delta\underset{\sim}{s}}}{\Delta\underset{\sim}{s} \cdot \Delta\underset{\sim}{s}} \tag{30a}$$

where:

$$'\underset{\sim}{s} \cdot \Delta\underset{\sim}{s} = {}'\bar{s}_{11}\Delta s_{11} + {}'\bar{s}_{22}\Delta s_{22} + {}'\bar{s}_{33}\Delta s_{33} + 2'\bar{s}_{12}\Delta s_{12} + 2'\bar{s}_{13}\Delta s_{13} + 2's_{23}\Delta s_{23} \tag{30b}$$

$$\Delta\underset{\sim}{s} \cdot \Delta\underset{\sim}{s} = (\Delta s_{11})^2 + (\Delta s_{22})^2 + (\Delta s_{33})^2 + 2(\Delta s_{12})^2 + 2(\Delta s_{13})^2 + 2(\Delta s_{23})^2 \tag{30c}$$

The "contact" stress at the yield surface* is given as:

$$^{c}s_{ij} = {}'\bar{s}_{ij} - k\Delta s_{ij} \qquad i, j = 1, 2, 3 \tag{30d}$$

The plasticity requires some explanation here. The stress state at time t_0 is shown plotted at point 0 in Figure 2. During the time interval Δt, the pressure state is moved to position A. The elastic deviatoric increment $\Delta\underset{\sim}{s}$ would tend to locate the stress state $'\underset{\sim}{\bar{s}}$ shown at point B, but the "contact" stress $^{c}\underset{\sim}{s}$ is found instead at point C. The stress state then moves circumferentially around the paraboloid during the deviatoric plasticity process. This last stress "change" is independent

*Actually $^{c}s_{ij}$ need not be found. It is simply expressed for clarity.

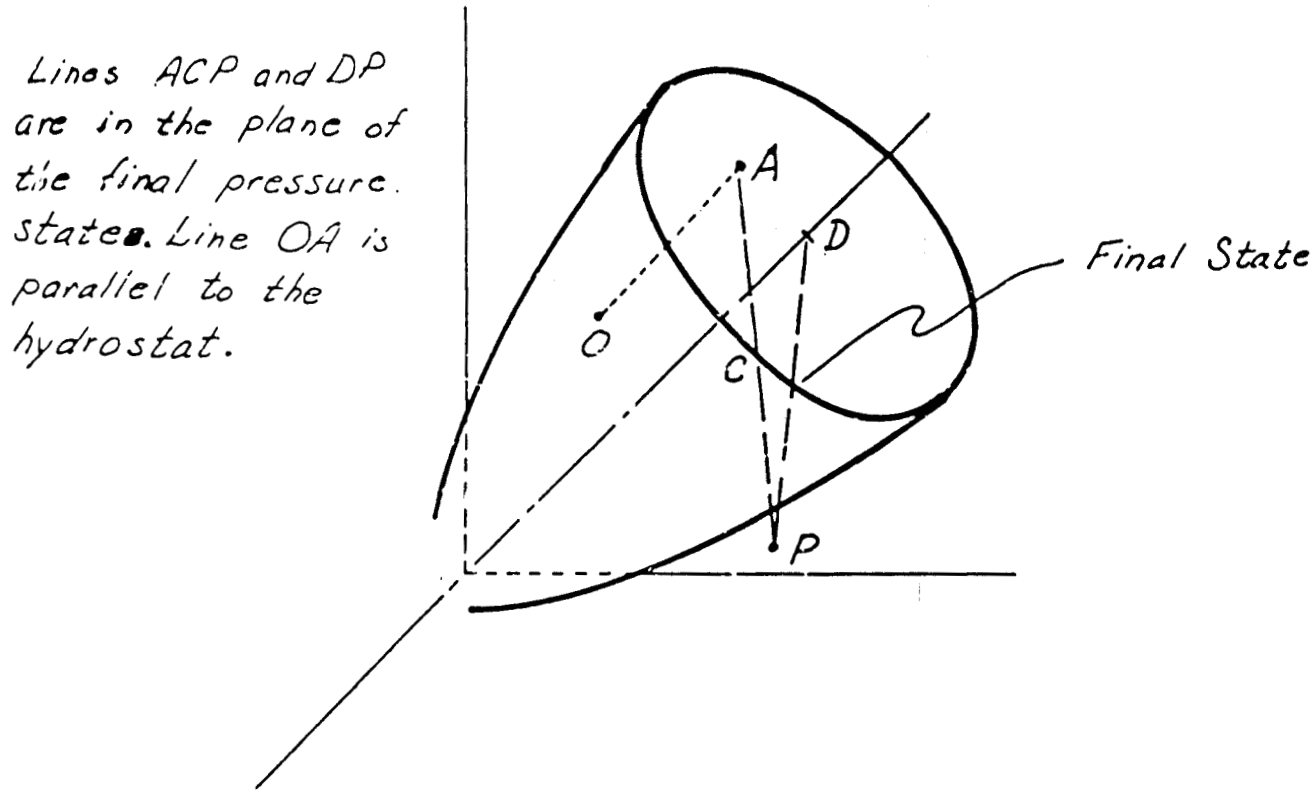

Figure 2. Stress states during the plasticity process. Initial stress state at 0, elastic pressure process to point A, contact stress at C, pseudo stress state at P and final stress state back at the plane through 'p.

of any pressure changes required to arrive at pressure 'p. Equation (18) for the deviatoric stress rate includes a pressure rate term. The purpose of this term is to account for the decreasing radius of the cross section in the above problem. Since this radius change has already been taken into account by the sequential spherical-deviatoric process, it must be dropped here. It must be included in the material stiffness matrix evaluation if it is required.

The plasticity process is now carried out over the remaining time ($k\Delta t$) from the contact stress state ${}^{c}s_{ij}$. In Appendix III the plasticity equations are integrated exactly using a constant $\dot{\epsilon}_T$ over a time increment Δt starting from the yield surface. The required parameters are:

$$\cos\psi = \frac{{}^{c}\underset{\sim}{s} \cdot k\Delta\underset{\sim}{s}}{|{}^{c}\underset{\sim}{s}|\ |k\Delta\underset{\sim}{s}|}$$

$$= \frac{{}'\underset{\sim}{s} \cdot \Delta\underset{\sim}{s} - k\ \Delta\underset{\sim}{s} \cdot \Delta\underset{\sim}{s}}{\sqrt{\Delta\underset{\sim}{s} \cdot \Delta\underset{\sim}{s}}\ R} \tag{31a}$$

where Eqn. (30d) has been used and

$$c = \exp\left(-\frac{k\sqrt{\Delta\underset{\sim}{s} \cdot \Delta\underset{\sim}{s}}}{R}\right) \tag{31b}$$

The reduction factor is then found as

$$r = \frac{R}{2c\ k\sqrt{\Delta\underset{\sim}{s} \cdot \Delta\underset{\sim}{s}}}\left[1-c^2 + (1-c)^2\cos\psi_0\right] \tag{31c}$$

The pseudo-stress state is then:*

$${}^{p}s_{ij} = {}^{c}s_{ij} + r\,\Delta s_{ij}$$

$$= {}'\bar{s}_{ij} + (r-k)\Delta s_{ij} \tag{31d}$$

*The pseudo-stress state ${}^{p}\underset{\sim}{s}$ need never be calculated.

The final stress state is found by bringing the stress radially back to the yield surface.

$$\frac{1}{2}\left({}^{m}P_{\underset{\sim}{s}}\right) \cdot \left({}^{m}P_{\underset{\sim}{s}}\right) = \frac{1}{2} R^2$$

$$m = \frac{R}{\sqrt{{}^{P}\underset{\sim}{s} \cdot {}^{P}\underset{\sim}{s}}} \tag{31e}$$

Equation (31d) is used to expand ${}^{P}\underset{\sim}{s}$ to give

$${}^{P}\underset{\sim}{s} \cdot {}^{P}\underset{\sim}{s} = {}'\underset{\sim}{\bar{s}} \cdot {}'\underset{\sim}{\bar{s}} + 2(r-k)\underset{\sim}{\Delta s} \cdot {}'\underset{\sim}{\bar{s}} + (r-k)^2 \underset{\sim}{\Delta s} \cdot \underset{\sim}{\Delta s} \tag{31f}$$

The final stress state is then:

$${}'s_{ij} = m\left({}'\bar{s}_{ij} + (r-k)\Delta s_{ij}\right) \tag{31g}$$

In summary, if the inequality of Eqn. (29a) is violated, then evaluate the following quantities in the order stated.

${}'\underset{\sim}{\bar{s}} \cdot \underset{\sim}{\Delta s}$	from Eqn. (30b)
$\underset{\sim}{\Delta s} \cdot \underset{\sim}{\Delta s}$	from Eqn. (30c)
k	from Eqn. (30a)
$\cos \phi_0$	from Eqn. (31a)
c	from Eqn. (31b)
r	from Eqn. (31c)
${}^{P}\underset{\sim}{s} \cdot {}^{P}\underset{\sim}{s}$	from Eqn. (31f)
m	from Eqn. (31e)
${}'s_{ij}$	from Eqn. (31g).

This sequence of nine easy steps gives an exact final stress state for the elastic-plastic deviatoric process under the assumption of constant $\dot{\underset{\sim}{\varepsilon}}_T$ over the time increment Δt. If a deviatoric stiffness matrix at the end of the time step is needed, then Eqn. (II.8b) is used to evaluate $[\overset{s}{E}]$ where the stress state $\overset{\prime}{\underset{\sim}{S}}$ is used in evaluation of the terms.

The final stress state at time t_1 is found from equations similar to Eqn. (19).

$$\left.\begin{array}{ll} \overset{\prime}{\sigma}_{11} = \overset{\prime}{s}_{11} - \overset{\prime}{p} & \overset{\prime}{\sigma}_{12} = \overset{\prime}{s}_{12} \\ \overset{\prime}{\sigma}_{22} = \overset{\prime}{s}_{22} - \overset{\prime}{p} & \overset{\prime}{\sigma}_{13} = \overset{\prime}{s}_{13} \\ \overset{\prime}{\sigma}_{33} = \overset{\prime}{s}_{33} - \overset{\prime}{p} & \overset{\prime}{\sigma}_{23} = \overset{\prime}{s}_{23} \end{array}\right\} \tag{32}$$

The final material stiffness matrix is given by

$$[E] = [\overset{v}{E}] + [\overset{s}{E}] \tag{33}$$

For codes which require the matrix [E], the stress may be found at time t_1 through an equation of the form

$$\overset{\prime}{\underset{\sim}{\sigma}} = \overset{o\wedge}{\underset{\sim}{\sigma}} + [E]\, \dot{\underset{\sim}{\varepsilon}}_T\, \Delta t$$

If this is the case, then the false stress state $\overset{o\wedge}{\underset{\sim}{\sigma}}$ must be found. This is given as:

$$\begin{bmatrix} \overset{\circ}{\sigma}_{11} \\ \overset{\circ}{\sigma}_{22} \\ \overset{\circ}{\sigma}_{33} \\ \overset{\circ}{\sigma}_{12} \\ \overset{\circ}{\sigma}_{13} \\ \overset{\circ}{\sigma}_{23} \end{bmatrix} = \begin{bmatrix} \sigma'_{11} \\ \sigma'_{22} \\ \sigma'_{33} \\ \sigma'_{12} \\ \sigma'_{13} \\ \sigma'_{23} \end{bmatrix} - [E] \begin{bmatrix} \dot{e}_{11} \\ \dot{e}_{22} \\ \dot{e}_{33} \\ \dot{e}_{12} \\ \dot{e}_{13} \\ \dot{e}_{23} \end{bmatrix} \Delta t \tag{34}$$

Conclusion

An elastic plastic model for cellular concrete has been presented together with details of implementation in a computer program. The yield surface is a surface of revolution centered on the hydrostat with the large end facing in the direction of increasing pressure. The large end is capped with a plane which is normal to the hydrostat and moves outward only. The spherical and deviatoric parts are handled separately, the pressure part with a deformation theory and the deviatoric part with an incremental theory. In the section on code implementation suggestions, the stress state can pass from an elastic to a plastic state during a time increment; the mixed process is handled exactly in both spherical and deviatoric parts. The process of deviatoric plasticity suggested for code implementation is an exact integration of the equations and has not appeared elsewhere. It is a considerable simplification over the usual methods in use today.

APPENDIX I

Volume Change Due to Surface Normality Condition

The plastic deviatoric strainrate vector was taken to be normal to the yield surface in the eight dimensional subspace of the deviatoric stress space rather than in the full nine dimensional space. In the eight space, no plastic volume change takes place. This is shown in this appendix and contrasted to the nine space result.

Equation (13) for plastic strainrate is written using Eqn. (17) for λ to give

$$\dot{\underset{\sim}{\varepsilon}}_p = \left(\frac{\underset{\sim}{s} \cdot \dot{\underset{\sim}{\varepsilon}}_T - \eta \dot{p}}{\underset{\sim}{s} \cdot \underset{\sim}{s}} \right) \underset{\sim}{s}$$

where $\eta = a_1 + 2a_2 p$.

The volumetric part of this strainrate is

$$\dot{\underset{\sim}{\varepsilon}}_p \cdot \underset{\sim}{n} = \left(\frac{\underset{\sim}{s} \cdot \dot{\underset{\sim}{\varepsilon}}_T - \eta \dot{p}}{\underset{\sim}{s} \cdot \underset{\sim}{s}} \right) \underset{\sim}{s} \cdot \underset{\sim}{n}$$

But from Eqn. (3a) we find

$$\begin{aligned} \underset{\sim}{s} \cdot \underset{\sim}{n} &= \underset{\sim}{\sigma} \cdot \underset{\sim}{n} + p\, \underset{\sim}{n} \cdot \underset{\sim}{n} \\ &= \underset{\sim}{\sigma} \cdot \underset{\sim}{n} - \left(\underset{\sim}{n} \cdot \underset{\sim}{\sigma}/3 \right) 3 \\ &= 0 \end{aligned}$$

where Eqn. (4) is used and $\underset{\sim}{n} \cdot \underset{\sim}{n} = 3$. Thus we have the result that

$$\dot{\underset{\sim}{\varepsilon}}_p \cdot \underset{\sim}{n} = 0$$

or there is no permanent volume change due to the plastic shearing process.

The full nine space analysis would have the following expression for $\dot{\bar{\underset{\sim}{\varepsilon}}}_p$.

$$\dot{\bar{\underset{\sim}{\varepsilon}}}_p = \bar{\lambda} \frac{\partial \varphi_s}{\partial \underset{\sim}{\sigma}}$$

$$= \bar{\lambda} \left(\frac{\partial \underset{\sim}{s}}{\partial \underset{\sim}{\sigma}} \frac{\partial \varphi_s}{\partial \underset{\sim}{s}} + \frac{\partial \varphi_s}{\partial p} \frac{\partial p}{\partial \underset{\sim}{\sigma}} \right)$$

$$= \bar{\lambda} \left(\underset{\sim}{s} + a\underset{\sim}{n}/3 \right)$$

$$= \left(\frac{\underset{\sim}{s} \cdot \dot{\underset{\sim}{\varepsilon}}_T - \eta \dot{p}}{\underset{\sim}{s} \cdot \underset{\sim}{s}} \right) \left(\underset{\sim}{s} + a\underset{\sim}{n}/3 \right)$$

where Eqns. $(4)_1$, (14), $(3)_1$ and $(3)_2$ are used to find the vector direction and Eqn. (15) is used as the starting point to find $\bar{\lambda}$.

As before, the volumetric part of this strainrate is

$$\dot{\bar{\underset{\sim}{\varepsilon}}}_p \cdot \underset{\sim}{n} = \left(\frac{\underset{\sim}{s} \cdot \dot{\underset{\sim}{\varepsilon}}_T - \eta \dot{p}}{\underset{\sim}{s} \cdot \underset{\sim}{s}} \right) \left(\underset{\sim}{s} \cdot \underset{\sim}{n} + \frac{a}{3} \underset{\sim}{n} \cdot \underset{\sim}{n} \right)$$

$$= \frac{a}{\underset{\sim}{s} \cdot \underset{\sim}{s}} \left(\underset{\sim}{s} \cdot \dot{\underset{\sim}{\varepsilon}}_T - \eta \dot{p} \right)$$

Except for the case where the plastic deviatoric strainrate vector $\dot{\bar{\underset{\sim}{\varepsilon}}}_p$ is identically zero ($\bar{\lambda} \equiv 0$) then there would always be a permanent volume change. It is also a peculiar volume change since it shows up in what was supposed to be a deviatoric component. This forces $\dot{\bar{\underset{\sim}{\varepsilon}}}_e$ to also have a volumetric part (which is recoverable) so that the volumetric part of $\dot{\underset{\sim}{\varepsilon}}_T$ is zero as defined.

A permanent volume increase due to shear is certainly not inconceivable. A densely packed granular material under finite shear might increase in volume since particle packing is disturbed. The above normality condition falls far short of explaining volume changes due to packing efficiency, however, since packing efficiency must remain within bounds. The normality condition produces no such bounds.

Since this type of volume change in cellular concrete is small compared to cell crushing, it is made zero in the mathematical model presented here.

APPENDIX II

Forms of Decomposed Stiffness Matrices

It is sometimes necessary to form a material stiffness matrix, $\underset{\approx}{E}$, such that

$$\dot{\underset{\sim}{\sigma}} = \underset{\approx}{E}\, \dot{\underset{\sim}{e}}_T \tag{II.1}$$

Due to the method of decomposition into volumetric and deviatoric parts, it is convenient to decompose $\underset{\approx}{E}$ also as

$$\underset{\approx}{E} = \overset{v}{\underset{\approx}{E}} + \overset{s}{\underset{\approx}{E}} \tag{II.2}$$

where

$$\dot{\underset{\sim}{s}} = \overset{s}{\underset{\approx}{E}}\, \dot{\underset{\sim}{\varepsilon}}_T \tag{II.3}$$

$$\dot{p}\underset{\sim}{n} = \overset{v}{\underset{\approx}{E}}\, \underset{\sim}{n}\, \dot{\gamma}_T \tag{II.4}$$

The matrix $\overset{s}{\underset{\approx}{E}}$ is found from the deviatoric material calculation and Eqn. (II.3) and the volumetric part is

$$\overset{v}{\underset{\approx}{E}} = k\, \underset{\sim}{n} \otimes \underset{\sim}{n} \tag{II.5}$$

where k is a constant which is found from the volumetric material calculation and Eqn. (II.4).

Note the following decomposition using Eqns. (II.1), (II.2), and $(5)_2$.

$$\dot{\underset{\sim}{\sigma}} = \left(\overset{v}{\underset{\approx}{E}} + \overset{s}{\underset{\approx}{E}}\right)\left(\dot{\epsilon}_T + \dot{\gamma}_T \underset{\sim}{n}\right)$$

$$= k\, \underset{\sim}{n} \otimes \underset{\sim}{n} \left(\dot{\epsilon}_T + \dot{\gamma}_T \underset{\sim}{n}\right) + \overset{s}{\underset{\approx}{E}} \left(\dot{\epsilon}_T + \dot{\gamma}_T \underset{\sim}{n}\right)$$

$$= k\, \dot{\gamma}_T \underset{\sim}{n} + \overset{s}{\underset{\approx}{E}}\, \underset{\sim}{\epsilon}_T + \overset{s}{\underset{\approx}{E}} \underset{\sim}{n}\, \dot{\gamma}_T$$

The properties $\underset{\sim}{n} \cdot \underset{\sim}{n}$ and $\underset{\sim}{n} \cdot \dot{\underset{\sim}{\epsilon}}_T = 0$ have been used. Now if $\overset{s}{\underset{\approx}{E}} \underset{\sim}{n} = \underset{\sim}{0}$ then the result using Eqns. (II.3), and (II.5) is

$$\dot{\underset{\sim}{\sigma}} = \dot{\underset{\sim}{s}} - \dot{p}\underset{\sim}{n}$$

The values of the entries in the matrices are found as follows. Eqns. (II.4) and (II.5) are used to give

$$k = -\dot{p}/\dot{\gamma}_T$$

or

$$\overset{v}{\underset{\approx}{E}} = -\frac{\dot{p}}{\dot{\gamma}_T}\, \underset{\sim}{n} \otimes \underset{\sim}{n} \tag{II.6}$$

This expression can be applied for either the elastic or plastic case.

The deviatoric stiffness matrix is not as straightforward. In the elastic case, the deviatoric stress rate is given as

$$\dot{\underset{\sim}{s}} = 2G\dot{\underset{\sim}{\epsilon}}$$

so that $\overset{s}{\underset{\sim}{E}} = 2G(\underset{\approx}{I} - \frac{1}{3}\underset{\sim}{n} \otimes \underset{\sim}{n})$ (II.7)

For the case where Eqn. (18) is used to find the deviatoric stressrate, the stiffness matrix is more involved. Equation (18) is rewritten as:

$$\dot{\underset{\sim}{s}} = \left[2G\left(\underset{\approx}{I} - \frac{1}{3}\,\underset{\sim}{n}\otimes\underset{\sim}{n}\right) - \frac{2G\;\underset{\sim}{s}\otimes\underset{\sim}{s}}{\underset{\sim}{s}\cdot\underset{\sim}{s}} + \frac{a\dot{p}\;\underset{\sim}{s}\otimes\underset{\sim}{\varepsilon}_T}{(\underset{\sim}{s}\cdot\underset{\sim}{s})(\dot{\underset{\sim}{\varepsilon}}_T\cdot\dot{\underset{\sim}{\varepsilon}}_T)}\right]\dot{\underset{\sim}{\varepsilon}}_T$$

By inspection, an acceptable form of the stiffness matrix is then:

$$\overset{s}{\underset{\approx}{E}} = 2G\left(\underset{\approx}{I} - \frac{1}{3}\,\underset{\sim}{n}\times\underset{\sim}{n}\right) - \frac{2G}{\underset{\sim}{s}\cdot\underset{\sim}{s}}\;\underset{\sim}{s}\otimes\underset{\sim}{s} + \frac{a\dot{p}\;\underset{\sim}{s}\otimes\dot{\underset{\sim}{\varepsilon}}_T}{(\underset{\sim}{s}\cdot\underset{\sim}{s})(\dot{\underset{\sim}{\varepsilon}}_T\cdot\dot{\underset{\sim}{\varepsilon}}_T)} \qquad \text{(II.8)}$$

For the case of a body of revolution where two of the shear strains are always zero, then the stiffness matrix need only be a 4 x 4 matrix and Eqns. (II.7) and (II.8) reduce to the following.

$$\overset{s}{\underset{\approx}{E}}_{\text{Elastic}} = 2G\begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{(II.9a)}$$

$$\overset{s}{\underset{\approx}{E}}_{\text{Plas}} = \begin{bmatrix} \frac{4}{3}G - bs_1s_1 + cs_1\dot{c}_1 & -\frac{2}{3}G\; bs_1s_2 + cs_1\dot{c}_2 & -\frac{2}{3}G - bs_1s_3 + cs_1\dot{c}_3 & -bs_1s_4 + cs_1\dot{c}_4 \\ -\frac{2}{3}G - bs_2s_1 + cs_2\dot{c}_1 & \frac{4}{3}G - bs_2s_2 + cs_2\dot{c}_2 & -\frac{2}{3}G - bs_2s_3 + cs_2\dot{c}_3 & -bs_2s_4 + cs_2\dot{c}_4 \\ -\frac{2}{3}G - bs_3s_1 + cs_3\dot{c}_1 & -\frac{2}{3}G - bs_3s_2 + cs_3\dot{c}_2 & \frac{4}{3}G - bs_3s_3 + cs_3c_3 & -bs_3s_4 + cs_3c_4 \\ -bs_4s_1 + cs_4\dot{c}_1 & -bs_4s_2 + cs_4\dot{c}_2 & -bs_4s_3 + cs_4\dot{c}_3 & -bs_4s_4 + cs_4\dot{c}_4 \end{bmatrix}$$

(II.9b)

where

$$b = 2G\left(s_1^2 + s_2^2 + s_3^2 + 2s_4^2\right)^{-1}$$

$$c = a\dot{p}\left(s_1^2 + s_2^2 + s_3^2 + 2s_4^2\right)^{-1}\left(\dot{\epsilon}_1^2 + \dot{\epsilon}_2^2 + \dot{\epsilon}_3^2 + 2\dot{\epsilon}_4^2\right)^{-1}$$

The volumetric part is:

$$\overset{v}{\underset{\approx}{E}} = -\frac{\dot{p}}{\dot{\gamma}_T}\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{(II.10)}$$

Equation (II.2) is used to combine Eqns. (II.10) and either (II.9a) or (II.9b). An observation which is useful here is that $\overset{s}{\underset{\approx}{E}}_{plas}$ can be expressed as the sum of $\overset{s}{\underset{\approx}{E}}_{elastic}$ and a matrix with entries:

$$\overset{s}{\underset{\sim}{E}}_{ij} = -bS_iS_j + cS_i\dot{\epsilon}_j \qquad \text{(II.11)}$$

This observation is used in the programmed version.

Each of the two terms of each entry in Eqn. (II.11) will require two multiplications. In the six space this will be a total of 42 multiplications involving b, and 72 multiplications involving c (where symmetry has been used). For the limited case where an approximate stiffness matrix is acceptable, the ratio of the relative sizes of the two terms can be calculated as:

$$\frac{\text{size of b terms}}{\text{size of c terms}} = \frac{2G(\underset{\sim}{s} \cdot \underset{\sim}{\dot{\epsilon}}_T)}{a\dot{p}}$$

If this ratio were large, then terms involving c would be dropped. If it were small, then all terms involving b would be dropped instead.

APPENDIX III

Integration of the Plastic Deviatoric Stress Rate

Equation (16) gives the deviatoric stress rate for a plastic process as

$$\dot{\underset{\sim}{s}} = 2G\left(\dot{\underset{\sim}{\varepsilon}}_T - \lambda \underset{\sim}{s}\right) \tag{III.1}$$

where λ is given by Eqn. (17). In the section on details of code implementation, the pressure is handled first and the deviatoric process is independent of the pressure. The pressure is only used to establish the radius of the hypersphere of the deviatoric von Mises surface. The value of λ of concern here does not involve the rate of pressure change so that the relevant form of Eqn. (17) is

$$\lambda = \frac{\underset{\sim}{s} \cdot \dot{\underset{\sim}{\varepsilon}}_T}{\underset{\sim}{s} \cdot \underset{\sim}{s}} \tag{III.2}$$

In a code, Eqns. (III.1) and (III.2) must be integrated over a single time step Δt during which the strainrate is kept fixed. The purpose here is to perform that integration in closed form.

An important observation is that the stress path from $\underset{\sim}{S}_0$ at the beginning of the time increment to $\underset{\sim}{S}_1$ at the end is a geodesic path in the direction of the local projection of $\dot{\underset{\sim}{\varepsilon}}_T$ on the hypersphere of the deviatoric yield surface. This observation will allow the necessary generalization at the end of this development.

For simplicity, picture a biaxial deviatoric stress and strainrate where no shears are developed. Then the inner product of the coplanar vectors $\underset{\sim}{s}$ and $\dot{\underset{\sim}{\varepsilon}}_T$ is

$$\underset{\sim}{s} \cdot \dot{\underset{\sim}{\varepsilon}}_T = |\dot{\underset{\sim}{\varepsilon}}_T| \; |\underset{\sim}{s}| \cos\psi \tag{III.3}$$

where ψ is the angle between them and where the magnitude of the stress vector is constant if the stress state always lies on the von Mises surface. From Eqns. $(4)_1$ and (2), the radius of the circle is taken as

$$|\underset{\sim}{s}| = \sqrt{\underset{\sim}{s} \cdot \underset{\sim}{s}} = \sqrt{2\left(a_0 + a_1 p + a_2 p^2\right)} = R \tag{III.4}$$

Since $\dot{\underset{\sim}{\varepsilon}}$ is considered to be a constant vector over the time span of interest here, the time derivative of $\underset{\sim}{s}$ may be found using Eqn. (III.3) and equated to the inner product of Eqn. (III.1) with $\dot{\underset{\sim}{\varepsilon}}_T$.

$$\dot{\underset{\sim}{s}} \cdot \dot{\underset{\sim}{\varepsilon}}_T = -|\dot{\underset{\sim}{\varepsilon}}_T| R\dot{\psi} \sin\psi = 2G \, \dot{\underset{\sim}{\varepsilon}}_T \cdot \dot{\underset{\sim}{\varepsilon}}_T - \frac{2G\left(\underset{\sim}{s} \cdot \dot{\underset{\sim}{\varepsilon}}_T\right)^2}{R^2}$$

Equation (III.3) and a trigonometric identity are used, both sides are divided by $-|\dot{\underset{\sim}{\varepsilon}}_T| R \sin\psi$ and the result is given as:

$$\dot{\psi} = -\frac{2G|\dot{\underset{\sim}{\varepsilon}}_T|}{R} \sin\psi \tag{III.5}$$

This equation is restated in integral form as:

$$\int_{\psi_0}^{\psi_1} \frac{d\psi}{\sin\psi} = -\frac{2G|\dot{\underset{\sim}{\varepsilon}}_T|}{R}\int_{t_0}^{t_0+\Delta t} dt$$

The integrals are evaluated and the result reduced to

$$\tan\frac{\psi_1}{2} = \exp\left(\frac{-2G|\dot{\underset{\sim}{\varepsilon}}_T|\Delta t}{R}\right)\tan\frac{\psi_0}{2}$$

If an elastic stress increment is defined as

$$\Delta\underset{\sim}{S} = 2G\,\dot{\underset{\sim}{\varepsilon}}_T\Delta t \tag{III.6}$$

then the result can be restated as

$$\tan\frac{\psi_1}{2} = \exp\left(-\frac{|\Delta\underset{\sim}{s}|}{|\underset{\sim}{s}_0|}\right)\tan\frac{\psi_0}{2} \tag{III.7}$$

The angle between $\underset{\sim}{\varepsilon}_T$ and $\underset{\sim}{s}$ is exponentially approaching zero with time. This is observed in the plots by Barr[8] of the plasticity process.

Equation (III.7) can now be used to find the final stress state. A simple method of finding a final stress state is the method whereby a pseudo stress state is found as

$$\hat{\underset{\sim}{s}}_1 = \underset{\sim}{s}_0 + f\Delta\underset{\sim}{s}$$

and the final state is simply a radial movement back to the yield surface, i.e.,

$$\bar{\underset{\sim}{s}}_1 = c_1\hat{\underset{\sim}{s}}_1 \text{ where } |\bar{\underset{\sim}{s}}_1| = R$$

Since the radial motion is such a simple operation, a solution is sought of the form

$$\underset{\sim}{s}_1 = \alpha\left(\underset{\sim}{s}_0 + \beta\Delta\underset{\sim}{s}\right) \tag{III.8}$$

Two scalar equations which allow evaluation of α and β are:

$$\underset{\sim}{s}_1 \cdot \underset{\sim}{s}_1 = R^2 = \alpha^2\left(R^2 + 2\beta\, \underset{\sim}{s}_0 \cdot \underset{\sim}{\Delta s} + \beta^2 \underset{\sim}{\Delta s} \cdot \underset{\sim}{\Delta s}\right)$$

and

$$\underset{\sim}{s}_1 \cdot \underset{\sim}{\Delta s} = R|\underset{\sim}{\Delta s}| \cos\psi_1 = \alpha\left(\underset{\sim}{s}_0 \cdot \underset{\sim}{\Delta s} + \beta \underset{\sim}{\Delta s} \cdot \underset{\sim}{\Delta s}\right) \qquad \text{(III.9)}$$

The first equation is solved for $\alpha\beta$ and substituted into the second equation. The trigonometric identity $\sin^2\psi = 1-\cos^2\psi$ is applied for both ψ_1 and ψ_0 and the equation solved for α.

$$\alpha = \frac{\sin\psi_1}{\sin\psi_0}$$

This is now substituted back into Eqn. (III.9) to give the result:

$$\beta = \frac{R}{|\Delta s|}\left(\frac{\sin\psi_0}{\tan\psi_1} - \cos\psi_0\right)$$

The trigonometric identity for $\tan\psi_1$ in terms of $\tan\frac{\psi_1}{2}$ is applied and then Eqn. (III.7) used to express β in terms of $\sin\psi_0$, $\cos\psi_0$ and $\tan\psi_0/2$. The double angle formula for $\tan\psi_0/2$ in terms of $\cos\psi_0$ is then applied and the result simplified to:

$$\beta = \frac{R}{2|\Delta s|}\left[1 - c^2 + (1-c)^2\cos\psi_0\right]\frac{1}{c} \qquad \text{(III.10)}$$

where:

$$c = \exp\left(-\frac{|\Delta s|}{R}\right) \qquad \text{(III.11)}$$

The entire process is pictured on the von Mises surface in principal stress space in Figure III.1. An initial stress state

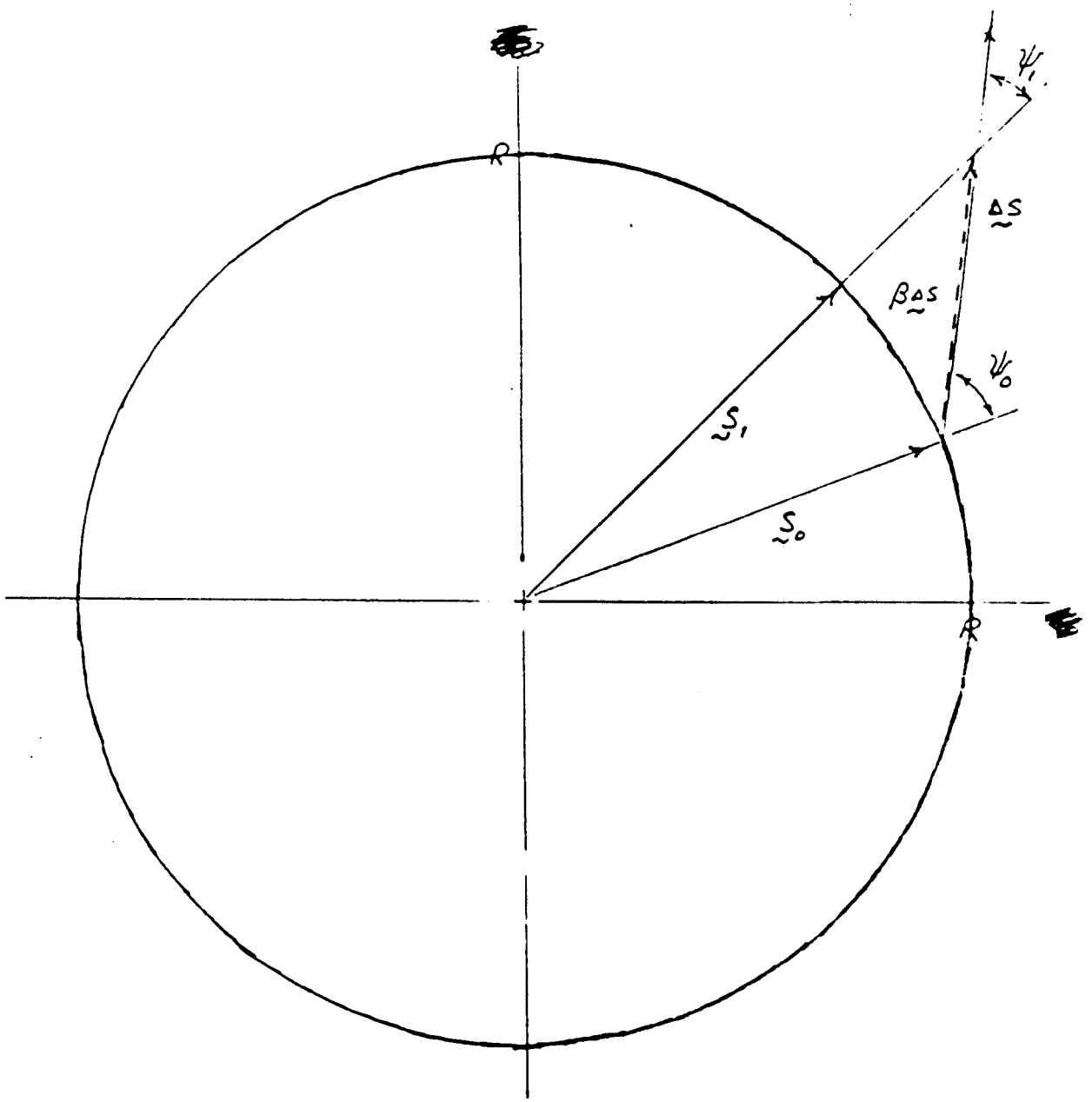

FIGURE III.1 . The

$\underset{\sim}{s}_0$ and an elastic stress increment $\Delta\underset{\sim}{s}$ are shown with some angle ψ_0 between them. The scalar β is determined from Eqn. (III.10) and the pseudo stress state $\underset{\sim}{s}_0 + \beta\Delta\underset{\sim}{s}$ found. This state is brought radially back to the yield surface at $\underset{\sim}{s}_1$.

Now to return back to the original problem where $\underset{\sim}{s}_0$ is some general state and $\Delta\underset{\sim}{s}$ some general increment. Since this cannot be visualized in a simple two-dimensional picture[1], the angle ψ becomes somewhat hazy. This is only a difficulty with our imagination but not with the mathematics. Equation (III.3) can still be used in the eight dimensional Cartesian subspace. The remaining development follows directly and can be considered abstractly with no recourse to geometrical visualization.

In particular, the pseudo stress state $\underset{\sim}{s}_0+\beta\Delta\underset{\sim}{s}$ and the subsequent radial path back to the yield surface are only crutches to simplify calculations. The true stress path was actually along the yield surface. The answers determined are exact under the assumption of a constant $\dot{\underset{\sim}{\varepsilon}}_T$ over the time increment Δt.

1. A picture of a moving stress state in a principal stress space is difficult to illustrate, since the changing orientation of the axes must somehow be included.

$$
\underset{\approx}{E}^{s}_{Plas} = \begin{bmatrix}
\frac{4}{3}G - bs_1s_1 + cs_1\dot{\epsilon}_1 & -\frac{2}{3}G\ bs_1s_2 + cs_1\dot{\epsilon}_2 & -\frac{2}{3}G - bs_1s_3 + cs_1\dot{\epsilon}_3 & -bs_1s_4 + cs_1\dot{\epsilon}_4 \\
-\frac{2}{3}G - bs_2s_1 + cs_2\dot{\epsilon}_1 & \frac{4}{3}G - bs_2s_2 + cs_2\dot{\epsilon}_2 & -\frac{2}{3}G - bs_2s_3 + cs_2\dot{\epsilon}_3 & -bs_2s_4 + cs_2\dot{\epsilon}_4 \\
-\frac{2}{3}G - bs_3s_1 + cs_3\dot{\epsilon}_1 & -\frac{2}{3}G - bs_3s_2 + cs_3\dot{\epsilon}_2 & \frac{4}{3}G - bs_3s_3 + cs_3\epsilon_3 & -bs_3s_4 + cs_3\epsilon_4 \\
-bs_4s_1 + cs_4\dot{\epsilon}_1 & -bs_4s_2 + cs_4\dot{\epsilon}_2 & -bs_4s_3 + cs_4\dot{\epsilon}_3 & -bs_4s_4 + cs_4\dot{\epsilon}_4
\end{bmatrix}
$$

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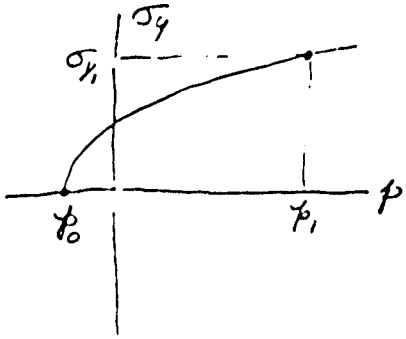
The equation has the form:

$$J_2 = a_0 + a_1 p + a_2 p^2 = \frac{1}{3}\sigma_y^2$$

This can represent the following curves and would be fitted as shown.

Parabola



$$\sigma_y^2 = C_1 + C_2 p \Rightarrow \frac{1}{3}\sigma_y^2 = \frac{C_1}{3} + \frac{C_2}{3} p$$

$$C_1 = -C_2 p_0$$

$$\sigma_{y_1}^2 = C_1 + C_2 p_1$$
$$= C_2 (p_1 - p_0)$$

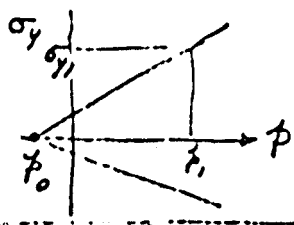
$$C_2 = \frac{\sigma_{y_1}^2}{p_1 - p_0}$$

$$C_1 = \frac{-p_0 \sigma_{y_1}^2}{p_1 - p_0}$$

note $p_0 \leq 0$

$$\boxed{a_0 = \frac{-p_0 \sigma_{y_1}^2}{3(p_1 - p_0)} \qquad a_1 = \frac{\sigma_{y_1}^2}{3(p_1 - p_0)}}$$

Cone



$$|\sigma_y| = C_1 + C_2 p \Rightarrow \frac{1}{3}\sigma_y^2 = \frac{1}{3}\left[C_1^2 + 2C_1C_2 p + C_2^2 p^2\right]$$

$$C_1 = -C_2 p_0 \qquad a_0 = \frac{1}{3}C_1^2$$

$$\sigma_{y_1} = C_1 + C_2 p_1 \qquad a_1 = \frac{2}{3}C_1C_2$$

$$C_1 = \frac{-p_0 \sigma_{y_1}}{p_1 - p_0} \qquad a_2 = \frac{1}{3}C_2^2$$

$$C_2 = \frac{\sigma_{y_1}}{p_1 - p_0}$$

$$\boxed{a_0 = \frac{p_0^2 \sigma_{y_1}^2}{3(p_1 - p_0)^2} \qquad a_1 = \frac{-2p_0 \sigma_{y_1}^2}{3(p_1 - p_0)^2} \qquad a_2 = \frac{\sigma_{y_1}^2}{3(p_1 - p_0)^2}}$$

note $p_0 \leq 0$